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Mathematics K–10 Literature Review
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This review was completed over a very short time-frame (about a month during which university resources were closed for one or two weeks). However, we believe we have still indicated the main thrust of both the research and implementation literature in the different areas. We could not possibly be comprehensive in the time but we have attempted to give enough details to whet the appetites of readers to make positive changes to syllabuses and their implementation in NSW. We could not have achieved so much without the assistance of Mella Cusack who searched for literature, ordered inter-library loans before we could begin writing, and whose fingers flew across the keyboard for hours on end as she entered the bibliographic data into Endnote. We also wish to acknowledge and thank several contributors to Mathematics Education Research in Australasia 1996-1999 (Owens & Mousley, 2000). Their efforts to summarise and review the literature made our task in certain areas possible. They are Elizabeth Warren on algebra, Steven Nisbett and Ian Putt on problem solving, Tom Lowrie and Kay Owens on geometry, measurement and visualisation, Shelley Dole and Alistair McIntosh on early arithmetic strategies, and Gary Asp and Barry McCrae on technology. We would also like to thank Geoff Morgan for his email contributions, John Pegg, Peter Howard, and colleagues from various interstate and international education authorities for collecting and sending us materials, and Beth Southwell for reading a draft.
Key Findings

The following are some of the significant findings from the literature review.

1. Children begin school with a range of knowledge, skills, and attitudes. Teachers need to build on what children already know or quickly learn to know. The use of early task-based interviews and observations followed by open tasks helps develop students' learning appropriately in their first year at school.

2. The importance of language in the development of concepts is emphasised in the case of place value. Irregularity in some of the English number names before 30 may cause some difficulties for students developing the ideas of place value and 10 as a composite unit.

3. Concrete materials are used initially to use counting to establish number and basic operational concepts, and to encourage students to develop mental visual imagery.

4. An emphasis of recent research has been the importance of students learning arithmetic reasoning and not just counting with concrete materials to answer a problem.

5. Visual imagery is an essential part of conceptual development.

6. Materials, including money, that can be manipulated may help students to establish the concepts of composite units and place value.

7. Game-like activities should be enjoyable and used for establishing concepts and not be just drill and practice games.

8. Materials and diagrams, both physical and mental, are useful if students can establish the abstract structure underlying the mathematics. For example, students must develop the notions of composite units and part-whole relationships to work with numbers.

9. Students have intuitive knowledge about multiplicative relationships when they enter school. However, these need to be developed and not lost by an emphasis on counting and additive relationships.

10. Making fractions through appropriate problems involving sharing of objects like pancakes involving continuous fractional sharing, may help the development of fraction knowledge. This can be begun as early as Year 1, with most development in rational number occurring in Years 3–5, including practical representations and solutions for operations on fractions.

11. Calculators support the use of materials in establishing number size, operational, and place value concepts. These should be used from an early stage. Computer programs may assist.

12. Discussions in pairs or small groups and as a class group help students to develop more efficient strategies.

13. Students find processes for subtraction more difficult to comprehend than addition.

14. Mental computation is a key component of the development of concepts of number and operations with numbers. Students should use mental computation supported by their own writing to work with larger numbers.
15. Sharing of different thinking strategies is encouraged by an emphasis on mental computation. This self-generated mathematics is a powerful tool in learning and requires a different presentation of activities.

16. Big ideas like composite units, parts and wholes, ratio, and chance need to be introduced informally in early primary school.

17. Diversity was found in Syllabus documents with some having only a few expectations across the years for each aspect of mathematics (e.g., NCTM, 2000) which are elaborated in terms of general expectations and illustrated by text and a few examples. Others present as a page of behavioural outcomes presented as dot points or in tables (e.g. Department of Education, Training and Employment, South Australia, 2000: primary school, p. 12 of 41, p. 25 of 41).

18. Syllabuses can be dynamic, leaving room for the addition of new topics, removal of old topics, and encouragement for the needs of the learners and the teachers in their local contexts to be taken into consideration.

19. Complexity of mathematical tasks can be perceived in terms of linguistic complexity, contextual complexity, representational complexity, operational complexity, conceptual complexity, and intellectual complexity.

20. Describing, explaining, and justifying, are part of argumentation, which is a key fore-runner of proof.

21. Experiencing, mathematising, and compressing or reifying ideas are key to mathematical thinking. Mathematics has a unique way of summarising the many constructs that link together (e.g., fraction notation).

22. In projects like Count Me In, assessment tasks may not necessarily be major learning tasks (e.g. backward counting), and the learning of number needs to focus on teachers helping students to mathematise rather than just doing the activities.

23. The traditional steps in measurement starting with informal units has been challenged by research in measurement. Other areas of research have supported the emphasis on composite units and grid imagery for area.

24. In order to bring students past the counting by ones, skip and jump counting should be encouraged. This can be encouraged through mental computation and coin or other counting. Recognising nearness to whole numbers on open number lines, and equal halving can also be introduced towards the end of Stage 1. Both these activities are fore-runners to proportional reasoning.

25. Physically representing (making and drawing) fractions and operations on fractions can be achieved in Stage 3 and Stage 4. Sound proportional reasoning needs to be the focus, rather than symbolic manipulations.

26. Proportional reasoning, taking tenths, will assist in the development of decimal number sense. Linear representations and number lines are effective for establishing these ideas. As with other areas, the ideas should be embedded in investigations as recommended in Realistic Mathematics Education. Percents can be a useful link between fractions and decimals with vertical double number lines recommended as a visual representation.

27. Everyday experiences, for example coins, can be used to establish composite units, and strategies such as jump counting.
28. Spatial reasoning is developed through the development of visual images. Real-life and computer experiences such as Geo-logo can encourage students to establish ideas about shapes and angles from an early age.

29. Current space and geometry lessons are not encouraging students to think adequately about shapes and class inclusion.

30. Pattern making and analysis is essential for a range of areas including number sense, algebraic reasoning, inductive reasoning, and spatial visualisation and concept development.

31. Negative numbers will develop as a concept in Stage 3, again partly due to the valuable extension of number concepts that comes from using calculators as a learning tool. The concepts, especially of operations that develops in the next Stage is best developed through a realistic investigation where concrete materials support the active thinking.

32. Algebraic reasoning develops from naming relationships embedded in physical, tabular and graphical representations. This requires the earlier introduction of different kinds of functions besides linear functions and this is achievable with using graphing programs on calculators and computers.

33. Many students arrive in primary school with an embryonic probability sense which can be developed throughout the Years K-10.

34. Data sense can be developed through collection, representation and analysis beginning in Stage 1. This exploratory data analysis can form a basis for early arithmetic strategies as well as provide necessary knowledge in statistical thinking.

35. No matter how comprehensive, well-structured and up-to-date, in terms of both research and practice, a syllabus might be, its successful implementation will depend, to a very large extent, on the skills, knowledge and expertise of the teachers involved.
**Recommendations**

1. The syllabus in mathematics K-10 should be built on the major curricular themes which have been identified in this literature review.

2. The syllabus in mathematics K-10 must take into consideration the emotional, cultural, linguistic and social needs of the students as well as their mathematical needs.

3. A syllabus that will bring about change in teaching and learning must focus on how teaching can bring about the mathematical reasoning and profound understanding recommended by this literature review.

4. Syllabus development is informed by consideration of students' ways of knowing. An arbitrary hierarchy of mathematics (cf. NSW Department of Education, 1969/1972), or a piecemeal spiral curriculum is not recommended.

5. The syllabus should be written in terms of teaching strategies for bringing about change in mathematical reasoning. Oral communication and sharing of ideas will become a major feature of mathematics lessons. Syllabus details will be in terms of activities and rich open-ended investigations that are suitable for such thinking. The content details will be details of students' thinking that teachers should encourage in the investigation and class discussions.

6. Continuity and progression will be brought about by indicating to teachers the level of problems and argumentation expected by students. The syllabus should not be expressed in terms of itemised or atomised content or indicators of success.

7. Mathematical reasoning should become the major focus of the syllabus and of mathematics lessons. Lessons will consist less than before of transmission models such as students following a teacher's model of working out some mathematics. Succinct procedures can be arrived at or shared by both students and teachers. The students will be expected to represent and share their own mathematical thinking as it develops. The syllabus must reflect this by indicating how any piece of content could be part of an investigation.

8. Profound understanding of fundamental mathematics reduces the need for practice. Greater emphasis can be placed on selecting and using mathematical tools such as calculators and formulae. Making sense of the results of data analysis, algebraic manipulations, and calculations is also fundamental.

9. The same kinds of mathematical thinking are expected of all students, that is algebraic reasoning, geometric reasoning, proportional reasoning, and probabilistic reasoning. These will be expressed and developed through description, explanation, justification, and argumentation. However, some students will be able to tackle problems that are more complex or to develop an investigation to a greater extent. This applies across Stages and within Stages. One implication of this, is that a syllabus for a particular Stage may describe, for example, the aspects of proportional reasoning expected of students, and activities that will generate such thinking. However, ways of extending and assisting students would also be given.

10. Number sense, spatial sense, measurement sense, data sense and probability sense are relatively imprecise but this does not detract from precision being a goal of mathematics. Sense is couched in experience (real and imagined) and experiment which are the starting points for attending to properties and
developing proofs. Experience is the place for evaluating solutions and conclusions. Sharing a variety of strategies provides for checking a solution by another means.

11. Syllabuses for different Stages should not be repetitive in activity nor broken down into small pieces that lose the continuity of the curriculum.

12. The research background and approach of the Department of Education and Training’s programs, Count Me In, Count Me In Too, Counting On, Count Me Into Space, and Measuring Up should be considered as examples in the development of the K-10 syllabus. The Stage 5 Syllabus could be further extended to emphasise investigation in establishing concepts, communication in classrooms, and give less emphasis on the hierarchy of content pieces.

13. The Syllabus should recommend ways of using investigations, materials, graphing and other computer packages to establish generalisations and basic algebraic reasoning. Functions other than linear functions should be introduced early in order for difference to highlight basic algebraic concepts.

14. Proportional reasoning should begin with halving, and be developed through sharing, multiplicative relationships, and fractional operations on real objects (or images).

15. The Syllabus should provide activities that will challenge students’ common misconceptions, especially in the areas of algebra, decimals, graphs, and negative numbers. These challenges can be overcome by negotiated classroom sharing of strategies and solutions for different investigations.

16. Everyday and computer experiences can be used to establish important mathematical concepts.

17. Students can develop geometric and logical reasoning such as class inclusion from Stage 3 if they have experienced a range of geometric activities which encourages recognition of properties, comparisons of parts, and relationships between parts.

18. Activities to facilitate the development of probability sense and data sense need to be introduced into primary school, especially in Stages 2 and 3.

19. No matter what the form and content of the mathematics K-10 syllabus, there will be a great need to develop in the teachers implementing it a profound understanding of the mathematics involved and ways in which students might learn that mathematics.
Theoretical Frameworks Underpinning the Development of Mathematical Knowledge, Skills and Understanding Across Years K–10

Plus ça change, plus c’est la même chose.

At the beginning of an exercise such as this literature review, it is instructive to remind ourselves of where we have been and what has gone before. We often hear that there is nothing new in education and, yet, there appears to be much change in mathematics education. In 1969, the then NSW Department of Education published *Curriculum for Primary Schools: Mathematics* (NSW Department of Education, 1969, 1972). In this document (p. 1), the following were listed as the aims of mathematics in NSW primary schools.

- To assist the child to understand and interpret his (sic) environment.
- To satisfy the present mathematical needs of the child.
- To lay a sound foundation for future mathematical studies.
- To create favourable attitudes towards and to stimulate interest in mathematics.
- To develop understanding of fundamental ideas of numbers, measurements and shapes
  - knowledge of language and relationships
  - skill in computation and problem solving
  - through the provision of opportunities to
    - explore, discover, describe and record relationships
    - create patterns and make statements about the relationships contained in them
- To show the contribution which mathematics has made and is making to our present civilization.

In 1989, a revised document, entitled *Mathematics K–6* (NSW Department of School Education, 1989, p. 8), provided the following aims for what was now called the syllabus.

- To create in students a sound understanding of mathematical concepts, processes and strategies and the capacity to use these in solving problems.
- To develop in students the ability to recognise the mathematics in everyday situations.
- To develop in students the ability to apply their mathematics to analyse situations and solve real-life problems.
- To develop in students an appreciation of the applications to mathematics of technology, including calculators and computers.
- To encourage students to use mathematics creatively in expressing new ideas and discoveries and to recognise the mathematical elements in other creative pursuits.
- To challenge students to achieve at a level of accuracy and excellence appropriate to their particular stage of development.

At the secondary level, the current courses—there are four of them—also list aims which are worth considering here. The *Syllabus Years 7–8: Mathematics* (Board of Secondary Education, 1990, pp. viii–ix) contains the following statements for “those [students] whose levels of cognitive development and interest encourage a more formal approach”. (There is a similar, but less demanding, set of statements concerning the development of all students.)

- To develop habits of effective thinking and intellectual independence, … students should:
  - acquire a background of mathematical concepts and symbolic representation;
• develop facility in formal operational techniques;
• develop an understanding of how (a) given mathematics may be applied to situations and (b) situations may be taken and analysed into the essential mathematical features, leading in turn to the solution of existing problems;
• develop the ability to communicate and interpret within mathematical systems;
• develop thinking strategies to problem solving in unfamiliar situations;
• develop a creative and imaginative approach to mathematical situations.
In the achievement of the aims stated above, students should be encouraged to develop (a) self confidence in handling mathematics and (b) an awareness and appreciation of its value in society.

For Years 9–10, the three courses (Board of Studies, New South Wales, 1996a; 1996b; 1996c) contain the same statement of aims (p.10 in each document).

The Mathematics 9–10 Syllabus aims to promote students’ appreciation of mathematics and develop their mathematical thinking, understanding, confidence and competence in solving mathematical problems.
This aim is to be achieved through developing students’ capacities to:
• acquire the mathematical knowledge, operational facility, concepts, logical reasoning, symbolic representation and terminology appropriate to their stage of mathematical development and in preparation for further study of Mathematics
• interpret, organise and analyse mathematical information and data
• apply mathematical knowledge and skills to creatively and effectively solve problems in familiar and unfamiliar situations
• communicate mathematical information and data
• justify mathematical results and give proofs where appropriate, making connections between important mathematical ideas and concepts
• value mathematics as an important component of their lives.

A quick comparison of these aims shows that there are many similarities but quite a number of differences. For example, as might be expected, similar statements about content and skills are made, although the purpose of students knowing these things is different. In the 1969/1972 document emphasis is placed on meeting the present and future mathematical needs of the student whereas problem solving has taken a much greater role in the reasons for learning mathematics in the later documents.

While the 1969/1972 aims mention using mathematics to understand and interpret the environment, the later syllabuses more strongly make the point that students need to see the mathematics in everyday situations and to use mathematics to analyse and solve problems in these situations. (A number of mathematics educators, e.g. Hughes, Desforges & Mitchell, 2000, p. 16, suggest that such aims fit best with the situated cognition perspective on the nature of knowledge in mathematics. What one believes about the nature of such knowledge will have a profound effect on how one believes students should learn it, but more of that later.)

The main change in the secondary syllabus is not the content but "the focus ... on students thinking about and doing mathematics, rather than just rote learning formulae and procedures" (Cooper, 1997, p. 5). For example, geometry requires "intuitive understanding, the ability to describe geometrical situations ... and the use of more investigative approaches" with more deductive solutions being only expected of Advanced students with Circle geometry being removed from the Core. There is more on ratio and practical rates, with variation and exponential relationships being introduced into the Advanced course. Investigative approaches lead to some number theory for Advanced students, and calculators are not treated separately. Unless trigonometry formulae can easily be derived, they are not expected to be memorised,
e.g. cosine rule. In Chance and data students are expected to analyse data resulting from practical applications and experiments. Spreadsheets, stem-and-leaf plots and box-and-whisker graphs emphasise the importance of exploratory data analysis. The purpose of the standard deviation as a measure of spread is more important than hand-calculating it. There is a continuation of Years 7-8 for algebra with an emphasis on patterning, appropriate use of the equals sign, the explaining of solutions and the recognition of difference between an expression and an equation. The Advanced course, includes the quadratic formula and roots and zeroes (Cooper, 1997). To some extent, these changes pick up some of the literature that is being reviewed here but we have to consider if it is bringing about real change in classrooms given the lack of the big picture changes we will be describing.

Not surprisingly, technology—calculators and computers—are not mentioned in the early syllabus document but do feature in the later ones. Such technology has the potential not only to assist in the teaching and learning of the mathematical content contained in the current syllabuses (Hembree & Dessart, 1992; Ruthven, 1996) but also to dramatically change the content of future syllabuses and, in particular, the approaches to be used (Groves, 1996; Kissane, 1999; National Council of Teachers of Mathematics [NCTM], 2000). (Groves, 1996; Kissane, 1999; National Council of Teachers of Mathematics [NCTM], 2000, pp. 24–27). Some have suggested that technology might even change the mathematics which can be learned (NCTM, ; Waits & Demana, 2000), 2000, pp. 26–27; Waits & Demana, 2000, p. 54). However, it is merely a recommended usage.

All of the syllabus aims listed have recognised the importance of students developing positive attitudes to the mathematics they are experiencing, although the extent to which this seems to take on a major component of the syllabuses seems to diminish as the students get older. Positive attitudes have long been lauded as important for learners of mathematics (Australian Education Council [AEC], 1991; Cockcroft, 1982; NCTM, 1989) and it is clear that the syllabus documents reflect this.

Other issues which appear to a greater or lesser extent in the syllabus aims include challenge and creativity in mathematics, the need for students to value mathematics as important in their lives, and issues of communication both within students’ mathematical learning and in the more general arena. This latter area includes notions of justification of mathematical results and reasoning as well as mathematical proof.

So, what we have in NSW currently is a series of mathematics courses which clearly see mathematics as a body of knowledge and skills—usually along the lines of a study of patterns and relationships—and processes such as problem solving, observation, investigation, representation, generalisation and abstraction. The content of all the courses is broken into traditional areas of mathematical knowledge and skills such as number, geometry, space, data and chance. Applications to and from the students’ real world are dealt with through a ‘working mathematically’ strand. The sequence of topics tends to be defined by the nature of the topics while the details tend to contain issues that may cause difficulty if not covered by the teacher. The fractured nature of the courses is further exacerbated by the introduction of outcomes and indicator statements which are used by teachers for both programming and assessment. It may be that these arrangements are the best for the successful learning of mathematics by students in K–10 classes in NSW schools. The purpose of this literature review is to investigate what the evidence from NSW, Australia and beyond says about this.
Approaches to learning

Every syllabus implies a philosophy of learning and teaching. The syllabus should be supportive of the general opinion about learning and teaching. A similar analysis of the current NSW syllabuses in mathematics in terms of what they say about approaches to learning shows a not surprising adherence to the principles espoused in the *National Statement on Mathematics for Australian Schools* (Australian Education Council [AEC], 1991). In the syllabuses, the nature of mathematics learning is described as follows (see, for example, NSW Department of School Education, 1989, pp. 4–5).

**Students learn best when motivated.**
- Mathematics learning is more effective when it is interesting, enjoyable and challenging.

**Students learn mathematics through interacting.**
- Mathematics learning should involve interaction with the physical and social environment, leading to the abstraction of particular mathematics ideas encountered.
- Mathematics learning is promoted by the appropriate use of a variety of materials, equipment and personnel.

**Students learn mathematics through investigating.**
- Mathematics learning should involve the investigation of mathematical patterns, relationships, processes and problems.

**Students learn mathematics through language.**
- Mathematics learning is promoted by the appropriate use of language. Language, including symbols and diagrams, plays an important part in the formulation and expression of mathematical ideas and serves as a bridge between concrete and abstract representation.

**Students learn mathematics as individuals but in the context of intellectual, physical and social growth.**
- Mathematics learning is promoted when individual differences of students are taken into account.
- Mathematics learning should be appropriate to each student’s current stage of development and should build upon previous experiences and achievement.

While it is unlikely that many teachers and mathematics educators would disagree with these statements in broad terms, we do know a lot more about students’ learning of mathematics in 2001 than we did in 1989. As well, we are much more aware of the importance of the cultural and affective aspects of mathematics learning which were barely given lip-service, even in the more detailed exemplifications of these principles contained in the syllabus documents. In the remainder of this section, we shall endeavour to give an overview of the important aspects of this new knowledge.

The draft Queensland Curriculum Framework (Queensland School Curriculum Council, 2000a) is based on a ‘learner-centred’ approach to learning and teaching which is defined as follows:

A learner-centred approach enables students to develop and demonstrate outcomes showing their understanding of their world. It enables them to continue making personal meaning throughout their lives through the complex thinking, creativity, investigating, communicating, participating and reflecting practices and dispositions developed during their schooling (p. 57).
Such an approach is based on certain understandings about learners, learning and learning environments. In the context of this review, we consider the following statements from the draft Queensland Curriculum Framework (Queensland School Curriculum Council, 2000a, p. 57).

No two learners are alike. They all differ in terms of inherited and acquired characteristics, approaches to learning and in terms of the values, understandings and interests brought from their participation in a variety of social and cultural groups. Such differences need to be recognised, and any learning disadvantage that may result from membership of a particular group is attended to so that all learners have the opportunity to develop and demonstrate the core learning outcomes of the curriculum.

Meaningful learning requires learners to actively process information in some way. Each learner perceives and interprets new information in a unique way based on their interests, motivation and social and emotional factors, and relates it to and builds upon their prior understandings. Therefore no two learners ever reach exactly the same understanding. Effective learning also involves a monitoring by learners of what has been learned, how it has been learned, how well it is understood and how they can organise themselves to improve their learning. If learners understand the constructed nature of knowledge, they can further manage their own learning by developing a critical approach to information and generating their own questions to probe for deeper understandings. Despite different student approaches to learning, the principle of equity requires that all learners be given opportunities to demonstrate the core learning outcomes of the curriculum.

Under the heading of “Approaches to teaching and learning in mathematics”, the Board of Studies, Victoria makes the following statement (Board of Studies, Victoria, 2000)

While there is no preferred way to teach mathematics, many of the most successful approaches share the following characteristics:
• teaching from a base of concrete experience;
• recognising mathematics as abstract and general;
• using a variety of modes of classroom activity;
• recognising individual differences and different learning styles and needs;
• emphasising the sensible use of mathematics, with attention to checking the reasonableness of results, choosing and using tools appropriately and effectively and being alert to finding reasons why ideas do, or do not, work;
• allowing time for growth.

The Western Australian Curriculum Council makes the following statements in its Curriculum Framework under the heading “Learning and Teaching” (Western Australian Curriculum Council, 2000)

**Opportunity to learn**
Learning experiences should enable students to engage with, observe and practise the actual ideas, processes, products and values which are expected of them. Students should practise, (that is, “do”) mathematics, but doing mathematics involves much more than the repetition of facts and procedures; it also involves working mathematically across all the strands.

**Connection and challenge**
Learning experiences should connect with students’ existing knowledge, skills and values while extending and challenging their current ways of thinking and acting. Learners’ interpretations of new mathematical experiences depend on what they already know and understand.

**Action and reflection**
Learning experiences should be meaningful and encourage both action and reflection on the part of the learner. Mathematical learning is most successful when students actively engage in making sense of new information and ideas. If students face mathematical situations that are not inherently meaningful, then they are forced to conclude either that mathematics does not make sense or that they themselves are incapable of making that sense. Providing students with isolated facts and procedures which they are expected simply to imitate and remember, or with partial explanations of concepts disconnected from their other mathematical ideas forces them to resort to learning strategies based on the passive reception of mathematical concepts and processes and the rote imitation of procedures. The result is likely be short-term storage that needs to be topped up regularly, rather than effective long-term learning.

**Motivation and purpose**

Learning experiences should be motivating and their purpose clear to the student. Mathematics is often promoted to students as an investment in the future and for some students this is sufficient motivation to keep them working at it. For others, however, this is not persuasive and the mathematics provided in school must provide its own motivation if such students are to continue to participate actively. All students, however, should have opportunities to experience the satisfaction and pleasure that mathematics can bring. Effective learning requires that students feel able to risk making mistakes without fear of the consequences.

**Inclusivity and difference**

Learning experiences should respect and accommodate differences between learners. Linguistic, cultural, gender and class differences between students are often regarded as adequate explanations for differences in mathematical achievement. This Framework starts from the premise, however, that a common cause of many students’ failure to learn mathematics in a sustainable and robust way is an inadequate match between the curriculum and the experiences and understandings of students.

**Independence and collaboration**

Learning experiences should encourage students to learn both from, and with, others as well as independently. Collaborative learning can enhance mathematical learning in a number of ways. Firstly, by working together and pooling their ideas, students can develop ideas and solve problems which may be inaccessible to them individually. Secondly, students’ command of mathematical ideas and mathematical language is likely to improve when they try to describe, explain or justify. Thirdly, discussion is one of the ways students come to understand that others may not interpret things in the same way or share their point of view. Working individually is also important in mathematics. It should enable students to ensure a personal grasp of concepts, processes and procedures. In turn, they should develop confidence in their capacity to do mathematics for themselves, including the sustained effort needed to work through problems.

In their *Principles and Standards for School Mathematics* (NCTM, 2000, p. 20), the National Council of Teachers of Mathematics state the Learning Principle:

Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.

The Council provides much backing for this principle, a selection of which is reproduced here:

Being proficient in a complex domain such as mathematics entails the ability to use knowledge flexibly, applying what is learned in one setting appropriately in another.
Mathematics makes more sense and is easier to remember and to apply when students connect new knowledge to existing knowledge in meaningful ways. Conceptual understanding is an essential component of the knowledge needed to deal with novel problems and settings.

... most of the arithmetic and algebraic procedures long viewed as the heart of the school mathematics curriculum can now be performed with handheld calculators. Thus, more attention can be given to understanding the number concepts and the modeling procedures used in solving problems.

Learning with understanding can be further enhanced by classroom interactions, as students propose mathematical ideas and conjectures, learn to evaluate their own thinking and that of others, and develop mathematical reasoning skills.

...procedural fluency and conceptual understanding can be developed through problem solving, reasoning, and argumentation. (NCTM, 2000, pp. 20–21)

Another view of differences in ways of viewing mathematics learning is given by Stigler and Hiebert (1999, pp. 90–91).

If one believes that mathematics is mostly a set of procedures and the goal is to help students become proficient executors of the procedures, as many U.S. teachers seem to, then it would be understandable to believe that mathematics is learned best by mastering the material incrementally, piece by piece. This view of skill learning has a long history in the United States. Learning procedures occurs by practicing them many times, with later exercises being slightly more difficult than earlier ones. Practice should be relatively error-free, with high levels of success at each point. Confusion and frustration, in this traditional American view, should be minimized: they are signs that earlier material was not mastered. The more exercises, the more smoothly learning will proceed.

Japanese teachers appear to hold a different set of beliefs about learning ... One can infer that Japanese teachers believe students learn best by first struggling to solve mathematics problems, then participating in discussions about how to solve them, and then hearing about the pros and cons of different methods and relationships between them. Frustration and confusion are taken to be a natural part of the process, because each person must struggle with a situation or problem first in order to make sense of the information he or she hears later. Constructing connections between methods and problems is thought to require time to explore and invent, to make mistakes, to reflect, and to receive the needed information at an appropriate time.

Comparing each of the above inputs makes it clear that there are a great many commonalities in statements about how mathematics learning and teaching might be best undertaken. One thing that is clear is that we certainly know a great deal more about such teaching and learning than we did previously, even if what we know is not always consistent or conclusive. There seems to be reasonable agreement that students should be actively involved in the construction of their own mathematics, even if there are some differences about precisely what this might mean in any given circumstance.

We have come a long way beyond the application of Piaget’s stage theory to individual, and individualised, mathematics development and now believe that the critical components of teachers, learners, mathematical tasks and the socio-cultural contexts of the learning and teaching have tremendous impact on the learning. For example, the work of Ma (1999) points very strongly to the importance of teacher knowledge—not only the quantity of this knowledge but also its nature—to students’ learning of mathematics. The work of Boaler (1997), Perry, Dawe, Howard, and

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1 We would suggest that there is a similar long history of this in Australia.
Dengate (2000), Teese (2000), Wotley (2000), Young-Loveridge (2000), and Zevenbergen (1997; 1998), for example—at very different levels of schooling and national contexts—has shown clearly the impact on mathematics learning of issues of social class, socio-economic status and ethnicity. The extensive work of Sweller and his colleagues (e.g., Bobis, Sweller, & Cooper, 1994; Leung, Low & Sweller, 1997; Monsavi, Low, & Sweller, 1995; Mwangi & Sweller, 1998; Sweller, 1994; Sweller & Chandler, 1994) on the importance of cognitive load in considering activities, materials, task nature and methods of presentation of mathematics to students has been most instructive in assisting teachers and curriculum developers in providing optimal approaches to mathematics teaching and learning. Similarly, the work of the very active group from Queensland University of Technology (e.g., Boulton-Lewis, 1993; 1999; Boulton-Lewis, Cooper, Atweh, Pillay, Wilss, & Mutch, 1997b; Boulton-Lewis & Halford, 1992; English & Halford, 1995) has added to our perspectives on mathematics education and the cognitive demands of certain approaches. In particular, the processing load involved in using (unfamiliar) concrete representations and analogues needs to be considered very carefully.

One of the clear applications of our extra knowledge about how children learn and the cognitive load that this may place on them, along with the realisation that much of the algorithmic learning of the four operations may not be as necessary as it was previously, given the advent of technological devices which can relieve many of the routines of arithmetic, is the re-emphasis and re-definition of mental computation. It is not suggested that students should not become fluent in operations with whole, rational and irrational numbers. Rather, the suggestion is that students should be given the opportunity to investigate beyond the ‘standard’ algorithms in their quest for fluency, efficiency and accuracy. Central to this suggestion is the development of students’ number sense—

The ability to decompose numbers naturally, use particular numbers like 100 or _ as referents, use the relationships among arithmetic operations to solve problems, understand the base-ten number system, estimate, make sense of numbers, and recognize the relative and absolute magnitude of numbers.

(NCTM, 2000, p. 32)

Hence, we should now know a lot more about what it is we are trying to do when writing a mathematics syllabus than we have done previously. We investigate here, even more deeply, what might be meant by students being actively involved in the construction of their own mathematics. We begin with the notion of constructivism.

Constructivism is, perhaps, one of the most over-used and misunderstood terms in current mathematics education circles. There have been many attempts to explain its meaning (see, for example, (Cobb & Bauersfeld, 1995a; Ernest, 1994; Greenes, 1995; Gough, 1997; Hughes, et al., 2000; Mascolo, Kanner, & Griffin, 1998; Op’t Eynde, de Corte, & Verschaffel, 2000; Perry, 1996) with no definitive result. While this is perhaps not surprising, it is not particularly helpful to the classroom practitioner. Nevertheless, there is enough commonality to allow some useful characteristics to be determined.

Lerman (1989, p. 211), following the pioneering work of Ernst von Glasersfeld (see, for example, 1991), has provided the following key tenets of constructivism:

1. Knowledge is actively constructed by the cognizing subject, not passively received from the environment.
2. Coming to know is an adaptive process that organises one’s experiential world; it does not discover an independent, pre-existing world outside the mind of the knower.
The first of these tenets is well-accepted by most educators and results in the great variety of activity—both physical and mental—which is seen in mathematics classes across the world. The second is much more problematic and less well-accepted. However, it adds the important dimension that individual learners construct their own mathematics as the result of their own experiences. It challenges the nature of mathematics as a fixed—or even dynamic—body of knowledge which exists somewhere beyond the mind of the learner. It emphasises the nature of mathematics as a developing body of knowledge, skills and processes which only exist in the learners’ minds. Now, all of this may be of great theoretical importance, but the empirical evidence is such that there do appear to be accepted pieces of knowledge, skills and processes which fit within society’s views of mathematics. That is, there does appear to be some general acceptance of the nature of mathematics. How can this be?

There are many adherents to constructivist approaches to learning who also recognise that learning does not occur in a vacuum and that the socio-cultural context in which this learning occurs is of great importance to the learning. As Cobb (1990, p. 209) has put so elegantly, “learning is an interactive as well as a constructive activity”. From this point of view, learning occurs through interaction with the context in which learners find themselves—including the people in this context—as well as through individual construction. This position owes a great deal to the work of Vygotsky (see, e.g., Dockett & Perry, 1996; Rogoff, 1990; Vygotsky, 1978).

Another interpretation of the importance of the social context in the learning of mathematics—sometimes dubbed ‘situated cognition’—arises from a belief that knowledge is inseparable from the context in which it is learned. Hughes, et al. (2000, p. 16) suggest such an approach to learning considers that what is learned are “the working practices in which we [learners] operate”. They continue:

Novices must be inducted into the working practices of the domain in question through a form of apprenticeship. Pupils and students will not, from the perspective of situated cognition, learn to use and apply their knowledge met in classrooms if it is met in the form of ordinary classroom work structured by textbooks and work cards. If pupils are to be enabled to apply their knowledge beyond the classroom then they must, from this perspective, learn through authentic experience of the subject in question. As far as possible, it is argued, pupils and students must learn mathematics through being an apprentice mathematician, and meeting problems in the ways mathematicians might experience them. (Hughes, et al., 2000, p. 17)

From the view of the socio-cultural theories of learning, “individual processes are not merely influenced by social and cultural practices and meanings, but are fundamentally constituted and transformed by them” (Mascolo, et al., 1998, p. 33). Cobb & Bauersfeld, (1995b, p. 5) suggest that these theories make an important contribution, particularly in accounting for the regeneration of the traditional practices of mathematics instruction. They, therefore, have much to offer in an era of reform in mathematics education that is concerned with the restructuring of the school and takes the issue of ethnic and cultural diversity seriously.

In trying to extend the individually-oriented theories of learning originally propounded by Piaget to consider the role that the socio-cultural context might play, neo-Piagetian approaches have developed. Such approaches privilege individual development over social interaction but do not deny the latter (Mascolo, et al., 1998). Hence, “the focus is on the individual, autonomous learner as he or she participates
in social interactions” (Cobb & Bauersfeld, 1995b, p.7). Critics of these approaches generally question the relative importance ascribed to the social and the individual but suggest that a more even-handed approach to coordination of the approaches may prove fruitful. However, not even such a relatively moderate approach will result in a seamless theoretical framework. Instead, the resulting orientation is analogous to Heisenberg’s uncertainty principle. When the focus is on the individual, the social fades into the background, and vice versa. Further, the emphasis given to one perspective or the other depends on the issues and purposes at hand. Thus, ..., there is no simple unification of the perspectives. (Cobb & Bauersfeld, 1995b, p.8)

One thing which is important to remember in all of this discussion of theories of learning is that they are just that. They do not necessarily say anything about how mathematics might be taught or how mathematics syllabuses might be constructed. English and Halford (1995) note that the various psychological and socio-cultural approaches to mathematics learning have sometimes been reduced to two or three approaches to teaching: student use of manipulatives; group discussion and “inquiry-oriented learner activities that promote meaningful and independent learning, as well as opportunities for learner reflection” (p. 12). While it is generally accepted that all of these may prove very useful in students’ learning of mathematics, it must also be said that there is a lack of evidence showing that they are either necessary or sufficient for the development of mathematical understanding, although there is some evidence that their presence is more likely to result in genuine learning with understanding (Boaler, 1997). What would seem to be required is learning and teaching activities presented in a context which is supportive of the student’s learning in numerous ways and where there is “a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student’s experiential reality” (Gravemeijer, 1999, p. 156).

**Visualisation** is another aspect of student learning that has received both theoretical and research attention recently. A study by Hai (1999) showed how the use of visual imagery, imagination, and visual reasoning can enhance the cultivation of mathematical creativity. Lowrie (1997) investigated the type of imagery Year 6 students used when attempting mathematics problems over a two-year period. The students used several types of imagery that were identified by Presmeg (1986). Concrete (pictorial) imagery was used by all children. Such imagery was found to be particularly useful in novel or difficult problem-solving situations. Pattern and memory imagery allowed students to develop rich schematic understandings. Such imagery contained propositions, event schema (students recalled “when I did this”) and visual recollections that were rolled together to form an holistic type of imagery. These memories allowed students to access relevant knowledge for a variety of problem-solving situations. Dynamic (moving) imagery appeared to be a powerful form of imagery often used for specific tasks. Such imagery encouraged what-if thinking when promoted through activities that required students to use their imagination. It was suggested that students should be encouraged to evoke a range of different forms of imagery depending on the type of problem encountered. This study supports earlier work of Owens (1994) on different kinds of imagery.

Both visual and non-visual reasoning play an important role in problem solving. Students will often use visual processing in the initial stages of problem solving, and then move toward more analytic, non-visual strategies when relevant patterns have been identified, or when a more informed understanding of the problem has been established. Importantly, visual methods allow students to negotiate the difficulties
associated with a problem when conceptual limitations do not allow a student to complete the problem quickly and analytically (Lowrie & Hill, 1996, with Year 6 students). Four distinctive features: (a) task difficulty, (b) problem representation, (c) metacognitive thought, and (d) the ability to apply knowledge to other situations, influence how students interpret, represent, and undertake solving problems and applying knowledge and techniques used to other problem tasks. Lowrie and Hill (1996) also showed that learning preference, task difficulty, personal beliefs, and attitudes toward mathematics affected problem-solving performance.

By contrast, there was no statistically significant interaction effect of high-achievement and method in an experimental study by Adibnia (1996) with Year 6 students. When compared to a control group who had the problems, and another who had no problems, the problem-solving performance improved more in the experimental group who were taught by an instructional approach based on the cognitive-metacognitive framework of Garofalo and Lester (1985). Interestingly there was a significant difference in favour of the females in the experimental group on the Process Problems test and only the experimental group had significant positive gains in relation to beliefs and attitudes.

Lowrie & Hill (1996) emphasised the use of visual imagery in a training program which improved the students’ capacity to solve spatial problems. It is suggested that visual problem-solving methods are utilised best in novel or complex learning situations. Nonvisual problem-solving approaches were usually the most efficient when students could quickly identify and access the relevant knowledge required to complete a specific task. It was found that students will often use visual processing in the initial stages of problem solving, and then move toward more analytic, nonvisual strategies when relevant patterns have been identified, or when a more informed understanding of the problem has been established.

Mental visual representations may result in drawings. Instruction in diagram use in a study by Diezmann (1999) resulted in an overall improvement in (a) the frequency and autonomy of diagram use, (b) the quality of diagrams that were generated, (c) the appropriateness of the reasoning with the diagrams, and (d) the success rates for tasks in which a diagram had been used. Individual students’ abilities to use diagrams effectively may be moderated by their idiosyncratic profiles of spatial intelligence and the development of spatial intelligence within the classroom (Diezmann, 1999; Diezmann & Watters, 1996a, 1996b). Diezmann pointed out that there are several kinds of diagrams that are useful in mathematics. These are part-whole diagrams are conceptually-oriented diagrams, unlike networks, matrices and hierarchies which are spatially-oriented diagrams. With instruction on using diagrams, Diezmann found that there was an increase in the number of diagrams that were spontaneously drawn. However, not all diagrams that were produced were accurate representations of the problem structure. The qualitative changes that were reported included a more accurate representation of elements of the problem structure—represented as nodes—and the movement between these nodes (Diezmann, 1998).

Outhred and Shaw (1999) commented that greater emphasis should be given to integrating the use of diagrams into the teaching of statistics courses. Moreover, they argued that open-ended tasks that required students to interpret the same data from different perspectives may also encourage diagram use.

Diagrams in trigonometry are an important teaching tool. Kendal and Stacey (1996) showed that with a range of schools and teachers, the ratio method was more effective than the unit method for trigonometry on triangles. Some of the difficulties
arose from the visual difficulties such as rotating triangles for use with the unit circle. They concluded that while the unit circle was a valuable way of introducing trigonometric ratios, it was better to introduce the mnemonics for the right-angled triangle and allow students to use this method for this type of question.

In a study of nine kindergarten children’s drawings of word problems, Outhred and Sardelich (1998) found that physical representations and drawings were indicative of the ability of young students to solve quite difficult word problems. Over the two-month period, drawings developed from having no structure to showing their solution strategies (Outhred & Sardelich, 1997). Using theoretical perspectives about representation, the authors suggest that drawn representations give children a means of recording their thinking and explaining the solution process. Drawings assist students to focus their attention on similarities between two structures preparing them for the introduction of number sentences. When students wrote down number sentences to represent the problem, the strength of their knowledge was evident in their ability to explain these understandings in different ways.

Owens (1999b) also found that students used physical materials to develop ideas, check ideas, and create aesthetically pleasing paper sculptures (3D models) in much the same way that drawings and paper cutouts may particularly help 2D space problems.

**Problem posing** has become a way of encouraging students to think mathematically. One of the successful studies illustrates how Years 3 and 4 students were encouraged to take a mathematics story and write a couple of mathematics (addition and subtraction) problems based on it. They were encouraged to leave out one of the numbers. Students worked collaboratively and their self-selected best problems were shared with the rest of the class. Students considered **problem structures**, especially allowing them to decide if the word problem was (a) about something happens, (b) altogether problems, (c) compare problems or (d) multiple operations. The students’ problems were often with multiple steps. This group of students did fewer problems than the control class, nevertheless they solved problems much better (Rudnitsky, Etheredge, Freeman & Gilbert, 1995). Silver and Cai (1996) show a similar result for Years 7 and 8 students with multiplication and division problems. They also showed that lower achieving students were able to pose problems even if they were less complicated. They also found the nonmathematical questions posed by the students to show the importance of context for problems in mathematics.

Numerous studies have emphasised the importance of problem solving from K-10. It is interesting that a study of Years 11 and 12 students by Stillman (1998) showed that students engaged in problem solving more if the context and previous experiences were relevant to the students. **Engagement** resulted in higher performance.

*Using realistic situations* as problem contexts invited a multiplicity of interpretations and methods for defining quadratic functional relationships. By reasoning empirically from the context of the problem, the students conceptualised quadratics iteratively and in terms of summation, in contrast to the most common view of quadratics as a product of two linear variations. Requiring students to verify and justify their strategies by cross-referencing between multiple representations of functional relationships and problem context led them to construct viable schemes to characterise quadratics in terms of rate of change, dimensionality and symmetry (Afamasaga-Fuata’i, 1998).
Owens (Owens, 1999b; Owens & Clements, 1998) illustrated that actions such as manipulation and drawing were often focusing students' participation and responsiveness in problem solving. This multifaceted response to the learning situation, was often triggered by selective attention, that is noticing aspects of physical and mental representations. Developing and interpreting schema for geometrical concepts is suggested as critical for geometric problem solving (Chinnappan, 1998). High achievers tended to produce more relevant-to-solution schema than in the overall number of schema used. A fine-grained analysis of two students' work indicated that the high achieving student used schematic connections in later attempts at a difficult problem. The number of links between schema, for example area and number, was considered significant (Chinnappan, 1999).

Students' mathematical development involves establishing and using schema (cf. Chinnappan, 1998; Moss & Case, 1999; Owens & Clements, 1998; Owens, 1999b; Skemp, 1989; Sweller, 1994). Schema are a way of storing knowledge that integrates various ideas. They direct attention to certain aspects of a perceived situation or a particular aspect of a problem. Chinnappan (1998) noted that useful problem representations in the domain of geometry are facilitated by schemas that provide a degree of organisation of prior knowledge. High-achievers accessed more problem-relevant schemas than the low-achievers. Both groups accessed almost equal numbers of geometry schemas but Chinnappan suggested that high-achievers build schemas that are qualitatively more sophisticated than low-achievers which in turn helps them construct representations that are conducive to understanding the structure of geometry problems.

Wilson (1998) found that most upper primary school students always thought about what they already know and what the problem was asking them to do. They also made a plan to work it out and tried to remember if they had done a similar problem before. Wilson provided a model of metacognitive behaviour which acknowledges three functions of metacognition—awareness, evaluation and regulation.

The use of metacognitive strategies was also noted by Kaur (1996) who studied students in Years 5, 6, 7, and 8. Protocols of good novice problem solvers showed that they already used the steps: 1. Understand/represent the problem, 2. Find a way to solve the problem, and 3. Solve the problem with many also undertaking step 4. Check the solution. The frameworks supplied by the Year 8 students were more detailed, comprehensive and sophisticated than those supplied by the students in the other years.

Detailed examination of the protocols of five Year 7 students in Singapore indicated that students did demonstrate metacognition during problem solving although their strategies were not always efficient and successful (Yeap & Menon, 1996). When students reflected upon their plans of action to ensure completeness and correctness, it often led to solutions. A systematic approach seemed to facilitate monitoring and regulating of the problem-solving process.

Yeap and Menon, Wilson, and Chinnappan all suggest that it is important for teachers to incorporate more extensively the development of metacognitive processes and emphasise systematic thinking in their regular mathematics instruction.

Good Year 9 students (high achieving control group) in Singapore, taught in an expository way, with a strong emphasis on algorithms, continued to excel in standard achievement test items in Yeap and Kaur's (1996) study. However, the low achieving experimental group was taught the remainder theorem and binomial theorem using
pattern observation, a problem-solving heuristic. They showed better performance in the achievement test than the class of similar ability taught in an expository way. The results of this teaching experiment lend some support to an instructional approach that placed emphasis on pattern observation. It was claimed that the experimental group of students benefited in terms of mathematical processes through their mode of instruction although the authors were hesitant about claiming that these students gained a deep understanding of the topics from pattern observation.

Another Year 8 classroom study using Polya’s approach to problem solving was implemented by Troyahn (1998). Overall, there were significant differences in favour of the experimental class on measures of problem-solving performance when compared to the control class. The author was able to identify both the cognitive and metacognitive activity of the students and to show significant change in metacognitive activity in favour of the experimental group as a result of the intervention. There was only limited evidence that the intervention led to a change in affect in the experimental group.

The development of Year 7 students’ problem-solving abilities developed as they participated in an 11-week program of thought-revealing problem activities, comprising problem-posing and model-eliciting experiences (English, 1999). The study reported profiles of development of three students who displayed different levels of achievement in number sense and novel problem solving. Facility with problem structures, divergent thinking, and the processes involved in working on a model-eliciting task were key issues. The program was a positive and productive learning experience for the students.

The issue of context for problems is still open to debate. Sullivan, Warren and White (1999) studied Year 8 students’ responses to closed and open-ended questions. For the closed tasks, the context seemed to help for one set of tasks, but made no difference in tasks that included diagrams. For open-ended tasks, the context helped for the two sets of tasks, but seemed to make a third set (dealing with a complex concept) more difficult. In two sets of tasks, students found the open-ended tasks more difficult, suggesting that these required thinking above and beyond that required for the corresponding closed tasks. In the other case the open-ended task was easier. In explaining these differences the authors claim that the open-ended tasks which were more difficult required students to link two concepts and to use such links to conjecture and generalise. Such open-ended tasks may serve the role of stimulating students’ thinking to higher levels. Both the context and open-endedness seemed to effect the focus and student response to the tasks. Work on two-step tasks had previously been found to be harder because students could not readily focus on the necessary information to solve tasks quickly (Ayres, 1995, 1996; Sweller, 1994). However,

it seems that each type of task would contribute productively to classroom programs, and teachers could be encouraged to plan to use contexts and open-ended questions productively. In some cases the open-ended tasks may serve as a useful preliminary exploration of topics; in other cases, they may be better left until later. (Nibs & Putt, 2000, p. 111)

This emphasis on investigations and open-ended problem solving will contribute to national development.

There are many benefits in teaching mathematics through investigations. The following is a summary of some of these benefits.
1. It is more related to everyday life and therefore more realistic for students.
2. Its dynamic nature makes it more motivating.
3. It is geared towards the learner’s ability and interest.
4. It lends itself well to group work.
5. It allows for the learner’s creativity.
6. It integrates diverse areas of knowledge and different processes.
7. It puts the emphasis on the learner’s responsibility in learning.
8. The learner feels greater ownership of the process and knowledge.

(Southwell, 2001)

Implications of This Review of Learning for Curriculum

These studies emphasise the importance of social interaction and shared meaning, and of engagement in problem solving for learning. Problem solving can be assisted by students being encouraged to visualise and physically represent the problem, to pose problems and thus enable problem structure to be more evident, to evaluate their problem solving and learn other metacognitive tools. Reflection and discussion on different approaches to a problem can also enhance the mathematisation that needs to result from problem posing and solving.

Major Curricular Themes

In a later part of this review, materials from numerous overseas countries and from the states and territories of Australia will be considered in some detail. One of the results of this analysis will be to show that many of these jurisdictions do not segment their mathematics syllabuses into the small pieces which have characterised the syllabuses of NSW and many other countries. In this section, we use our knowledge of these other syllabuses and practices to attempt to synthesise some of the overarching themes or ‘big ideas’ which might form a foundation for the new NSW Mathematics K-10 syllabus.

Mathematisation

In the Netherlands, there is no unified mathematics syllabus, although more than 80% of primary schools subscribe largely to the approach known as Realistic Mathematics Education (RME)² (Heuvel-Panhuizen, 1999). The basis for this approach to mathematics education is the idea of mathematics as a human activity where “lessons should give students the ‘guided’ opportunity to ‘re-invent’ mathematics by doing it” (Heuvel-Panhuizen, 1999, p. 3). This involves students in the process of ‘mathematisation’—a term coined by the eminent Dutch mathematics educator, Hans Freudenthal, in the 1960s. Two forms of mathematisation are distinguished. The first is ‘horizontal mathematisation’, where “students come up with mathematical tools which can help to organize and solve a problem set in a real-life situation” (Heuvel-Panhuizen, 1999, p. 4). The other is ‘vertical mathematisation’ which “is the process of reorganization within the mathematical system itself” (Heuvel-Panhuizen, 1999, p. 4).

Connections

Even though it has often been suggested that mathematics has little connection to the real world or to the other learning being undertaken by students, this is not the case. If students are able to make these connections, they are much more likely to be successful in their learning.

² The Dutch translation of ‘to imagine’ is ‘zich REALISren’. It is this emphasis on making something real in your mind that gave RME its name. For the problems presented to the students, this means that the context can be one from the real world but this is not always necessary. The fantasy world of fairy tales and even the formal world of mathematics can provide suitable contexts for a problem, as long as they are real in the student’s mind. [Heuvel-Panhuizen, 1999, p. 4 #2]
When students can connect mathematical ideas, their understanding is deeper and more lasting. They can see mathematical connections in the rich interplay among mathematical topics, in contexts that relate mathematics to other subjects, and in their own interests and experience. Mathematics is not a collection of separate strands or standards, even though it is often partitioned and presented in this manner. Rather, mathematics is an integrated field of study. Viewing mathematics as a whole highlights the need for studying and thinking about the connections within the discipline. (NCTM, 2000, p. 64)

Connecting mathematics to other school subjects has emphasised inductive mathematics and inductive generalisations. Students have developed skills in argumentation and communicating observations but a balance is needed in the areas that distinguish mathematics as a discipline such as deductive proof (Kilpatrick & Silver, 2000).

Mathematics itself is a multidimensional discipline. It is a body of knowledge. It is a language. It contains algorithms and formulas. But if these are the only emphases in the classroom, then we have failed to concentrate on ... a primary characteristic of mathematics ... that mathematics is a sense-making activity ... that requires reasoning and justification. Each level of these curricula [e.g. National Science Foundation-funded curricula in USA] is full of verbs such as justify, demonstrate, explain, show, confirm, defend, and so on. (Robinson, Robinson, & Maceli, 2000, p. 115-116)

The notions of abstract-apart and abstract-general purported by Mitchelmore and White (1995) can well be linked to development in many areas of mathematics. In particular they point out that emphasising abstract-apart situations in which the mathematics is linked to one specific concrete example, may not assist students to make the links that are so powerful in mathematics. For example, students who can solve linear equations to solve a contextual problem have abstract-general concepts while those who can simply solve the symbolic cases have abstract-apart knowledge. 

**Algebraic Reasoning**

Students reason about their mathematical experiences. From an early age, students should be encouraged in their classroom discussions to use deductive reasoning, that is logically going from generalised statement to conclusions about particular cases. Students also examine particular cases, identify relationships, and generalise. They are thinking inductively (Tzekaki, 1996).

When students are discussing in this way they will be able to move to more formal and logical approaches as they go through their mathematical schooling.

Students represent ideas in their minds by using imagery. Pirie and Kieren (1991) particularly refer to primitive knowing, image making, image having, properties noticing, formalising, observing, structuring and inventising. Representations of mathematics in the physical form will both lead to this development and result from this thinking. Representations may be as tables, graphs, manipulatives, drawings symbols or in written form. Students learn from each in different and idiosyncratic ways. However, the representations can be a source of discussion and development of mathematical ideas. In particular, students’ algebraic reasoning will develop. Aspects of algebraic reasoning include the various roles of variables and the meaning of equality. Algebraic reasoning should be commenced early in a student’s school life (Greene & Findell, 1999; Schifter, 1999; Tang & Ginsburg, 1999).
Argumentation and Proof

The process of justifying one’s actions and one’s solution processes and results in a problem solving situation is a feature of everyday life. Children, as well as adults engage in such argumentation. Based on the work of Piaget, the ability to argue logically was placed within the realm of formal operations and so was considered beyond the realms of young children. Recent work, in mathematics education, and in other areas of cognitive development, suggest that this is not necessarily so (Dockett & Perry, 2000; Horn, 1999; Krummheuer, 1995; Perry & Dockett, 1998; Yackel, 1998; Yackel & Cobb, 1996).

For example, Yackel and Cobb (1996) have outlined the importance of creating a classroom context where students are encouraged to develop and present mathematical explanations and where the value of attempting to make sense of the explanations proffered by others is recognised. Krummheuer (1995) has developed an extensive analysis of a process for argumentation which he has observed in mathematics classrooms.

Argumentation, which has been shown to be a feature of mathematics classroom working within a socioconstructivist approach (Cobb & Bauersfeld, 1995b; Yackel, 1998; Yackel & Cobb, 1996), allows students to justify not only their own mathematical thinking but also allows them to distinguish between the strengths of arguments and whether or not the mathematics being constructed within the arguments is actually different from previous mathematical arguments which have been interactively constructed. This ability to evaluate the strength of arguments has been seen as a major difference between Japanese and USA mathematics classrooms (Inagaki, Morita & Hatano, 1999). It is also one of the foundations of mathematical proof that Ma (1999) has shown to be lacking in USA teachers.

The type of argumentation and proof used in a mathematics classroom will depend on the mathematical sophistication of the learners and the opportunities which are afforded them to justify their conclusions and methods (Yackel & Cobb, 1996).

Over the years of schooling, as teachers help students learn the norms for mathematical justification and proof, the repertoire of the types of reasoning available to students—algebraic and geometric reasoning, proportional reasoning, probabilistic reasoning, statistical reasoning, and so forth—should expand. Students need to encounter and build proficiency in all these forms with increasing sophistication as they move through the curriculum. (NCTM, 2000, p. 59)

One example of how this development can be achieved successfully through the provision of appropriate experiences and the guidance of a sensitive teacher is contained in Maher and Martino (1996).

Number Sense, Mental Computation, and Different Types of Numbers

While counting with understanding is a goal of early education, we find that students intuitively develop concepts about rational numbers and negative numbers. Mental computation and sharing ways of solving problems helps students develop their number sense. The work of McIntosh and his colleagues (e.g. McIntosh & Dole, 2000; McIntosh, Nohda, Reys & Reys, 1995; McIntosh, Reys & Reys, 1997) has clearly established the importance of number sense and mental computation in a student’s mathematical development. In these modern times with calculators and computers able to take much of the formal algorithmic work out of mathematical calculations, the importance of having a sense about where a calculation is going and about the reasonableness of a result is critical.
Multiplicative reasoning (see proportional reasoning) is integral to developing number concepts so that addition and counting should not be the only early experiences of number. Multiplicative reasoning helps the development of place value, especially with decimal fractions. The numerous subconstructs associated with common fractions are ideal for establishing the wholeness of arithmetic but they need to be embedded in practical experiences.

Nevertheless, negative numbers are to be grasped either by algebraic reasoning, rules, or by use of some form of practical embodiment. Some rules actually confuse the operations of addition and subtraction with position (e.g., two negatives make a positive). Hayes (e.g., 1996) has used cards with +1 and -1 and used dummy zeroes of +1, -1 together to solve operations with negative numbers building on basic counting notions, but this model is somewhat unreal, seems to require drill, and works for addition and subtraction and multiplication as equal groups if the commutative law is applied. Linchevski and Williams (1996) used two coloured discs on an abacus as an embodiment for entering and leaving the disco. The Year 6 students developed their own ideas of the operations on negative numbers. This study provides a good example of Realistic Mathematics Education in which the ideas are developed by intuition from a life-like experience.

Proportional Reasoning

Under this heading we include fractions. Common fraction notation usually represent rational numbers (Lamon, 1999) but there are many subconstructs associated with it. Decimal fractions, multiplicative reasoning, and ratio will also be considered. An emphasis on counting and addition, especially in developing multiplicative meaning, can reduce students developing proportional reasoning. For example, if students practice making equal groups for multiplication by counting by ones, then they are less likely to think multiplicatively (e.g., doubling or 3:1). Simple ideas of doubling and halving begin in the early years of school, and sharing has been a key to early development of proportional reasoning. Numerous studies have shown that, by Year 3, students show proportional reasoning in limited contexts (see summary in Hunting, Davis, & Pearn, 1996). Increasing and decreasing multiplicatively or in ratio, and the ideas of similarity for shapes can be well established by the end of Year 5. Proportionality connects many of the mathematics topics studied in Years 6-8 (NCTM, 2000).

Spatial and Geometric Thinking

Spatial thinking involves visual imagery processes such as recognition of shapes, transforming shapes, and seeing parts within shape configurations. It also involves spatial conceptualising and the interaction of visual imagery with these concepts. Students need to reason about both two dimensional shapes, three dimensional shapes, position in space, and different-sized spaces. Students in early primary school begin to reason about shapes by considering certain features of the shapes as well as using their prototypical images. Spatial thinking plays a role in making sense of problems and in representing mathematics in different forms such as diagrams and graphs. A degree of spatial awareness and related meanings are essential for using manipulatives in many aspects of mathematics. For example, separateness is necessary for counting and yet collections can assist with establishing composite units.

Data and Probability Sense

Data plays a critical role in our modern society. Much information is transmitted through graphs, tables and using statistical ideas. Students in primary and secondary
schools need to be able to deal with this data in a sensible way. In the same way that they need to develop a sense about number, they need to enjoy a sense about data. They need to be able to treat reports of data critically and to establish the veracity of claims for themselves—or, at least, to test this veracity when claims are made. The work of Watson and Jones and their teams (see, e.g. Jones et al., 2000; Watson & Moritz, 2000) has established in Australian contexts the need for students to develop such a data sense from an early age. Complementary work in other parts of the world has reinforced this notion of building data sense throughout the K-10 range (Cobb, McClain & Gravemeijer, 2000; McClain, Cobb, & Gravemeijer, 2000; Shaughnessy, 1997; Shaughnessy, Garfield, & Greer, 1996).

Chance (probability) experiences are part of almost everyone’s day. The language of probability is met regularly—we have heard two-year olds talk about the chance that it will rain, for example. Early introduction of probability language and experiences can assist in the avoidance of misconceptions in problems where intuition alone is insufficient to solve them (Bright & Hoeffner, 1993; Shaughnessy, 1992; Way, 1997). Further development of instructional strategies and learning programs can continue this development into later years (Jones, Langrall, Thornton, & Mogill, 1999; Linsell, 1997; Watson, Collis & Moritz, 1997; Watson & Moritz, 1998). There is a need to give students at all levels the opportunity to develop their thinking about chance and its quantification so that they are in a position to be able to make sensible decisions in situations of uncertainty (Borovcnik & Peard, 1996; Peard, 1996).

Models
While the use of manipulatives in mathematics education is well-established, particularly in the early years of school, there is a deal of evidence to suggest that such manipulatives are not automatically helpful in the development of students’ mathematical ideas (Baroody, 1989; Ball, 1992; Clements, 1999; Howard & Perry, 1999; Perry & Howard, 1994; Price, 1999). Part of the problem stems from the students’ inabilities to argue cogently from the analogies which are formed through the manipulatives or to be overcome by these analogies to such an extent that it is the manipulatives, not the mathematics which becomes most important (English, 1999). The Realistic Mathematics Education emanating from the Netherlands has suggested an alternative way of thinking about models. It is suggested, in contrast to the common approach where

the students are to discover the mathematics that is concretized by the designer,
... in the RME approach, the models are not derived from the intended mathematics. Instead, the models are grounded in the contextual problems that are to be solved by the students. The models in RME are related to modeling;
the starting point is in the contextual situation of the problem that has to be solved. ... The premise here is that students who work with these models will be encouraged to (re)invent the more formal mathematics (Gravemeijer, 1999, p. 159).

This approach to modelling allows a development of the notion of ‘model of’ mathematical activity becoming a ‘model for’ mathematical reasoning. For example, problems about sharing pizzas were modeled by the students by drawing partitioning of circles that signify pizzas (model of). Later, the students used similar drawings to support their reasoning about relations between fractions (model for) (Gravemeijer, 1999, p. 161).

The big idea here is to reconsider the notion of models and materials so that they become not the holders of the mathematics but both models for activity and models for reasoning, rather than simply bridges between concrete and abstract.
Profound Understanding of Fundamental Mathematics

In her outstanding book (Ma, 1999), Liping Ma introduces the notion of a profound understanding of fundamental mathematics and uses this to distinguish between the actions, knowledge and thinking of Chinese and USA teachers. We use it here as an indication of what teachers of mathematics in NSW K-10 classes should be striving for in their students. There are many challenges involved in this statement, some of which might be able to be addressed by a new K-10 mathematics syllabus. Others will need to be addressed by teacher education and professional development, school structures, and individual teacher commitment. However, it is salutary to end with a large quote from Ma as a clear challenge to the NSW mathematics education community to develop this big idea.

Profound understanding of fundamental mathematics (PUFM) is more than a sound conceptual understanding of elementary mathematics—it is the awareness of the conceptual structure and basic attitudes of mathematics inherent in elementary mathematics and the ability to provide a foundation for that conceptual structure and instill those basic attitudes in students. A profound understanding of mathematics has breadth, depth, and thoroughness. Breadth of understanding is the capacity to connect a topic with topics of similar or less conceptual power. Depth of understanding is the capacity to connect a topic with those of greater conceptual power. Thoroughness is the capacity to connect all topics.

The teaching of a teacher with PUFM has connectedness, promotes multiple approaches to solving a given problem, revisits and reinforces basic ideas, and has longitudinal coherence. A teacher with PUFM is able to reveal and represent connections among mathematical concepts and procedures to students. He or she appreciates different facets of an idea and various approaches to a solution, as well as their advantages and disadvantages – and is able to provide explanations for students of these various facets and approaches. A teacher with PUFM is aware of the “simple but powerful” basic ideas of mathematics and tends to revisit and reinforce them. He or she has a fundamental understanding of the whole elementary mathematics curriculum, thus is ready to exploit an opportunity to review concepts that students have previously studied or to lay the groundwork for a concept to be studied later (Ma, 1999, p. 124).
Curriculum Issues and Developments Elsewhere

Continuity and transition

Mathematics education for students should form a relatively seamless development across the Stages and within the Stages of a curriculum. Topics should be interconnected to provide the means for holistic development. Ideas may form early in school and expand in detail and character (e.g., using notational representations) through the years.

To cover the Years K-10, NSW currently has five mathematics documents. Leaving aside, for the moment, the specialisation undertaken by all students in Years 9, 10, this arrangement still results in each student being subject to different and discontinuous programs in the course of their compulsory years at school. As we have seen earlier, these courses have much in common but they also have many differences. Even the aims of the courses are slightly different, even more than one might expect just because the students are of different ages! What light can be shed by looking at other jurisdictions?

South Australia

A new curriculum framework is currently being trialled in South Australia. It covers the entire range from children’s birth to the end of secondary school—Year 12. This is a direct reflection of the fact that the one government department is responsible for both prior-to-school and school education. The draft framework—the South Australian Curriculum Standards and Accountability Framework—is broken into four bands: Early Years (Birth to Year 2 of school); Primary Years (Years 3-5 of school); Middle Years (Years 6-9 of school); Senior Years (Years 10-12 of school). The bands do not and are not meant to parallel school structures. The overall organisation of the draft South Australian framework provides a coherence across the bands from which curricula in each of the key areas can be developed. The draft framework provides a great deal of detail in the mathematics area and, linked to standards levels, assists teachers in developing their programs. The draft framework statements do not differentiate courses of study for particular groups of students, although there are statements made concerning Aboriginal and multicultural perspectives and equity issues related to gender, social class, geographical location, and disability (Department of Education, Training and Employment, South Australia, 2000).

Australian Capital Territory

The Mathematics Curriculum Framework from the Australian Capital Territory (Department of Education and Community Services, ACT, 1992) is clearly based on the National Statement on Mathematics for Australian Schools (AEC, 1991). It consists of four bands: Band 1 covers preschool-Year 1; Band 2, Years 2-4; Band 3, Years 4-7; and Band 4, Years 7-10. The framework is written as one document across the four strands with no differentiation for students of varying abilities, although the framework for Years 11-12 does suggest that “some of the more able students in Years 9 and 10 should be provided with courses which access some of the experiences from the post-compulsory years” (Department of Education and Community Services, ACT, 1992, p. 2).

Queensland

The preschool to Year 10 curriculum framework in Queensland is currently in draft form Queensland School Curriculum Council, 2000a). Similarly, the Years 1 to 10 mathematics syllabus is currently being developed Queensland School Curriculum...
Council, 2000b). Both documents suggest a coherent approach to the syllabus with no indication of separate courses across the years. However, there are certain outcomes which are marked as ‘discretionary’ and particular level. They indicate additional contexts or areas of learning and are considered desirable. “It is not expected that these discretionary learning outcomes will be demonstrated by all students” Queensland School Curriculum Council, 2000b, p. 12).

Victoria
The second edition of the Victorian Curriculum Standards Framework (Board of Studies, Victoria, 2000) provides a detailed frame for the development of a detailed mathematics curriculum in the Years Prep-10. It relies on the National Numeracy Benchmarks for reporting achievement in three aspects of numeracy - 'Number sense', Spatial sense' and 'Measurement and data sense' at each of Years 3, 5 and 7. The CSFII for mathematics is presented as a coherent framework with the standards providing six levels of achievement. It is expected that most students would demonstrate achievement at Level 6 by the end of Year 10. There are Level 6 extension outcomes for students who have met Level 6 outcomes before they have completed Year 10.

Western Australia
The Curriculum Framework in Western Australia establishes learning outcomes for all students from Kindergarten to Year 12. It is not meant to be a curriculum or a syllabus but a document guiding the development of learning and teaching programs according to the circumstances of the schools and teachers and the needs of the students. It is presented as a coherent whole with little indication of how different abilities of students might be catered for, except to say that it “does not prevent schools from offering programs that enable students to achieve outcomes additional to those specified in the document.

Tasmania
Tasmania basically uses the national statement as the basis of its mathematics syllabus. Hence, the details of the Tasmanian approach are not canvassed here.

Northern Territory
Since 2000, the Northern Territory Education Department has been working on the development of their Curriculum Framework (NTCF), with the first Pilot Version released in January, 2001 (Northern Territory Department of Education, 2001). One key feature of the overall NTCF is the notion of essential learnings which, to some extent can be aligned with the ‘new basics’ in Queensland. These learnings are described as follows.

**Essential Learnings** – These are the critical processes which all learners should develop as a result of their formal schooling and should enable learners to leave school equipped to be active participants and negotiators in a changing world. Essential Learnings are not additional or optional curriculum extras; they are central to all teaching and learning. They should form an essential part of every learner’s education, are developed in studies across all learning areas and can be used as a strategy for curriculum integration. (Northern Territory Department of Education, 2001, p. 2)

While these essential learnings provide the broad organisers for the syllabus, the learning areas, including mathematics, flesh this out. The following is a very useful summary of how that works in mathematics and also shows how links to national benchmarks will be made.
The outcomes of the mathematics component of the NT Curriculum Framework are organised globally into three strands from the start of schooling until Band 3 (which includes the Year 7 National Numeracy Benchmark). These are Number Sense, Measurement and Data Sense and Spatial Sense. For Bands 4, 5 and beyond, Number Sense separates into Number and Algebra while Measurement and Data Sense separates into Measurement and Chance and Data.

The entire process of Working Mathematically has been embedded in the Constructive Learner domain of the EsseNTial Learnings. The Constructive Learner focuses on the elements necessary for learners to become life-long producers and contributors and, in this case, using their mathematics appropriately and effectively to do so. In addition, to a lesser degree, Mathematics contributes to the development of the Creative Learner and the Inner Learner and provides ample opportunities to develop the Collaborative Learner.

National Numeracy Benchmarks are heavily embedded within the appropriate levels of the mathematics outcomes and indicators – learning experiences that will effectively develop numeracy in learners are those which are frequently accessed and used across the entire curriculum and within a wide range of real life contexts. (Northern Territory Department of Education, 2001, p. 79)

**Singapore**

The Singapore Mathematics Curriculum is currently being revised. However, there are certain characteristics of the present curriculum which are worthy of note here. The main one is the clear splitting of the mathematics courses from an early stage. At Standard 4—approximately Year 4 in NSW—there are two separate courses prescribed—normal and extended. In Secondary school, there are four courses—special, accelerated, normal academic and normal technical. Hence, the courses available in Singapore are even more diverse than in NSW. Another point of note in Singapore curricula is that in 1998, approximately 30% of the curriculum content in mathematics was removed to allow more time for problem solving and investigation. The major material removed was that which required rote learning and was not attached to ‘real-life’ situations. The consequences of this reduction in content are not reflected in the latest TIMSS results.

One point worth noting about the Singapore education experience is the high reliance on extra tuition outside normal school hours. In a recent report in *The Straits Times* newspaper, it was claimed that more than 1 in 3 school students (almost half of all primary students and a third of all secondary students) attend private tuition. The amount of money paid for this tuition amounts to about 5% of the annual education budget in Singapore (Quek, 2000).

**United States of America**

The key planning document for progressive mathematics education in USA schools is the *Principles and Standards for School Mathematics* (NCTM, 2000). In spite of the debate concerning what has been dubbed the ‘maths wars’, this document still provides the major hope for US mathematics education. It is based on the influential documents *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), *Professional Standards for Teaching Mathematics* (NCTM, 1991) and *Assessment Standards for School Mathematics* (NCTM, 1995). Six principles and ten standards are introduced to provide coherence and to organise the statements in each of the year bands, in much the same way as has been achieved in South Australia, Western Australia, Victoria and Queensland. There is no indication of a diversity of courses of study although it is well known that textbook series which claim to adhere to these principles and standards do encourage such diversity.
The Netherlands

There is no centralised decision-making system in the Netherlands concerning curriculum in schools. Schools can decide which textbooks they use and can develop their own curriculum. The three major determinants of mathematics education, particularly in Dutch primary schools, are:

- the mathematics textbook series
- the ‘Proeve’—a document recommending the mathematical content to be taught in primary school
- the key goals to be reached by the end of primary school as described by the government (Heuvel-Panhuizen, 1999, p. 10).

Recently, in 1998, a fourth determinant has been added gradually with efforts to produce longitudinal teaching/learning trajectories in each of the areas of mathematics to be learned. Each “teaching/learning trajectory provides the teachers with a more or less narrative sketch of how the learning process can proceed” (Heuvel-Panhuizen, 1999, p. 15). The trajectories differ in their form and intent from the traditional focus on clear goals and outcomes. They are designed to provide an overview of how students’ mathematical understanding might develop.

In no way, however, are the trajectory blueprints meant as recipe books. They are rather intended to provide teachers with a mental educational map which can help them, if necessary, to make adjustments to the textbook. Another difference with the traditional goal description is that there is no strict pre-requisite structure. In addition, the learning processes are not regarded as a continuous process of small steps, nor are the intermediate goals considered as a checklist to see how far students have got. … such an approach … does not take into account the degree to which understanding and skill performance are determined by the context and how much they differ between individuals. (Heuvel-Panhuizen, 1999, p. 15)

Japan

The following description of the Japanese education system is drawn from Hughes, et al., (2000, p. 95).

The Japanese education system is highly centralized, with a national curriculum that is revised every ten years or so. All teachers are required to follow this curriculum and textbooks can only be used in schools if they have been authorised by the Japanese Ministry of Education. However, individual teachers have a certain amount of discretion in deciding what teaching methods, materials and resources they will use in a particular lesson. …

Classroom practice is underpinned by two fundamental principles:

- all children are taught in same-age, mixed ability classes;
- the whole class proceeds at the same pace.

The first principle means that there is no setting or streaming according to children’s ability. The second principle means that, for any given lesson, all the children in the class are being taught the same part of the curriculum. Both principles reflect the fundamental aim of Japanese state education of ensuring that all pupils have reached a minimum level of attainment by the end of compulsory schooling at 15 years of age.

Running alongside the state education system is an extensive system of private education, most notably in the form of evening classes (Juku). A large number of Japanese children attend Juku, for up to two or three hours a time on three evenings a week.

In Japan, the curriculum is much smaller—in terms of the number of topics to be covered—than is typical in Western countries, including Australia. Hence, teachers
and students are able to spend a much longer period of time—sometimes, weeks—on the one topic. Another strong feature of Japanese schools, especially at the primary level, is a focus on the whole child. “Japanese elementary teachers see their job as ‘raising children’—as promoting children’s social, ethical, emotional, aesthetic, physical, and intellectual development” (Lewis, 2000, p. 17).

**England and Wales**

There is one mathematics curriculum for government schools in England and Wales and this is embodied within the National Curriculum. The National Curriculum was introduced in 1988 and has undergone numerous revisions since—the latest in conjunction with the National Numeracy Strategy is ongoing, with material for Years 8 and 9 currently in draft form. The curriculum is broken into four Key Stages with the latter split in mathematics to form two courses—foundation and higher. In each of the following areas, eight attainment targets (plus one each for exceptional performance) are specified: using and applying mathematics; number and algebra; shape, space and measures; and handling data. There is a great deal of information given in the mathematics curriculum and reference is made to detailed supplements of examples (Department for Education and Employment and Qualifications and Curriculum Authority, 1999).

In 1999, the British government implemented the National Numeracy Strategy (Department for Education and Employment, 1999) with the aim that 75% of all 11 year-olds would attain Level 4 in the mathematics National Curriculum tests by 2002. (In 1999, the equivalent percentage was 58%.) The strategy is quite prescriptive and has had a major impact on the teaching of mathematics in primary schools:

> The definition of numeracy adopted by the National Numeracy Strategy is centred on basic skills, i.e. facility in both mental arithmetic and traditional written procedures rather than application of numbers to real life problems. However because the broad UK national curriculum is statutory, the strategy has had to encompass all aspects of primary mathematics. (Brown, 2000, p. 3)

The National Numeracy Strategy has mandated a number of details concerning the lessons in mathematics which must be implemented. For example:

- schools will provide a structured daily mathematics lesson of 45 minutes to one hour for all pupils of primary age;
- emphasis on mental calculation throughout, and de-emphasis in the early years on ‘vertical’ written methods and calculator use;
- daily three-part lessons with each part of a specified length (mental / oral warm-up, direct teaching of the whole class or groups, plenary review) to significantly increase the amount of whole class teaching;
- a national framework of detailed and exemplified objectives specified for each group of lessons throughout each year of primary school;
- provision of regular mathematical activities and exercises that pupils can do at home;
- a systematic national programme to train teachers. (Collated from Brown, 2000; Department for Education and Employment, 1998; 1999)

The level of prescription of both the mathematics curriculum and the National Numeracy Strategy has meant that teachers and students have very little flexibility in their approaches to mathematics learning and teaching.

**Cross National Studies**

One purpose of cross-national studies is to highlight parts of mathematics that need to be addressed by a curriculum. For example, Cai and Silver (1995) showed that while Chinese students in Years 5 and 6 outperformed USA students on a long
division with remainder procedure, they did not do as well on interpreting and giving a correct solution in terms of the problem.

Another study (Cai & Santel-Parke, 1995) showed that the non-representative sample of Year 6 students in USA generally scored lower than the three non-representative samples from China, Taiwan, and Japan on being able to recognise different representations of the same question (pictorial and numerical). One Asian group did not score as well on visual representation of fractions. Cai and Santel-Parke (1995; 1997) noted concrete examples and manipulatives are used differently in Chinese classrooms where the linkage to abstractions is made. Generally there was less use of manipulatives in USA classrooms than Japanese and Taiwanese classrooms in which the abstract concepts are built up from the materials (Stigler & Stevenson, 1991). For example, the use of concrete materials allows proportional reasoning to be introduced into Year 4 in Japan.

The Dutch approach only recently paid a token gesture to a national curriculum by having 23 simple goals for primary education that are not a great deal of help on how to teach (Heuvel-Panhuizen, 1999). They do not include probabilities but they do include some geometry and calculator use thus supporting approaches recommended by mathematics educators and so assisting to see these aspects considered in teaching. However, the impact of the research institutes and regular publication of easy to read information on mathematics education has influenced textbooks that are commonly in use. The approach tends to have a number of basic principles including having whole class discussions of problem activities that students can readily imagine and that may have been attempted by students at a level suitable to their own ability. The initial problems are often supported by a model which is then able to be abstracted to other situations. For example, the stylized diagram of the number of people getting on and off a bus can be used for other addition and subtraction problem. The empty number line, that is used for jumps in simple operations like mental addition, is expanded later as a double line for ratios with the two sets of numbers on either side and for fraction and percentages.

One important message about the Dutch approach is that the levels that are used are didactical levels differing from so called cognitive levels of development of students' knowledge (e.g. Piagetian type concrete-abstract levels, the Count Me In Too approach, concrete materials to mental approaches), the size of numbers, or the type of problem (e.g. one step or two step problems). They provide the kind of teaching activity that will generate discussion and the context can be pushed back to develop the mathematical ideas.

"To be numerate in a particular situation a person needs a blend of mathematical, contextual and strategic knowledge" (Hogan, 2000, p. 19). The mathematical topics are linked to the students' life experiences by strategies such as identifying key features of the problem and where mathematics can help.

Key Issues in curricula

Key Issues Raised in NCTM Principles and Standards in School Mathematics

In General

- Conceptual understanding is an important component of proficiency.
- Learning with understanding is essential to enable students to solve the new kinds of problems they will inevitably face in the future.
• Assessment should also be done for students and be a routine part of ongoing classroom activity rather than an interruption.

• Technology is not a replacement for basic understandings and intuitions but it can dramatically increase the possibilities for engaging students with physical challenges in mathematics.

**On Numeration and Computation**

• Representing numbers with various physical materials should be a major part of mathematics instruction in the primary school.

• The same operation can be applied in problem situations that are quite different from one another.

• Developing fluency requires a balance and connection between conceptual understanding and computational proficiency.

• Part of being able to compute fluently means making smart choices about which tools to use and when.

**On Other Topics**

• Geometric ideas are useful in representing and solving problems. Students should gain experience in using a variety of visual and coordinate representations to analyse problems and study mathematics.

• Instructional programs should not repeat the same measurement programs year after year and students should learn to choose appropriate units for measurement.

• Many students have difficulty with understanding perimeter and area.

• Understanding that all measurements are approximations is a difficult but important concept for students.

• Probability is connected to other areas of mathematics.

• Students should learn what it means to make valid statistical comparisons.

**On Problem Solving**

• Problem posing is an integral part of all problem solving.

• The teacher's role in choosing worthwhile problems and mathematical tasks is crucial.

• Opportunities to use problem-solving strategies must be embedded naturally in the curriculum across the content areas.

• Reasoning and proof begin in the early years. A mathematical proof is a formal way of expressing particular kinds of reasoning and justification.

• Conjecture is a major pathway to discovery.

**On Communication**

• Communication is an essential part of mathematics and mathematics education, and it is intertwined with reflection as processes in mathematics learning.

• Written communication should be nurtured but it is important to avoid a premature rush to impose formal mathematical language.
On the Balance of Topics

All students should learn algebra. Over the spectrum of years, number and geometry are more and equally important in PreK-2 and 3-5 with measurement and data analysis / probability having about equal but smaller shares. Measurement decreased in Years 6-8 with geometry increasing. Number also decreases giving space for algebra to increase gradually across the years.

In Years 6-8 students should learn significant amounts of algebra and geometry and see them as interconnected. Mathematics should be substantive, engaging and meaningful learning.

Key Issues of South Australian Curriculum Standards and Accountability Framework from birth to Year 12

The following are general principles of education.

- Learning is the process of constructing knowledge.
- Learning is not linear, it involves learners extending, elaborating, reorganising, reformulating, and reflecting upon their own frameworks of knowledge.
- Learning involves building on prior knowledge.
- Learning is making explicit the implicit conceptions, the frameworks and explanatory systems in the minds of learners, which shape how they interpret and what they learn in fundamental ways. Learners' conceptions are embedded in their culture and tied to their use of language.
- Learning occurs in a context and the understandings about the context is part of what is learned.
- Learning involves learners communicating their questions, intuitions, conjectures, reasons, explanations, judgements and ideas in a variety of forms.
- Learning involves developing knowledge, skills and dispositions to think and act in ways which determine individual effort, the setting of personal goals, self-assessment, awareness of the uses (and misuses or abuses) of knowledge.
- Learning involves the progression of learners through cycles of growth.
- Essential learnings and the capabilities and dispositions they are designed to foster are
  - Identity; a sense of personal and group identity, the capacity to contribute to, critically reflect on and shape relationships amongst individuals and groups;
  - Thinking; a sense of creativity, wisdom and enterprise and the capacity to contribute to, critically reflect on and shape ideas and solutions; interdependence; a sense of being connected with others and their world and the capacity to contribute to, critically reflect on and shape their local and global communities;
  - Futures; a sense of optimism about their ability to shape the future, and the capacity to contribute to, critically reflect on and shape possible futures;
  - Communication; a sense of the power and potential of literacy, numeracy and information and communication technologies, and the capacity to contribute to, critically reflect on and shape their present and future through powerful uses of literacy, numeracy and information and communication technologies literacy.
Students use emergent literacy and numeracy skills to represent, describe, analyse, question, hypothesise, recall and recount experiences.

This includes learning:

- Literacy skills of identifying and using symbols and patterns, and understanding the relationships between these;
- Skills to use symbols, images or words to represent objects;
- Skills in recognising familiar signs, labels and names;
- Skills and disposition to experiment with emergent writing;
- Engagement in and display of a disposition to enjoy literacy and numeracy activities.

Perhaps more could be said on valuing numeracy in these overarching statements.

**Exploring, Analysing and Modelling Data**

- Students pose questions to explore and investigate situations that interest them, to make decisions and to plan actions. They organise and represent data they collect, or data provided by other sources.
- Students interpret data using methods of exploratory data analysis.
- Students develop skills in making and evaluating predictions, inferences and arguments that are based on the data.
- Students engage with data to develop, understand and apply notions of chance and probability.

**Measurement**

- Students understand attributes, units, and systems of measurement.
- Students apply a variety of strategies, techniques, tools and formulas for determining measurements.

**Number**

- Students understand numbers, ways of representing numbers, relationships amongst numbers and number systems.
- Students understand the meaning of operations and how they relate to each other.
- Students use computational tools and strategies fluently and estimate appropriately.

There are no changes in the early and primary school outcome statements for number since the last revision.

**Pattern and Algebraic Reasoning**

- Students learn to create, analyse and generalise numeric and visual patterns, and they learn ways of describing a variety of relations using words, and where possible, symbolic rules.
- Students analyse mathematical structures, and use symbolic algebra to represent situations. They solve problems involving linear relationships.
Students use mathematical models to make connections and analyse how things might change in both real and abstract contexts.

Spatial Sense and Geometric Reasoning

- Students explore, identify and analyse relationships between spatial attributes of two and three dimensional shapes.
- Students explore and analyse ways of representing locations, pathways and arrangements.
- Students analyse and use flips, slides, turns and dilations to explore geometric relationships.

Key Issues in the Queensland School Curriculum Years 1 to 10 Mathematics Syllabus-in-Development

- Students are expected to be critically aware and appreciate the nature of mathematics, its value and power, its uses and how it assists in judging, for example size.
- Students should have confidence to tackle problems.
- Students are expected to have knowledge and understanding of facts and concepts.
- They are expected to be creative to design methods and adapt procedures, to study, interpret and solve problems to investigate, discover, hypothesise and use logical arguments with initiative, persistence, and originality.
- Communicating and appreciating others’ arguments is important in learning mathematics.

Queensland suggests around 200 hours per year in the first 3 years, around 180 over the next four years, around 80 over the next 3 years.

They present the curriculum as level statements. They draw together statements similar to the overarching statements of NCTM with more specific appropriate content for each of the levels.

They list the number strand under number concepts; addition and subtraction; multiplication and division. Money becomes the third part of the elaborations of number concepts along with (a) count, pattern and order, and (b) quantity and place value.

By the end of level 4, expected by the end of the 7th Year at school, students can multiply decimals by 10, recognise different formats for division and divide by 1 digit whole numbers giving both decimal and common fraction remainders. They can also give common fractions and whole percentages of numbers. They also mention the role of inverse and identity (0, 1) in operations and sequences of operations when making connections between multiplication and division. They recognise the importance of every day rates such as words per minutes and simple direct proportion problems. It is only at the end of level 5 (end of Year 8) that students are expected to multiply and divide with decimal numbers and common fractions and to use ratio language, calculate rates, and more complex problems.

In Algebra, students are expected to notice and later record patterns as a precursor to algebra.

Measurement is given in three areas (a) time, (b) mass, length, area and volume, and (c) angle.
There is a chance and data strand.
The spatial concepts and visualisation strand is divided into (a) shape, (b) location and movement, and (c) transformation and symmetry.

Aspects of geometry such as naming points and angles, finding angles on sets of lines is not done until level 5. No mention of proof is given, and even congruence of triangles is not covered formally, until beyond level 6.

**Key Issues in the Singapore Curriculum, 1990 Revision of 1981 Curriculum**
The key aims are:

- Acquire the necessary mathematical knowledge and skills, develop thinking processes and apply them in mathematical situations that they will meet in life;
- Use mathematics as a means of communication;
- Develop positive attitudes and a sense of personal achievement in mathematics; and
- Appreciate the importance and power of mathematics in the world around them. (p. 2)

This is centred around a mathematical problem-solving framework that emphasises concepts, skills, processes, attitudes, and metacognition. Succinctly the concepts are numerical, geometrical, algebraic, and statistical. The skills begin with estimation and approximation; mental calculation; communication; use of mathematical tools and then manipulations. The processes are deductive reasoning (including logical thinking, deducing new information from existing and drawing conclusion), inductive reasoning (including recognising patterns and structures and forming generalisation), and heuristics for problem solving (including using diagrams, tables, making suppositions and so on). Words such as *estimate, calculate, visualise, use* are common among the objectives.

Some points can be raised about the early stages in which students might first do things. However, the notes also specifically exclude some items. For example, in Year 1 students are to develop their multiplication tables but they are to exclude learning them. With fractions they are to know about halves and quarters but notation is excluded in Year 1, then they talk of fractions represented by areas but exclude discrete objects. Later they solve fraction problems of additions and subtractions, first with no more than 2 different denominators and calculate fractions of discrete objects in a whole.

By Year 3, students are doing four operations with money in decimal notation as they do with other measures.

By Year 4, students are discussing ratio and proportion and solving real life measurement problems excluding compound units.

The recommended texts are intended to be about problems solving with one objective being about visual thinking. Each year is a teacher's book, a textbook supplemented by a workbook and computer mathematics for further practice. They should begin with concrete materials. The colourful textbook tries to encourage several answers, telling stories about a problem, encourages expressions, for example, of subtraction and comparative statements about the one diagram. Diagrams are deliberately chosen to encourage a visual picture. Money is introduced in the sense of changing money and what can be purchased. One thing that seems to happen is that teachers end up using just a textbook (without a teachers' guide) so only a teacher who appreciates the full context of the lesson will use the textbooks well. There are also "Let's Explore" sections such as how to make up different
numbers or the quite difficult tangram puzzle of using the pieces to make 2 equal sized squares. By Year 2 they are presented with standard right-left algorithms for addition and subtraction. The visual clarity of the coloured summary diagrams is important. For example, place value charts always have circles with 10 in the tens column. There are also subtle points; for example, multiplying 7 by 2 is presented with a diagram of two strips of dots, 5 in one colour and 2 in another. It is important that there be class discussion on the solution here. Interestingly, many number ideas are related to measures. For example, they look at the dials of different scales given in both kilograms and then grams.

By Year 3, the students are solving a number of problems supported by diagrams in which units are mixed, number sizes are relatively large. There is a large section on fractions and matching notation to meaning. Angles are made, areas divided into units are calculated and perimeters formed by thread and measured.

By Year 4, the use of the place value charts extends to decimals with extensive work on decimals and operations on decimals such as adding decimal numbers with numerous place values or multiplying and dividing by whole numbers. Symmetry and isometric drawings of 3D shapes are the major geometry focus.

By Year 5, the links between different representations, for example, percentages, decimals and fractions, is emphasised. Diagrams still abound especially in the introduction to equivalent ratios.

By Year 6, proportions and percentages are compared using a linear representation. Angles, volumes of cuboids, areas, nets are all covered with complex diagrams used.

Key Issues in a USA Program

The *Investigations* program from the Teacher Education Research Council (TERC, 1995–1999) recommends even for Year 2, investigations that may take one to 3 hours over the equivalent number of sessions. Each book points out how to plan ahead, e.g., the materials to be prepared, the mathematical emphasis, and the assessment resources. In general, resources include literature, computer programs, concrete materials and household goods, while assessment tasks are also the learning experiences. The mathematical emphases varies from broad ideas such as recognising combining and separating situations to activity such as moving along a path, visualising and then representing a path. Nevertheless a number of questions are provided for teachers such as how do students make direct comparisons of length? Or How do students describe movements along a path? Dialogues are regularly provided to illustrate how the investigation will encourage students to mathematise or where they may have difficulties. The types of natural language teachers can use is also illustrated and specifically mentioned. Student sheets are provided to make games and assessment in particular easier to prepare. The following provides a sample of the investigation books that are used. The first group are intended for Year 2, sometimes recommended for either year 1 or Year 1 or Year 3 as well.

1. **How far? And How long?** This unit introduces students to physically measure by comparison and make paths that form shapes and then using the simplified logo language to follow mazes and make shapes. Teachers are given considerable guidance to the program. Interestingly, 12 turns are required to make a circle.

2. **Coins, coupons, and combinations** introduces skip and jump counting, using coins to help. Numbers are generally two digit.
3. Does it walk, crawl or hop? Provides students with the chance to classify and use two attributes to form Venn diagrams. Fantasy figures and fuzzy real word situations are used. These investigations can be supported by a computer program.

3. How many pockets? How many teeth? Is an introduction to collecting and representing data but, for example, pocket towers are compared. It is also a good way of collecting two unrelated sets of data such as number of students at school and number of pockets each day.

4. Timelines and rhythm patterns are interesting ways of teaching sequencing, and of introducing the number line in which numbers a little smaller than 3 (e.g., I was not quite 3 when my brother was born) can be considered. Rhythms also introduces notations and timing.

5. Putting together and taking apart reminds teachers of good strategies and misleading strategies for different story problems, and the use of the equals sign or line diagrams to help with representations that link positively to later uses of such representations. A 100 square is used to make number combinations by adding in jumps.

6. Thinking mathematically for grade 2, is a smorgasbord of activities, numerous counting games that stress relationships like groups that make 10 and usually set out so students can make a choice of activity, making and analysing 3D shapes from cubes, and using a computer program for tessellating and patterning.

The Investigations program from TERC (TERC, 1995-1998) recommends for Year 4 making multiplication tables and doing clusters of problems. Mental calculations include how to look at the strategies of using landmark numbers (multiples of 10s and 100s), skip or jump counting on, and working from left to right for additions. Interestingly, they also give numerous symmetry activities using pattern blocks and geoboards and they introduce grids including positive and negative numbers (e.g. for sunken ships). Some of the materials for Year 5 include exploring landmarks in the number system, including a large 1 000 chart, and simple reasoning about numbers that are added and subtracted and rearranged to make zeroes. Making a hundred, factor pairs, solving number puzzles, and getting close to 1000 or 0 in a number game (positive and negative numbers may result). Another book provides for extensive analysis of tables and graphs as patterns of change. Investigations on shapes include making shapes that follow rules, finding angles and turns together, and building similar shapes. These multifaceted investigations are assisted by the Geo-Logo computer package. Measurement considers different tools, mapping 100 meters and how far products travel.

In Year 5, weight and volume are explored with units used in shops and the real world, and time is part of an investigation on TV commercials and lifestrips. Common fractions and decimal fractions and percents are investigated in Year 5 with models, grids, and graphs as sources of discussion. Reasoning about multiples, what to do with extras, multiplication and division clusters, millions, and estimation are key areas on number in Year 5. Between Never and Always uses common games to establish the ideas of probabilities and comparing them as well as fair and unfair ideas while an exploration of data allows students to use fraction, decimals, and percents in dealing with "Sampling Ourselves", reading newspapers, and analysing and presenting data on playground safety.
American Mathematical Association of Two-Year Colleges

This group emphasises the importance of preparing students for the future with an emphasis on algebra and real-life contexts. However, it seems that some areas fall back into being a list of topics.

**Number and Data** has sections on mental estimation e.g., anticipated costs, distances, times; different kinds of numbers, units, magnitudes, extreme numbers and number sense; calculations of different kinds such as mental, paper and pencil and calculator; coding systems and their patterns; index numbers e.g. for stock markets; information systems for geographic and management information systems, and visual representations of data.

**Growth and Variation** has sections on linear change with rate of change as a constant, contrast examples with nonlinear change; proportions with situations modelled by similarity and ratio, disproportional change, calculating missing terms, and mental estimation using proportions; exponential growth such as population growth, compound interest, doubling, ordinary and log-scaled graphs; normal curve distributions for heights, repeated measures and manufactured goods and nonexamples such as incomes; parabolic patterns such as falling bodies, optimisation problems; cyclic functions for time of sunrise, sound waves.

**Chance and Probability** with elementary data analysis, quality control charts, outliers, data = pattern + noise; probability, chance and randomness, binomial probabilities, simulations, two-way tables; risk analysis

**Measurement and Space** includes direct and indirect means, estimation, instruments, plumb lines, calculated measurements, accuracy, tolerance, detecting misalignments; measurement formulas for plane figures, right triangle trigonometry and applications of Pythagorean theorem; dimensions as a factor in multivariable phenomena; geometric relations and proofs and constructions; spatial geometry on shapes in space, volumes, angles in three-dimensions (roof trusses), interpreting construction diagrams; nominal vs true dimensions (e.g., of 2 x 4s); tolerances and perturbations in constructing 3D objects

**Reasoning and Inference** statistical inference, scientific inference, mathematical inference, and verification

**Variables and equations** with algebraic reasons and ideas; equations, graphs, and algorithms

**Modelling and Decisions** with financial mathematics, planning, mathematical modelling, scientific modelling, and technological tools.

These areas of the curriculum take the early high school mathematics through to useful mathematics that should be considered in Years 9–12.

**Curriculum Expectations**

It is interesting to compare the big pictures of different curricula. For example, NCTM (2000) give a few major goals across pre-K to 12. They are given as

- Understand numbers, ways of representing numbers, relationships, among numbers, and number systems
- Understand meanings of operations and how they relate to one another
- Compute fluently and make reasonable estimates
Each of these is expanded into a few expectations such as count with understanding, develop understanding, develop a sense of size, develop and use strategies for computations.

A number of studies have shown that students begin school with a range of mathematics knowledge (e.g. Aubrey, 1993; Suggate, Aubrey, & Pettitt, 1997). Students’ beginning knowledge needs to be explored by teachers. Recognition of cultural background, for example, Indigenous cultural knowledge, may suggest ways to approach new experiences in number and other subjects (Jones, Kershaw, & Sparrow, 1995; Perso, 2001; Willis, 2000). Teachers are surprised at how well some students do on entry to school and in their first year (Bobis, 1997, 1999a; Clarke, 2000; NCTM, 2000). These differences should be taken into account in curriculum documents.

The provision of a sequence of learning experiences is not necessarily in the same hierarchical order for all classes or students. For more advanced students, it is not just a matter of speeding up activities. Nor do all students necessarily develop the same strategies. For example, some students may not spend time with counting-on strategies and many never need to count backwards more than one or two numbers. "However, flexibility in thinking about numbers ... is a hallmark of number sense" (NCTM, 2000, p. 80) and this should be reflected in curriculum documents. Curricula should cater for individual differences.

After reviewing the curriculum situation in the USA and showing that changes were needed if secondary schools were to prepare students mathematically for the future, Lott and Souhrada (2000) provide the following advice first offered by Moore at the start of the 20th century.

1. Integrate the mathematical concepts so that discrete courses of algebra, geometry, trigonometry, statistics, and so on, do not exist.
2. Determine the mathematical topics necessary for the new century and delete those topics that are no longer integral to the curriculum.
3. Use the tools of the day, including technology, to enhance the learning of mathematics.
4. Teach mathematics with a variety of techniques to meet the needs of all students.
5. Adopt new ways of assessing students' learning and methods of teaching.
6. Accept the reality that the mathematics curriculum is not fixed and will change frequently.
7. Ensure that a continuous, systematic effort is made to keep the public informed about what is needed in mathematics and that mathematical illiteracy at any level should not be tolerated.
8. Change teacher-preparation programs to meet the demands of new curricula.

(Lott & Souhrada, 2000, p. 108)

Numerous articles emphasise the importance of related mathematics, of investigations within meaningful contexts, and mathematics that prepares students for the workforce (e.g. Cumming, 2000; Forman & Steen, 2000). Forman and Steen discuss a range of curricula and provide appendices illustrating these curricula. They emphasise that employers and citizens have a stake in mathematics education. In particular, they point out the importance of mathematical sense in using mathematics, and the need to be literate in mathematics as citizens. They contrast, for example, NCTM standards that say

Represent situations that involve variable quantities with expressions, equations, inequalities, and matrices
with the California Academic Standards Commission that say solve linear equations and inequalities with rational coefficients; use the slope-intercept equation of a line to model a linear situation and represent the problem in terms of a graph

and the American Mathematical Association of Two-Year Colleges that suggest

The study of algebra ... must focus on modeling real phenomena via mathematical relationships. Students should explore the relationships between abstract variables and concrete applications and develop an intuitive sense of mathematical functions. Within this context, students should develop an understanding of the abstract versions of basic number properties ... and learn how to apply these properties. Students should develop reasonable facility in simplifying the most common and useful types of algebraic expressions, recognising equivalent expressions and equations, and understanding and applying principles for solving simple equations.

Rote algebraic manipulations and step-by-step algorithms which have received central attention in traditional algebra courses, are not the main focus ... Topics such as specialized factoring techniques and complicated operations with rational and radical expressions should be eliminated. The inclusion of such topics has been justified on the basis that they would be needed later in calculus. This argument lacks validity in view of the reforms taking place in calculus and the mathematics being used in the workplace. (cited in Forman & Steen, 2000, pp. 152-153)

This last statement provides a more rounded view of mathematics. This last group provides a functional mathematics model as illustrated by the content summary given earlier.

Teacher Reform

A key issue in change for effective mathematics learning is that of changing teacher knowledge and practice. Ball's (2000) approach is to encourage teachers to prepare activities but concentrate on what mathematising students will be doing in the lesson. Empson (1999) has shown that teachers implementing the TERC *Investigations* (TERC, 1995-1998) begin to think about how students might think. Part of the study asked teachers to write down solutions that students might give to a task. This was relatively effective when teachers were implementing the curriculum tasks in *Investigations*. At least the teachers developed concerns about teaching that were centred on students' thinking and grounded in the alternative conceptual trajectories of the curriculum. The teachers had acquired a predisposition to elicit strategies from children and to expect a variety of responses, and they believed that they and the students were benefiting. This means that a curriculum can at least generate tasks that elicit fairly predictable patterns of reasoning, and to alert teachers to the variety and meanings of those strategies. (See also Carpenter, Franke, Jacobs, & Fennema, 1998; Fennema, Carpenter, Franke, Levi, Jacobs, & Empson, 1996.)

The long-running project on counting and arithmetical strategies in NSW indicates the significance of changing teacher's content knowledge, pedagogical knowledge, and recognition of knowledge sources (Bobis & Gould, 1998). In particular, it is clear that resources and time for individual assessments must be addressed. Further, having a consultant or other teacher to show how to use an activity seemed important in building teachers' confidence in the approach. In particular, the use of groups was a significant change that teachers made to their teaching.
Clarke (2000) and his colleagues also found significant changes in teachers' professional growth when they participated in the Early Numeracy Research Project in Victoria. The individual assessments made teachers more aware of their students. The discussions between teachers and researchers assisted teacher development and recognition of important changes in teaching. For example, identified themes were:

- More focused teaching (in relation to growth points)
- Greater use of open-ended questions
- Giving children more time to explore concepts
- Providing more chance for children to share strategies used in solving problems;
- Offering greater challenges to children, as a consequence of higher expectations
- Greater emphasis on "pulling it together" at the end of a lesson
- More emphasis on links and connections between mathematical ideas and between classroom mathematics and "real life mathematics"
- Less emphasis on formal recording and algorithms; allowing a variety of recording styles. (Clarke, 2000, p. 22)

Teachers' also observed that students were better at explaining their reasoning and strategies, enjoying mathematics, and "having a go", experiencing success at all levels, and being aware of their learning. Teachers were more aware of how students' knowledge was going to develop and to use that knowledge in their teaching.

In general these three projects closely linked to curricula and showed positive changes in teaching in primary schools.

Gould (1997) provides high school teachers in particular with ideas for starting lessons other than the use of worked examples. These include the use of review of previous learning, or a short statement of goals. He gives an example for geometry with steps of (a) the teacher poses the task, (b) clarifies the task, (c) students work on the task individually or in groups, (d) the teacher leads summaries of reports from each group, and (e) these are reviewed and synthesised.

Haimes (1996) illustrates how teachers do not necessarily follow the intended Western Australian high school curriculum. The curriculum provided ways of approaching the topic but the teacher in the case study reverted to showing procedures. The teacher had only limited appreciation of the idea of algebra for problem solving and the importance of algebraic reasoning. This big idea is not necessarily clear if the curriculum becomes itemised. Professional development experience is really needed if teachers are to see algebra as more than procedures.

Over a four year period 18 of 21 teachers participating in a Cognitively Guided Instruction program in USA had fundamental changes in beliefs and instruction that the teacher's role evolved from demonstrating procedures to helping children build on their mathematical thinking by engaging them in a variety of problem-solving situations and encouraging them to talk about their mathematical thinking. Changes in the instruction of individual teachers were directly related to changes in their students' achievement. For every teacher, class achievement in concepts and problem solving was higher at the end of the study than at the beginning. In spite of the shift in emphasis from skills to concepts and problem solving, there was no overall change in computational performance. The findings suggest that developing an understanding of children's mathematical thinking can be a
Nevertheless implementation of *Cognitively Guided Instruction* has also met with mixed procedures. For example, Knapp and Peterson (1995) report that teachers fell into three groups (a) those who changed their theoretical perspective on teaching, encouraged students to correct each other, and enjoyed teaching with the students and (b) those who ended up turning the ideas of the workshop into new procedures such as using manipulatives first, then solving problems often individually in writing. The third group aspired to the theory behind the reforms but felt rather guilty that pressures and lack of experience meant they did not teach in this way. Frid and Malone (1995) also show that in fact the teachers who were wanting to have students work more autonomously did most of the negotiation and so ratification and endorsement really characterised the classrooms. The beliefs, expectations, and norms were too dominant for true negotiation to take place. In another study, 18 Year 1 teachers were observed using the USA-based *Everyday Mathematics* curriculum using the framework of eliciting children's solution methods, supporting children's conceptual understanding, and extending children's mathematical thinking. Though some teachers supported students' mathematical thinking, they were less often eliciting or extending it (Fraivillig, Murphy, & Fuson, 1999).

A study of 28 USA students from reform and traditional classrooms suggested that those from reform schools understood functions as models of real contextual problems more but both groups had limited depth of conceptual knowledge about functions (Williams, 1998).

Nevertheless, secondary teachers do succeed in implementing new teaching approaches. Williams (2000a, 2000b, 2000c) has investigated the role of collaborative groups in different school years including Years 7 and 8. She details some of the important aspects of her teaching. She has a clear purpose of letting students explore a concept through a task that begins with an introductory task or explanation that will allow them to work as a group. The groups are set up carefully to ensure positive support in the group when a more negative student may disrupt the group. The groups are given time to prime the reporter who then answers the specified question. Each group has a turn, the teacher makes sure students can hear and are listening to others. The order for reporting is usually in reverse for the next question.

Teachers need to be able to choose appropriate problems, especially those that will draw out possible solution strategies and reasons from students so that these ideas may be discussed and refuted by logical reasoning. While teachers judiciously provide, for example useful terminology, and counter examples, nevertheless, “teachers need to initiate, manage, and participate in mathematical discourse without becoming the sole authority. ...Shaping open and fair examinations of those ideas suggests a different view of mathematics to students- a view that mathematics is dynamic, growing, and created by people.” (Smith, 1996, p. 397).

Helping children to articulate their thought processes (Tickle, in press a) and share and discuss their various methods for solving number problems can help a class to select the best or more efficient method. For example, when adding $7 + 5$ most classes of students will suggest that it can be done by (a) starting at 7 and counting on, (b) breaking 5 into 3 and 2 so $7 + 3 = 10$, then $+ 2$ to get 12, (c) recognising the 5 in 7, i.e. $7 = 5 + 2$, so $5 + 5 = 10$, then $+ 2$ to get 12, (d) $7 + 5 = 6 + 6$ by compensating for the near double. It is then the teacher’s role to lead a discussion that would suggest that the second method (b) works more often and is quick and so
is more effective but the teacher and students still value all solutions. Most of these strategies will be developed by students through the use of concrete materials, especially 10-frame cards (Tickle, in press b).

Questions used for informal classroom assessment have been categorised by Liyanage, Irwin and Thomas (2000). Questions were surmised to fall into the following categories: (a) questions on subject matter that do not require any mathematical operation or transformation; (b) questions of one or a few steps without transformation; (c) questions with transformation from one representation system to another without interpretation or extrapolation or connection of two or more subconcepts, or with or without practical references, (d) questions with transformation and one of the points excluded in the previous level, and (d) questions requiring students to make generalisation with or without judgement. Reflecting on classroom teaching with these standards, it should be relatively easy to improve.

Selecting real life problems is one role for teachers. Students in Year 6 in Singapore did not always relate their solutions to the “real life” situation but were sometimes prone to indiscriminate and mechanical application of procedures (Menon, 1996; 2000). An inappropriate question particularly showed this. Menon argued that the curriculum needs to include more open-ended questions as well as questions with insufficient or extraneous information to allow students more flexibility in answering questions. This should lead to a narrowing of a perceived gap between the planned and implemented curriculum. This is crucial (NCTM 2000, p. 52) and illustrated by Williams (2000b, 2000c). In addition, appropriate problems for drawing out key ideas in mathematics are useful to teachers.

Another role for teachers is that of assessment through task-based interviews or lessons. A study of how well teachers rated students on the Count Me In assessment tasks, indicated there were some discrepancies but generally teachers agreed. Interestingly, these discrepancies were in the more straightforward areas of Forward and Backward Counting Sequences in which hesitation was taken as lack of knowledge. The areas requiring more complex development, such as place value and early arithmetic strategies, gave little variation between raters. More is said on this issue under assessment.

Changes in Research

Research into teaching and learning has changed in that researchers are not equating learning with immediate recall, retention, and transfer nor understanding with achievement. Learning is no longer an end product but an ongoing activity and the importance of social interaction on learning is recognised. (Kieren, 1994)

Recent mathematics education research has focused on classroom or actual learning situations so the qualitative approach to research has been showing what makes a difference (Cobb & Bauersfeld, 1995a.; Lambdin, Kloosterman, & Johnson, 1994; Steffe, Cobb, & von Glasersfeld, 1988 ). There are still many unanswered questions but even forming those helps change the approach to curricular development. For example, in geometry, Lambdin et al. pose the questions of what activities are most motivating and challenging and how can geometry be approached from a multicultural perspective? For fractions, they posed the question of what methods work best for helping students attach meaning to and determine reasonableness of operations on fractions? Interestingly, these questions are addressed in the literature in the years following 1994.
Assessment

By 1998, teachers in Victorian schools were reducing the examinations and tests used for assessment (Clarke, Gronn, & Clarke, 1998). Nevertheless, tests and examinations were more prevalent in secondary school than primary school and the use of journals and one-to-one interviews was greater in primary schools. Nevertheless Williams (2000b) illustrates how she used journals with a geometry investigation in Year 8.

Validity involves

how well assessment fits with educational goals, about the relationship between the demands on and delivery of assessment, about the adequacy of assessment methods, instruments and conditions, and about the consequences of assessment both for individuals and for institutions (O'Brien, 2000, p. 25).

Teachers, students, time available, timing, and student awareness of decision-making processes are all part of valid assessment. Validity is supported by reliability. Reliability is seen to be broader than the older psychometric measurement approach. Reliability is sustained by contextualised judgements based on firm evidence, and generalisation across tasks and readers. Teacher involvement in a syllabus that provides the criteria and standards for the assessment assists validity. Assessment must cover the full gamut of the curriculum. Formative assessment must be consistent with the expectations of the syllabus in terms of the nature of mathematical thinking. Time-constraints for tests make them unsuitable as sole venues for this kind of thinking, and alternative procedures are necessary. Nevertheless, creative ideas built on preparatory work can develop appropriate test tasks (cf. Burton, 1996).

In standards-based assessment, curricula provide standards for different levels of achievement. For each task, teachers can develop task-specific criteria, which are declared in advance of implementation of the task. Sometimes multiple criteria for a task means that clear guidance is needed for trade-offs.

Criteria and standards that are explicit, well-articulated, well-understood, and easily internalized and applied by teachers ... enhance the comparability and portability of results, and help students (and their parents) understand assessment decisions. ...

High-quality assessment:
• is integrated with learning,
• is adequate, comprehensive and authentic,
• produces fair and comparable results,
• involves processes that allow for reflection,
• is cost effective,
• and is accountable.
• ... is implemented and moderated in partnership between schools and the Board,
• is school-based, ...
• values contextualized teacher judgments,
• is based on criteria and standards, ...
• makes use of student portfolios as evidence of achievement. (O'Brien, 2000, p. 28, original emphases not included).

These points are also raised by others writing on assessment (e.g., Burton, 1996; Lee, 1998; Smith, 2000). Lee (1998) points out that assessment should come early to inform the teaching and learning and to assist students to celebrate their learning.
Mulligan (1996) provides examples of problems suitable for assessment of students' thinking in K-6. Clarke (1996) gives guidelines for assessing standards from observation. In particular, he builds on teachers' current observational practices to include not only the content but also mathematical disposition and a range of processes. Mansfield (1996) indicates how a teacher can practically implement suitable continuous assessment and recording. She summarises ideas on good questions, rubrics on understanding, solving and answering the problem, and the use of an analysis system whereby a teacher can find out if it is the reading, interpreting the problem, selecting procedures, carrying these out, or interpreting the results. She also gives an example of turning a multiple-choice question, "Which of the following is true about 87% of 10?" to asking for an explanation showing that only about half the reasons given by students were correct. More importantly, the answers were more informative for teachers on how students' were thinking. Newman's (1983) error analysis scheme is still important for teachers today.

Southwell (2000) bases her ideas on Angelo and Cross (1993) and discusses the types of assessment tasks that can be used to assess knowledge and skills such as focused listings, analysis and thinking such as pro and con grids, synthesis such as concept maps and one-sentence summaries, and attitudes such as classroom opinion polls. In practice, Smith (2000) shows how the sharing of goals and purposes of assessment of open-ended tasks could be achieved by students' self-reflection through self-assessment reports. The sharing of assessable work samples in a class also provides students and teachers with insights into mathematical learning.

An example of a rich assessment task on decimals at upper primary school is presented by Stephens and Sullivan (1997). The task includes some closed items requiring analytical thinking, open-ended items and an item requiring communication of thinking processes. Developing these tasks takes time and consideration of the assessed students' competence. They begin with the curriculum focus, the expected learning outcomes, how the ideas can be made into a tangible reality, how students can be active, and enjoy it, and how it can be made investigative, relevant, challenging and engaging.

Complexity becomes a major issue for selecting mathematical tasks. Williams and Clarke (1997) suggest that this complexity runs along several avenues. These are linguistic complexity, contextual complexity, representational complexity, operational complexity, conceptual complexity, and intellectual complexity. They provide a number of examples for secondary schools from enhanced multiple-choice questions, numerical response questions, "good" questions, challenging problems, and open-ended exploratory questions.

Several general criteria for assessing responses to tasks are presented in different articles. Tasks should be developed to promote these different kinds of responses. For example, Clarke (1996) lists:

- Problem solving—careful choice of strategies, thinking mathematically, self-monitoring of the problem-solving process, problem posing.
- Communication—can the student communicate mathematics at a high level of quality in written and oral form to the teacher and other students? Does the student reflect on and clarify thinking about mathematical ideas? Does the student use multiple representations appropriately in given situations?
- Reasoning—can the student make and test conjectures? Does the student understand the nature and purpose of counter-examples? Can the student construct simple arguments and carefully evaluate the arguments and conjectures of others?
• Connections—does the student see the links between these and other similar problems? Does the student demonstrate an understanding of links between mathematical topics (for example, the various forms of rational number)? Does the student see the link between a mathematical activity and its real-world application? Does the student see appropriate connections between mathematics and other disciplines?

Mansfield (1996) gives details for assessing criteria as follows:

• Exemplary response ... is complete and includes a clear and accurate explanation of the techniques used in solving the problem... accurate diagram, ... all important information, ... full understanding of the ideas and mathematical processes.
• Competent response ... is fairly complete and includes a reasonably clear explanation of ideas and processes used, ... solid supporting arguments.
• Satisfactory with minor flaws ... is completed satisfactorily, but the explanation is lacking in clarity or supporting evidence. The underlying mathematical principles are generally understood, but the diagram or description is inappropriate or unclear.
• Nearly satisfactory, but contains serious flaws ... is incomplete, ...major computational errors,... misuse of formulas, ... does not show full comprehension of the mathematical concepts.
• Begins problem, but fails to complete solution ... is incomplete, ... little or no understanding,... diagram or explanation is unclear.
• Fails to begin effectively ... [as] problem is not effectively represented ... no solution ... attempted, pertinent information ... not identified. (Mansfield, 1996, p. 20).

A similar 5 point set is used in the Victorian Exemplary Assessment Project - Mathematics (Beesey, Clarke, Clarke, Stephens & Sullivan, 1998). These include (a) fully accomplished the task and goes beyond specified level, (b) task accomplished, (c) substantial progress, (d) attempt at the task makes some progress, (e) little progress or understanding evident. These general rubrics can then be used by teachers to specify precise expectations for a problem.

Peressini and Webb (1999) look at foundational knowledge (a) concepts, facts, and definitions, (b) procedures and algorithms, (c) misconceptions. They also give details about the solution process - (a) analysis - understanding, exploration and verifying, and (b) reasoning-inductive (pattern), deductive, spatial, proportional, and abstracting/generalizing. A third area is that of communication, (a) language, (b) symbols/notation, (c) dimensions/labels, and (d) argument. They provide analysis of responses in these terms for two students who undertook the four investigation tasks that they give in the article.

This section would not be complete without reference to the extensive work of Ellerton, Clarkson, and Clements (2000) on assessment, language, and curricula. The language of the curricula impacts on teaching and learning. A significant proportion of responses to multi-choice and short-answer tests mismatch interview responses to questions indicating that conceptual development is often not assessed well by tests.

Using Technology

None of us would query the advantage of using a calculator to work out a problem like $123.45 \times 34.76327$ but we must recognise too the advantages of using them for working out complicated functions in order to use graphing for an analytical purpose.
Before calculators we studied calculus (applications of the derivative) to learn how to obtain accurate graphs. Today we use accurate graphs produced by a graphing calculator to help us study the concepts of calculus. (Waits & Demana, 2000, p. 57)

We may well be challenged by the achievements of the *Everyday Mathematics* curriculum (USA) which uses technology extensively. Year 4 students develop symbolic formulas, and in high school, matrices, graphs, and curve fitting play a role, largely made possible by technology. The effects of changing parameters in functional expressions is emphasised more than factoring and simplifying rational expressions (Robinson, Robinson, & Maceli, 2000).

Technology facilitates the observance of pattern and relationships, to create a virtual environment for exploration and conjecturing (as with geometric drawing utility), to create simulations, to provide an effective means for using mathematical tools and operations, ... to implement some algorithms or procedures, ... to access or organize data, to support a conjecture or general statement with experimental evidence, to check paper-and-pencil calculations, to facilitate the teaching of programming fundamentals, and to highlight the limitations of technology (e.g., a crucial error introduced by rounding, or the loss of information that occurs in digitized pictures). In addition, technology allows for the elimination or reduction in emphasis of some topics or skills such as complicated long division done by paper and pencil. ...to simplify some rational functions,...to focus on the concept of a linear asymptote. Technology also suggests new content such as computer graphics, dynamical systems, and fractals.

In summary, technology affects what students learn and how learning is accomplished. Teachers need to understand and be able to use technology in an ever-growing number of ways consistent with how people use it outside the classroom. (Robinson, Robinson, & Maceli, 2000, p. 123)

Using computers can change teaching. A study of the effect of introducing computers into mathematics teaching involved 12 New Zealand teachers who were involved in the action research (Thomas, Tyrrell, & Bullock, 1996). The resulting change in perspective, which for some teachers took as long as a year to achieve, was characterised by:

- a lessening of control and greater use of guided discovery learning that made use of discussion and group work;
- a willingness to learn along with the students;
- a desire to plan lessons involving the computer where its role is a tool for learning;
- an ability to make mathematics and its implications, and not the computer, the focus of concern.

Two important influences in bringing about the shift were the constant availability of the computer in the classroom, and the encouragement, motivation and assistance provided by the support staff.

Teachers need support to link curriculum to on- and off-screen computer work, strategies for linking on-screen mathematics to the curriculum, knowledge and understanding of the content of computer programs and the mathematical implications of this knowledge for students, and an understanding of how and what students will learn from their use and interaction with a computer program (Rollo & Bobis, 1996).
For primary school students (aged about 8) computer environments are satisfactorily dynamic and interactive to help students coordinate spatial and numeric concepts to novel problems (e.g., Geo-logo, see for example Yelland & Masters, 1997), and to link screen images to real three-dimensional space to develop understandings of perspective, proportion and angle (e.g., Lowrie, 1998). However, in these studies teachers played a significant role in encouraging teacher-student and student-student discussion, and this encouraged students to use higher-order thinking skills and metacognition. The computer provided immediate graphical feedback on the reflective modifications children made to their thinking about aspects of the problem tasks and how to solve them. This enabled the children to build and test ideas for themselves and, in so doing, make sense of mathematical ideas and problem-solving strategies.

The main finding of Turner's (1999) study of high-school teachers' use of Logo microworlds was that the formal requirements of the mathematics curriculum eventually "overwhelmed" individual teacher interest in the constructivist potential of Logo: "Teacher preference was for computer use as an 'efficiency tool', directly applicable to mathematical concepts as pre-determined within the subject's curriculum topic list, over supporting students learning to program as a basis for student-directed mathematical exploration or problem solving" (p. vii). This supports Yelland's (1997) claim that teachers will not use technology as a learning tool for their students until its use becomes embedded in curricula.

Dynamic geometry software, such as Cabri Geometry and The Geometer's Sketchpad, has the potential to return geometry to greater prominence in school mathematics curricula. A crucial factor when working in a dynamic geometry environment is that geometric constructions based on visual appearance alone—so called by-eye methods—are not drag-resistant, that is they do not retain the required geometric properties when individual components are dragged. Hence, dynamic geometry software forces students to think very carefully about the properties of the figure they wish to draw. Visual constructions provide an essential scaffolding for Cabri constructions at the start of an investigation. The use of Cabri enhances the use and understanding of appropriate geometric language, particularly when students actively discussed their constructions with each other (Cashman, 1997; Vincent & McCrae, 1999a). However, design of worksheets and teacher intervention and whole class discussion may be essential for the software usage to be effective (e.g., Stewart, 1999). In one study, the students using dynamic geometry software improved in the van Hiele geometry levels (Vincent, 1998). Studies on angles have been particularly successful (Brown, 1997; Dix, 1999; Redden & Clark, 1997). The angle sum of a polygon investigation by Brown involved primary school students. Engebretsen (1997) found conceptual knowledge, including vectors, was strengthened by the use of conjectures and trial and error methods with dynamic geometry software. O'Connor (1999) showed how it improved visualisation skills.

In Israel a traditional course for Year 9 students on functions began with a definition of a function and its graphical presentation and emphasised the linear function (Schwarz & Hershkowitz, 1999). The next version of the course included quadratic functions but the most recent version included the use of graphical function software, open-ended investigations and collaborative group work. Students' responses to a test on functions were classified in terms of (a) having and using prototypical images which may be an advantage or disadvantage, (b) part-whole reasoning by matching different representations with a particular representation of a function, and (c) understanding attributes in different representatives and types of representations. The number of idea units given in justifications were determined and used in
analyses. The prototypical image of a function is a straight line which may be modified by having a parabolic image if this has been introduced in school. For example, to illustrate the function that passes through 3 noncollinear points, students employed their prototypical image. There was less change from a prototypical image in earlier versions of the course but with the introduction of technology and the new collaborative approaches for teaching espoused in the latest version of the course, there were changes to curves. The researchers emphasised the importance of introducing many different kinds of functions in Year 9. The manipulations that were possible in the interactive learning environment encouraged a greater diversity of mental images.

Various studies have shown that students can use spreadsheets as a way of finding number patterns. The computer assists students to focus on patterns and to develop abstractions that are represented by algebra. One article appears in the primary school magazine *Teaching Children Mathematics* (Battista & van Auken Borrow, 1998).

Spreadsheets have a major role for high school mathematics by being a tool for data collection and manipulation and as a catalyst for the exploration and examination of patterns emerging from the data (Geiger & Goos, 1996). The nature of the task students were given to perform was identified as the major determinant for the differences, indicating that if computer tasks are to elicit high level verbal reasoning these tasks need to be carefully structured. The teacher's role should be to place the task in context, to ensure its problematic nature is not concealed and to direct discussion towards conjecturing and argumentation rather than simply information exchange. Software and task design features must be such to ensure student engagement with the software (see also Price, 1997; Lewis, 1996). Many studies indicate student attitudes are positive to spreadsheet or other program usage at different levels (Jun , 1997; Lockwood & Boland, 1998; Looi & Tan, 1996; Lowrie, Hill, & Hemmings, 1996; Richards, 1997; & Yoong, 1997).

The use of graphing technologies will broaden the curriculum. Both the local and global properties of functions and relations can be studied, classes of functions will be more completely studied, and functions that are not members of presently recognised classes can be studied (Harvey, Waits, & Demana, 1995). Furthermore, time may not be wasted on learning different techniques for finding the zeroes of different classes of polynomials and so. At this stage, the questions can vary to allow for a more general approach. For example, a box without a top is to be constructed from a rectangular sheet of cardboard that is 30 cm by 20 cm by cutting squares out of each corner and turning up the resulting sides, explore the possible volumes of this box. Texts should not present worked examples but rather discuss ways to solve problems and then present some limited examples that could be further developed by students. Students can be more self-directed, for example, by being able to check graph and algebraic solutions. Paper and pencil graphing should emphasise accuracy and understanding, and not be about speed or frequent usage.

Computer systems that can be used flexibly for teaching algebra show they improve facility in algebraic manipulation and in use of different representational forms (Arnold, 1996). However, while students liked to use graphical representation, its use was often automatic and superficial. A table of values was more difficult for students to interpret, and less frequently chosen. Computer algebra tools, enabling symbolic manipulation, were most effective for investigations and development of extended algebraic processes such as equation solving. However, some algebraic forms (including simple expressions and tables of values) fail to provide clear signals to
provoke meaning and action strategies (Arnold, 1996). The success of students using a computer algebra package to develop the lines of equations successfully, especially for equations of the form $5x + 12 = 3x + 24$, overcame both the limited perspective of students about equations and the gap between arithmetic and algebra (Thomas & Hall, 1998). Geiger (1998) found students said that graphics calculators enabled and supported mathematical learning and exploration in unfamiliar contexts. However, having to learn to use a graphics calculator, as well as having to learn the mathematics, is seen by some students who are already struggling with the mathematics as an additional and unwanted burden (Faragher, 1999; Tynan & Asp, 1997). Nevertheless, the different presentations of a topic assist students to understand the topic (Kissane, 1997, 1998; Tobin, 1997, 1998) and it is clear that careful development of their use to improve students’ comfort level is necessary. For students aged 12–14, Graham and Thomas felt that, by programming predictions on the home page of the graphic calculator and by trying out predictions easily, students developed "an understanding of letters in algebra as stores with labels and changeable contents" (Graham & Thomas, 1997; pp 3–13). Anderson, Bloom, Mueller, and Pedler (1997), Dunne and Brown (1996), and Howlett (1998) illustrate that many teachers use graphics calculators.

The development of computer algebra systems, especially on calculators, and their potential for linking numerical, graphical and symbolic representations is significant and is expected to have an impact on curricula, assessment, and teaching because it challenges the algorithmic nature of current teaching of algebra and graphing. In particular, key concepts will need to be a focus and students will have to know what a procedure is for, what input is required, when it can be used and what the software output means and whether it is sensible (Stacey, 1997).

Computer Algebra Systems (CAS) help students focus on choosing an appropriate sequence of algebraic manipulations leading to the solution of an equation and contribute to the formalisation of the approach. When required to solve word problems the CAS students were more likely to construct and solve an equation than attempt to use arithmetic-based backtracking or trial and error procedures. Since Stacey and MacGregor (1999a) have shown that backtracking does not help students to understand the variable concept or bridge the arithmetic-algebra gap, then CAS may help overcome some difficulties here. However, in Tynan and Asp's (1998) study, CAS students were no more successful in completing the tasks than the non-CAS students. However, they were "enthusiastic about using CAS as a dynamic checking device, and CAS use did not reduce student's by hand-algebra skills" (Tynan & Asp, 1998; p. 627).

Webpages (Shield, 2000) are a good, real-life relevant application of mathematics. Webpages for banks provide calculators that can be used to work out repayments and value added monies. These can be used to explore the effect of changing one of the variables. Sets of data can be generated and then placed in a spreadsheet like Excel to draw graphs. Currency conversions provide a practical example of ratio. Annuities provide exponential functions. Housing applications provide interesting spatial experiences, to be able to position a house on a given site, to consider boundaries, northerly aspects, and mirror images. A watering system site and industrial award sites provide many practical calculation problems with supporting images and information.

When emails or web-conferencing are available to students, it seems that getting people (including students thinking they have nothing to offer) to actually respond is one essential issue but then many students did value the opportunity for group
collaboration afforded by the web conferencing, and the access it gave to the views of others, but that they had difficulty with accessing and using the technology and working in groups (e.g., Foley & Schuck, 1998; Kidman & Nason, 1997). With primary school classes, enthusiasm and comments were dampened by the lack of face-to-face contact, being able to express one’s idea in writing and limited computer drawing facilities.

Using a sociocultural perspective on learning, the proposed roles for technology should be recognised by curriculum developers and researchers and teachers. The lowest level is that of technology as master. Here the complexity of usage limits student activity to a few operations over which they have competence and the student, without sufficient mathematical understanding, blindly accepts the output produced. In the next level, technology as servant, the user is in control, applying the technology as a reliable fast mechanical aid to complete mathematical tasks whose output is largely regarded as authoritative, and not questioned. In the third level, technology as partner, the technology is seen as a companion with which to explore, and not just a tool for doing something. There is also an awareness that an outcome needs to be judged against criteria other than the technology-produced response, with a consequent recognition of the need to balance the authorities of mathematics and technology. At the highest level, technology as an extension of self, the technology provides an extension of the user's mathematical abilities, becoming an essential part of the students' mathematical repertoire, something that shares and supports their mathematical argumentation (Galbraith, Renshaw, Goos, & Geiger, 1999).

Technology Rich Mathematical Assessment Tasks

Currently Western Australia has distinguished tasks for their final school examination that are especially for graphics calculator use, not for calculator use, and those which are not influenced by calculator use (Kissane, 1997). Victoria has also permitted the use of graphics calculators in final examinations. Victoria has provided extensive professional development for teachers on their use. The mathematics associations of NSW and of Queensland have encouraged the use of graphics calculators. In NSW, they may be used for School Certificate Examinations but not the Higher School Certificate. The point is that their use in senior levels will encourage their use at lower levels where they can be advantageous.

Assessing such tasks, as for the Victorian curriculum, requires both the selection of quality tasks and a scoring rubric which focuses on qualitative differences in how students are using, say, a graphics calculator (Lindsay, 2000). For example, at a very high level, the student accomplishes the task, correctly and accurately uses mathematics, uses the technology to give full, detailed and elegant explanations and interpretations, correctly links graphical, symbolic and numeric representations including any use made of algebra, and is able to move to generalisations. From a curriculum perspective indicators of quality must be included.
Developments in Specific Areas

Mathematical and Algebraic Reasoning

Reasoning and Proof

NCTM give the following across pre-K to Year 12.

- Recognise reasoning and proof as fundamental aspects of mathematics
- Make and investigate mathematical conjectures
- Develop and evaluate mathematical arguments and proofs
- Select and use various types of reasoning and methods of proof

These statements suggest that all school mathematics can and should be understood. Geometry provides many opportunities for giving a chain of reasons which is part of students' conjecturing and discovery.

In Years 3–5, students are putting forth their own ideas for examination. They must explain. They must listen to other explanations and reason how these fit together or show a difficulty that needs resolving. By Years 6–8, students need to know the limitations of inductive reasoning as well as its possibilities (NCTM, 2000).

Argumentation is seen as important as early as Year 2 (Krummheuer, 1995; Yackel, 1998; Yackel & Cobb, 1995; 1996). Proving is to be seen as separate from formal finished proofs. Proving is a fairly rigorous form of argumentation. In both, people tend to use their intuitions to move to a new step in the argument. Metaphors (e.g. Lakoff & Nunez, 1997) are either created or assist with the argument (Douek, 1999). However, proof requires logical constraints and establishing of validity of the argument by drawing on the body of related knowledge. Transforming knowledge is much like a metaphor. Its argument is usually non-linear and non-deductive. Transformational reasoning needs nurturing in school mathematics (Simon, 1996). Nevertheless, intuition, as Burton (1999), explains is important to mathematicians and essential for beginning investigations and justifications in school mathematics.

Clements (1997) provides an historical tour of why proof and proving died in secondary schools and concern has grown, especially since the National Curriculum in UK. Clements particularly argues that making generalisations and producing mathematical arguments should not be left till levels 7 and 8 of the Mathematics Profiles (Australian Education Council, 1994). Hoffer as far back as 1981 argues that geometry had to change and that geometry can develop reasoning skills rather than rote proofs. Logical arguments require being aware of properties from diagrams, and aware of ambiguities and connectives in language. For example, a trapezium with three sides, the length cut off the fourth by the altitude from the opposite vertex is drawn (without parallel or right-angle indicators). Students are asked what other information is needed to find the area. Such examples of logic are necessary before proof. (Hoffer goes on to show the van Hiele levels and how these apply to visual, verbal, drawing, logical and applied skills.).

As early as 1988, the Shell Centre for Mathematics Education in the UK was encouraging teachers to appreciate the ideas of proof. In a course book, prepared by Kaye Stacey, called Describing, Explaining, Convincing (Stacey, 1988), teachers are encouraged to consider a variety of investigations using patterns, counter examples, words, and clarity. A pattern and some examples might be enough to convince the person themselves but more was required to convince others of different
mathematical ideas. In a sense, this looked at the issue of rigour. A proof would need to be more rigorous again to convince a mathematician. Rigour has been an issue raised by, for example, Barnard (1996). He raised a number of issues because he felt that proof was more important than students doing investigations which he saw, at best, being speculative and pre-inductive thinking. Among his various arguments, he raised the issue of compression. He linked this to reification espoused by Sfard (1991), and procepts by Gray and Tall (1994). On this, there is no problem but it seemed that this was to encourage massive amounts of procedural mathematics. He felt students needed to develop and compress ideas in order to use them. For example, having a visual image of the equilateral triangle with line of symmetry and the half triangle having sides 1, 2, and \( \sqrt{3} \) could be unpacked for various purposes such as trigonometric values. He felt irrational numbers expressed as surds would be needed and not calculator generated number approximations. Similarly, fractions often expressed richer mathematical expressions than a decimal representation (see later comments on Lamon, 1999, for the multiple constructs for fractions). Barnard pointed out that mathematical expressions can be quite succinct and useful. While we can agree with his sentiment, his emphasis on rigour of expression may at times have been inappropriate since some mathematical expressions are complex and cause the mental load he seemed to wish to avoid (not to mention the difficulties of using these in computer webpages—some are just too complex). He quoted an example of a multiple-choice algebra question which was reported as more students in Australia selecting the wrong answer than before. However, the particular item contained a visual distractor which students looking for easy ways would choose. This is a point in itself, because in problem solving and answering multiple-choice questions in particular, students look for the easy way and not the correct way (see also arguments for students being able to consider two-step problems, not just by algorithmic approaches but as an expectation from early primary school). Barnard (1996) does point out, however, that context can sometimes make the application of mathematics less certain, whereas the mathematics itself is precise. It seems students have lost this sense of rigour but it is hard to accept the rigorous notion of only right and wrong in these days of stochastics. Students need to be aware of how rigorous an answer should be for a particular circumstance. Clarity of expression does need to be developed.

Slomson (1996) continues the debate and points out the different kinds of proofs for different students. Less formal is expected at schools and patterns may lead to a general statement even if the full induction is not followed through. Carrying out a calculation in a clear method can be a proof of one kind. Slomson points out that the Pascal's triangle relationship can be proved by combinatorics but also by algebra. While he does not advocate that students need prove everything, it is still important that students follow logical arguments and not learn bad habits. Neither the didactic teaching of many proofs nor investigations are likely to teach students to be able to write good proofs (it is like expecting everyone to be good musicians rather than appreciating good music). MacKernan (1996) uses Bertrand Russell's arguments to show Euclid did not prove well. Arguments can be of many kinds and not just deductive proof and students should realise that not all patterns will lead to a true statement. Pope (1996) continues by pointing out that non-algebraic, visual proofs are possible. Reid (1996) and Hanna and Jahnke (1996, 1999) both illustrate that deductive argument and thinking which explores and explains are legitimate mathematical, deductive activity, often used by scientists. Data collection, looking for pattern, making the next assumption are all part of mathematics. In one sense, it is the peers who agree whether a particular deduction is sound. For example, we
usually all agree with lines of equations that follow one from another, or the use of metaphor for an argument but the record of empirical results and patterns is a good start.

This starting point, according to Hewitt (1996), is the essential step that students must be aware and focus on the important properties. He believes that proof should be a part of all students curriculum at all ages because it focuses on properties but using properties is not without problems. For example, the cutting up or folding of a triangle to make the three angles form a straight line makes students aware of the fact that the angle sum of a triangle is $180^\circ$. Similarly, trying to find a triangle for which this does not work may make students aware of the property. This is, in Stacey's (1988) terms, convincing oneself. Another example that he gives is the use of properties of addition for writing $x + (x + 1) + (x + 2) = 3x + 3 = 3(x + 1)$ but different awarenesses arise if we focus attention on the $3$ or on the $(x + 1)$ in the final expression. Hewitt goes on to argue that finding a rule for the numbers that came from an investigation is not enough. There is often more than one rule but only one will fit the particular situation of the investigation. Attention should remain with the original problem and properties, and not lost in generating a table of values and finding the number rule for the investigation (he used examples of the leapfrog game and the dimensions of a rectangle with a given area and perimeter). "If students are to know about proof, then they will have to come to know about it through using their own awareness rather than trusting what someone else says is true" (p. 31). Over time, students can be aware of the meta-issues such as recognising the underlying assumptions, knowing that particular cases is not enough, convincing arguments may not be proof, and original problems can be transferred to equivalent problems. Ball (1996) and Anderson (1996) continue the debate by showing classroom examples of explanations, insights, proving and precision of students from upper primary school.

Hanna and Jahnke (1996, 1999) illustrate that physical experiments can be shown by experimentation or the use of physical laws. For example, the theorem that the midpoints of sides of a quadrilateral are the vertices of a parallelogram can be most easily proved by applying the laws of the lever or the notion of centre of gravity. Using this approach to proof will reduce the predominance of step-by-step procedures. Dynamic geometry software also provides insights into traditional geometry theorems through experimentation and rigorous construction (see also De Villiers, 1995; Mason, 1993). Hanna and Jahnke (1996) provide a comprehensive survey of proof in mathematics and its different roles in education where the relationship of mathematics to reality to meet pedagogical complexity is also an issue. They give the following aspects of proving, the first group coming from De Villiers (1990), and they follow it with three different proofs that explain different mathematical ideas for the one theorem. They list:

- verification (concerned with the truth of a statement)
- explanation (providing insight into why it is true)
- systematisation (the organisation of various results into a deductive system of axioms, major concepts and theorems)
- discovery (the discovery or invention of new results)
- communication (the transmission of mathematical knowledge) (p.18)

One should add to this model the functions of:

- construction of an empirical theory
- exploration of the meaning of a definition or the consequences of an assumption
- incorporation of a well-known fact into a new framework and thus viewing it from a fresh perspective. (Hanna & Jahnke, pp. 902-903).
O'Daffer and Thornquist (1993) provide definitions such as

Mathematical proving is a process that uses definitions, postulates, previously proven statements, and deductive reasoning to produce a sequence of true statements providing a valid argument that a statement to be proved is true ...

Proof by counter example involves finding at least one example in which a generalization is false...

Indirect proof involves assuming that the negation of the statement to be proven is true, and showing that this assumption leads to a contradiction ...

Proof by induction is by far the most complex type of proof ... 'if a property is true for 1, and if for all n>1, the property being true for n implies it is true for n + 1, you can conclude that the property is true for all natural numbers.' ...Inductive reasoning might be used to discover a generalization about natural numbers, while mathematical induction would be required to prove it. (O'Daffer & Thornquist, 1993, pp. 49-50)

There is no doubt that empirical approaches to geometry, for example through dynamic geometry software, help students to discover properties and gain insights into the property and what needs to be considered in a proof. Battista and Clements (1995) illustrate a proof using dilation for areas of similar triangles that gives insights which are confounded by trying algebraic proofs and another on circle geometry. They point out that few students reach a formal level of thinking during high school which they see as necessary for formal proof but that students at the first levels of geometry can still justify a property about one case. While O'Daffer and Thornquist (1993) summarise well what research was saying about students achieving proof, it should be remembered that poor performance of students in Senk’s (1985, 1989) studies was largely resulting from the formal proof teaching including that on similarity and complex embedded triangle diagrams.

Ironically, the most effective path to engendering meaningful use of proof in secondary school geometry is to avoid formal proof for much of students' work. By focusing instead on justifying ideas while helping students build the visual and empirical foundations for higher levels of geometric thought, we can lead students to appreciate the need for formal proof. Only then will they be able to use it meaningfully as a mechanism for justifying ideas. (Battista & Clements, 1995, p. 53)

Algebraic Reasoning

A number of curriculum documents deal with finding patterns and representing patterns from an early age. There has been a continuing push for algebra to be introduced as a summary of the relationship between numbers that are forming a pattern.

Sfard (1991) refers to conceptual development having three stages. The first is the process of interiorisation, in which the concept can be analysed without performing the processes. At the condensation stage the learner compares concepts, generalises and moves between representations. In the next stage, reification, the students is able to perceive the whole structure and the parts, often seeing this as a metaphor or mental image with a definite structure. The argument for abstraction is also presented by Mitchelmore and White (e.g., 1995). Symbols play a role in representing the relationship. Nevertheless, students continue to struggle with developing patterns (MacGregor & Stacey, 1993). If students are to look at patterns, the building up of the pattern and then the naming of the numbers and pattern is important as illustrated by Milton & Reeves (2001). Verbalising the relationship seems to become an important precursor to being able to use symbols satisfactorily
(MacGregor & Stacey, 1995, Quinlan, 1996). Bishop (2000) showed that students tend to follow a developmental path with the following categories of strategies.

**Concrete:** Students can model concrete patterns but do not seem to understand the number pattern and can only use symbols for simple evaluations.

**Proportional:** Students seem to be aware that there is some relation between the number and its position, but express this relation as a single multiplication.

**Recursive:** Students tend to focus on the relation between successive numbers in the pattern.

**Functional:** Students tend to focus on the relation between the numbers in the pattern and their position in the sequence. (Bishop, 2000, p. 122)

Algebra is a language for generalisation, abstraction and proof; a tool for problem solving through equation solving or graphing; and used for modelling with functions. Algebraic symbols and ideas are used in other parts of mathematics and other subjects and currently is a gateway for further studies. Secondary school mathematics involves algebra with real variables.

Recent research has focused on a number of approaches for developing meaning for the objects and processes of algebra. These approaches include, but are not limited to, problem-solving approaches, functional approaches, generalisation approaches, language-based approaches, and so on. Problem-solving approaches tend to emphasise an analysis of problems in terms of equations and a view of letters as unknowns. Functional approaches support a different set of meanings for the objects of algebra; for instance, the use of expressions to represent relationships and an interpretation of letters in terms of quantities that vary. A somewhat different perspective is encouraged by generalisation approaches that stress expressions of generality to represent geometric patterns, numerical sequences, or the rules governing numerical relationships—such approaches often serving as a basis for exploring underlying numerical structure, predicting, justifying and proving. Some algebra curricula develop student algebraic thinking exclusively along the lines of one such approach throughout the several grades of secondary school; others attempt to combine facets of several approaches. (Stacey, 1999).

If algebra begins from mathematical investigations, then teachers need to focus on establishing the recognition of pattern and the various ways of representing the data. This data may be gained by using other mathematical knowledge such as Pythagoras' theorem, and the investigation may highlight other areas of mathematics such as symmetry. For example, a table of values for the size of squares and inscribed squares can be converted to a graph, and the images generated by the investigation are related to mathematical reasoning and conclusions (Chapin, 1998).

Based on a correlational study of scores on tests of reasoning and algebra, English and Warren concluded that:

Logical reasoning, analogical reasoning, and spatial visualization appear to have some bearing on success in early algebraic learning. Yet little opportunity seems to exist in our present curriculums for the development of these reasoning processes. ... An understanding of the variable notion is not a natural progression from generalizing from a pattern. Yet generalizing from tables of data was significantly correlated with an understanding of the variable construct. ... Once students acquire an elementary understanding of the variable notion, they can use this knowledge in drawing generalizations from patterns (English & Warren, 1995, p. 14)
The development of students' understanding of the variable concept is reflected in the following levels: letter is ignored, letter is evaluated, letter as object, letter as specific unknown, letter as generalised number and letter as variable (Küchemann, 1978). Stacey and MacGregor (1997) confirmed the existence of a further two categories, these being, \( x \) as multiple referent and \( x \) as an alphabetical label.

A longitudinal study of students studying algebra from Years 7 to 9 in Queensland indicates some current problems (e.g., Boulton-Lewis, Cooper, Atweh, Pillay, Wilss, & Mutch, 1997a, 1997b; Boulton-Lewis, Cooper, Atweh, Pillay, & Wilss, 1998; Boulton-Lewis, Cooper, Pillay, & Wilss, 1998; Cooper, Boulton-Lewis, Atweh, Pillay, Wilss, & Mutch, 1997; Pillay, Wilss, & Boulton-Lewis, 1998). By the end of Year 9 most students had moved from not being able to explain the commutative law for addition and multiplication to satisfactory explanations. The distributive law continually caused some students problems. Most did not exhibit difficulties with inverse operations or order of operations throughout the three-year period. In line with other studies, some students continued to hold the notion of equals in the incomplete solution as denoting the answer. For equals in the complete solution, algebraic understanding moved slowly to recognising an equal or balanced relationship in Year 9. The writers explained these poor results as cognitive load requiring a much more step by step approach through the framework of arithmetic, pre-algebra, to algebra.

Reviewing students' responses to some basic algebra questions for the Higher School Certificate, Pegg and Hadfield (1999) found

- An inability to recognise the difference of two squares;
- A tendency to want to reinterpret questions so that they became equations to be solved;
- An inability to understand the place of numerators and denominators in fractions; and
- Difficulties in monitoring basic algebraic and arithmetic operations while undertaking the solution process. (Pegg & Hadfield, 1999, p. 423)

The first two have been commonly reported errors. However, it is disconcerting to think that errors were made with questions that would have been encountered many times and should not have caused excessive load on working memory. There is a need to question what is happening and also to see that qualitative assessments are needed for these examinations and not just norm-referenced scores.

Assessment of students in Year 9 or 10 and again when they were in Year 10 or 11, revealed that the failure to solve the problems did not lie in the students' failure to comprehend the written information, to understand the problem structure, or to see how the parts were related to each other, or to the whole. The main obstacles were the incorrect use of algebra syntax, and the failure to integrate the given information as an equation or set of equations (MacGregor & Stacey, 1996b).

Interestingly, in MacGregor and Stacey's (1995) study, when students were asked to select a pattern from physical models that were then recorded in tables, students would happily provide a range of patterns that were not necessarily consistent. While it is valuable for students to have to think logically, the approach to find a recurrent pattern does not seem to encourage the idea of a functional relationship if this is the only introduction to algebra that is received. Further they found students could not communicate in words what the patterns were, highlighting the importance of teachers getting students to talk about their mathematics. The data was ambivalent on whether the concrete approach to algebra in which a letter stood for the number of objects in a container assisted students' learning or not.
A study of the typical errors that students make with algebraic symbols, illustrates how persistent the errors with implicit exponent is for products such as \((3a)(2a)\). For example, students may overgeneralise from addition and call this \(6a\). Although teachers may verbally emphasise this meaning, it is recommended that students should explain their solutions and the meanings more in shared classroom conversations as a means of reducing the meaninglessness and errors associated with algebraic symbols. (Anderson, 1995).

Solving algebraic equations and following rules for simplifying expressions is not the way to start an algebraic course. Rather it should follow through from flexible arithmetic thinking to algebraic thinking in K–8 involving formalisation of patterns, functions, and generalisations, and representations of quantitative relationships (Silver, 1997). A simple arithmetic problem can be extended so it has no one answer. For example, find a way to decide how many cards will be in various numbers of boxes. Teachers can probe by asking how many in a certain number of boxes and then ask students to say in words what they did in each case and to write a general explanation to describe what they did (Day & Jones, 1997). A simple activity such as how many dots has a die can be extended by asking students what related questions could be asked. After much discussion of all the paths that students suggest, students will develop and generalise patterns using words and then symbols (Curcio, Nimerofsky, Perez & Yaloz, 1997). This verbal discussion is an essential step in developing algebraic thinking (Quinlan, 1996). Natural language description of number patterns seems to be a necessary prerequisite for representing the patterns in algebraic notation as both were correlated in a large study by Redden (1996a, 1996b). Nevertheless linguistic skills do not necessarily predict students' ability to work with algebra (MacGregor & Price, 1999). Some students (aged 11 to 15) with good metalinguistic skills had many of the algebra items incorrect, while for others, poor metalinguistic skills did not affect their ability to learn algebra. Nevertheless, few students with metalinguistic awareness of symbol, syntax, and ambiguity could achieve high algebra scores. Students need to recognise the syntax of algebra.

A study involving 286 students aged 14–16 years showed that different cognitive models were underlying algebraic and non-algebraic solutions to unequal partition problems (MacGregor & Stacey, 1998). The structure of certain word-problems can be perceived in different ways, depending on the grammatical form of presentation of the problem and the student’s expectation of how it will be solved. The data provide support for the hypothesis of Nathan, Kintsch, and Young (1992) that in the solution of algebra word-problems there are three components of interpretation and modelling: a propositional text base, a cognitive model of the situation, and a formal model of the mathematical relationships. Although there are two equally valid cognitive models of the situation, only one links to an algebraic representation of relationships.

However, considerable early work with algebraic generalisations does not mean delaying symbols per se, rather it requires a means of providing a bridge for students to use symbols when they are ready, not just to label but to show a relationship (Schwartz & Whitin, 2000). Algebraic thinking for a young student may be investigating the table of basic addition facts and noting symmetry across a diagonal, recognising the commutative property. It may be that an older student notes that the sum of odd natural numbers generates square numbers or a spreadsheet is showing exponential growth (Moses, 2000). An integrated course allows algebra to be in a range of contexts from different areas of mathematics and students will not forget parts of the mathematics nor lose confidence (Begg, 2000). For example, a function
box can be used with shapes such as squares and cubes and circles to introduce squares, cubes and roots as well as formulae (Sulzer, 1998).

The continuing use of real world problems for lessons on functions is recommended. Carter (1998) illustrates examples on taxi fares, birdseed eaters, and debts for linear functions and a guitar for exponential functions from data presented as tables and graphs.

In this functional approach to algebra, the results of a large study using a written test and interviews indicated that success relies on an array of thinking processes not generally believed to be needed for early algebraic understanding (Warren, 1997; 1998). In particular, spatial visualisation, that is the skill of mentally manipulating, rotating, twisting and inverting pictorially represented stimuli and viewing from differing perspectives seemed important for generalising from visual patterns. Students also quickly changed their approach to reaching a generalisation once they realised the generalisation was not applicable to all the presented cases; that is, they exhibited an ability to think flexibly.

Students need to reason by analogy if they are to develop algebraic thinking in which the objects of algebra can be treated without reference to earlier experiences (English & Sharry, 1996). Nevertheless, this needs to develop from experiences in different contexts to an abstraction in each case and these need to be consolidated (White & Mitchelmore, 1996). Students tended to treat variables as symbols to be manipulated rather than quantities to be related. From previous mathematical experiences students (at university in White's study) seemed to have learned to operate with symbols without regard to their contextual meaning.

If students are solving the problem with a table or a graph then they need to be asked what is the pattern if you go beyond your table or graph. Younger students up to Year 8 seem to think algebraically if they reason from tables and graphs rather than from formulation of equations for a problem (Day & Jones, 1997; Harvey, Waits, & Demana, 1995). The use of graphics calculators especially is useful for generating these ideas (see earlier section on technology). Both single (e.g. x 3) and more (x 3 + 7) operational functions should be generated (Willoughby, 1997). These are especially useful if some variations are made and graphs are generated. Non-linear relationships should also be introduced in the early years in non-formal ways (Day & Jones, 1997; Willoughby, 1997; & McCoy, 1997). Extensive examples that can be generated in a mathematics laboratory are given by McCoy (1997). From these, formulae can be built up (van Reeuwijk & Wijers, 1997), and spreadsheets introduced (see also Battista & van Auken Borrow, 1998). Some studies have shown that students are more successful in solving word problems if they are taught from the beginning to use two variables when there are two unknowns in a problem (e.g. Mathews, 1997). Physical models can also involve two variables and this provides interesting algebraic understandings that may change, for example, slopes of graphs or starting points (McCoy, 1997; Patterson 1997). For example, the distance a car travels down a slope when the slope is changed. These change patterns become the focus of many of the experiments. Fouche (1997) makes the point that starting early with this exploratory, diagrammatic thinking is important before any algebraic solving of problems is generated formally. Much of this can be generated by extending the problem with questions such as did anyone do it differently? What do you think helped you decide how to get your answer? What would happen if ...? Will what you did always work that way? How do you know? Do you see a pattern in this? How could it be done a shorter way? What other numbers work or don't work? Why do you want to change your answer? Does it make sense to you? Such questions
encourage students to think about their mathematics, predict, justify, and apply mathematics (Rowan & Robles, 1998).

Confrey and Smith (1995) also introduce the idea of covariation when establishing an understanding of functions. Rather than see a function as a rule to link two variables, often expressed algebraically, two sequences of numbers, presented in tables, are seen as covarying.

The exponential function represents a critical site for investigations because of its importance in modelling population growth, radioactive decay, compound interest, musical scales, and so on. It has a rich variety of representational forms: for example, tree diagrams, embedded figures, and logarithmic spirals, as well as the common trio of graphs, tables, and equations. In addition, it requires the student to engage deeply in issues of repeated multiplication. (Confrey & Smith, 1995, p. 67)

Diagrams in the form of graphs continue to play an important role in work on functions. Through the graph of, for example, a polynomial function in which the slopes are noted and the zeroes of f(x) are marked, students can begin to reason about the function (Douady, 1999). Kieran and Sfard (1999) also see the graph as a way into meaningful algebra for students in Years 7 and 8 because they are able to develop their own generalisations from the graphs and note relationships because of the available visual medium.

A review of a coordinate geometry straight-line question in the NSW Higher School Certificate revealed that 70% could give an equation when two intercepts and another point are marked on a graph. This is currently Years 7 and 8 work. Most students selected to use point-slope or two point formulas although seven out of eight possible ways were used by the small number of students (75 out of 26 000) whose scripts were examined. Little comment can be made about failures except that errors were made in working through the number substitutions. With a much harder question for which the midpoint and an endpoint were given and they had to find the other endpoint, it is interesting that while about 30% were able to do it by substituting in the midpoint formula and solving, this process, along with attempts to use the distance formula, also led to some arithmetical errors. Interestingly this novel question also encouraged students to use a variety of strategies, and often sound spatial information. However, the question in which they had to show a certain angle was 90°, only 38% of students could do it with most selecting the product of gradient equals -1 formula with some using Pythagoras' theorem but many selected one of a number of strategies, often using gradients or a formula but making errors in application/calculation.

According to MacGregor and Stacey (1996a) representing a problem algebraically is not an intuitive process. It needs to be explicitly taught. They also suggested that selecting word problems that were trivial, that is those which could simply be solved by arithmetic means, could contribute to the solution paths chosen by the students. Different uses of a letter to symbolise the unknown affected students' attempts to use an algebraic method. For example, x referring to different quantities in the one equation and x referring to different quantities at different stages of the solution affected the algebraic method chosen (Stacey & MacGregor, 1999b).

New learning (MacGregor & Stacey, 1997) interfered with students' algebra in one large study of difficulties with algebra. For example, with older students misuse of exponentials occurred while others confused the notion that the coefficient of x being one means x is one. The way that a problem is presented interferes mainly when the
syntax really dominates (e.g. Lopez-Real, 1997) but otherwise students seem to be able to reinterpret the problems successfully (MacGregor & Stacey, 1998).

The materials used when teaching algebra need to be critically examined. One particular difficulty is the use of a variable which becomes associated in the students mind as a label for the container rather than a variable for the number in the container (MacGregor & Stacey, 1996b, 1997a). This error appeared to be fairly persistent. Stacey and MacGregor (1999b) also expressed concern with regard to curriculum increasingly based on arithmetic thinking. They suggested that in many texts there is a predominance of problems that do not really need algebraic thinking for their solution and that equation methods that are based on arithmetic thinking are being retained and promoted in some instances up to Year 10. Boulton-Lewis et al (1997a), and Cooper and Williams (1997) present findings from their research which is that students do not appear to connect the teacher’s material representation to their own mental representation. A gap seemed to exist between the concrete and symbolic representations. This is claimed to be due to the additional cognitive demand the materials bring to the learning. By contrast, Wright (1999, cited in Warren, 2000a) concluded from the results of his case studies involving a variety of teaching approaches, that concrete materials appeared to have considerable merit in assisting students to focus on the arithmetic structure behind the rules which they had developed.

Algorithms are used merely to reduce effort so that the sense-making activity is promoted. A key issue in some studies (see Ayres, 1996) is that cognitive load needs to be reduced, while other researchers show that self-generated algorithms are the best approach for doing this (see section on computation). Real-world contexts are recommended for problem solving, often focussing on mathematical modelling and combining geometry, trigonometry, algebra, and other areas of mathematics (Robinson, Robinson, & Maceli, 2000) To do this, teachers need solid foundations in geometry and statistics, calculus of single variables, algebra and functions, and discrete mathematics (Robinson, Robinson, & Maceli, 2000).

Stacey and MacGregor say that essential foundations for learning algebra are:

- Seeing the operation, not just the answer
- Understanding the equals sign
- Understanding the properties of numbers
- Being able to use all numbers, not just whole numbers
- Working without a practical context (1997, p. 253)

They provide a number of suggestions for achieving these algebraic foundations in keeping with the NCTM Standards (1989) and their earlier work. They propose that teachers observe and discuss in class the various methods that students use to solve problems, encouraging students to perceive the more sophisticated methods that use, for example, the main operations of division, multiplication as well as addition and subtraction rather than more idiosyncratic methods. Calculators can help students to see patterns when large numbers are involved. Ask problems that express the operations in real-life terminology. (e.g., compare, older than, for subtraction). Continue using number pattern work, especially encouraging tables to illustrate relations in a way that focusses students’ attention on the relationship across the variables rather than on one variable. Number pattern investigation is to develop facility with numbers, flexibility in the way numbers are manipulated and thought about and knowledge of their general properties such as multiplication being the inverse of division. Encourage students to see the equals sign as not just getting an answer and not as a next step sign. This can be done by writing equalities in
different ways. Finally students need to work out the meaning behind a written statement that contains a mathematical situation. Non-routine problems provide an important source of developing number pattern strategies and flexible thinking (Curcio, Nimerofsky, Perez & Yaloz, 1997).

Carpenter and Levi (1999) also tried true-false statements and open number sentences with multiple variables to try to establish the notion of variable. They successfully introduced these ideas to Year 1 and 2 students who were used to talking about different strategies for solving contextual word problems. In particular, the students reasoned about number identities and generalised these (e.g. $78-0=78$). If the first exposure to variables is only through equations with one solution, then students have a hard time developing the variable concept, especially if they use generalisations and pronumerals to describe properties of number such as the commutative law of addition. Similarly the study avoided introducing the equal sign in its traditional format as before an answer. When it is used only in this way, students would see $50 + 50 = 99 + 1$ as false because $50 + 50 \neq 99$ whereas $50 + 50 = 100 - 7$ was seen as true. Open number sentences can also be used for conjectures with this age group. Interestingly, in this study, students tended to consider examples as sufficient justification for a generalisation. They can also accept that a single example is sufficient to show that a conjecture is not true.

**Number Sense**

*Beginning Numeration.*

For some time the effects of research by Piaget and so-called readiness for number may have encouraged an emphasis on students not being presented some mathematical experiences before they could conserve number or other attributes. In addition, an emphasis was placed on pre-number activities like making patterns. The perceived need for classification and for seriation to precede other ideas on number is also questioned by researchers (Davies, 1991; Young-Loveridge, 1987; Gifford, 1995). Unfortunately, these activities may have limited the possible approaches that could have been taken. Hughes (1986) for example has shown clearly that preschool children could understand concepts such as subtraction and they could use numbers when closely related to an activity such as blocks hidden in a box. Furthermore, it was clear that students could develop logic through using numbers. Even one-to-one correspondence of number names and objects was not essential for students to be able to make logical statements such as more or less. Betelli, Joanni and Martlew (1998) show a similar result by taking children from different backgrounds and showing that they could reason about number independently from the knowledge of counting. This happened even when objects were spread out or clustered together. Their tasks related to a teddy whose buttons were on a plate and dealt with needing one more or having enough. Although the three year-old children from the preschools which did not use informal experiences with number did not function as well when larger sets were presented, it was clear that counting was not being used. This conclusion was based on the fact that there was only a slight age effect. If counting was used the larger number of buttons would have influenced success.

Culture and context are likely to influence children's entry knowledge of number. Children may experience considerable everyday use of numbers, but an emphasis on age sometimes leaves number experiences as relatively limited and unexplored by the children. Finger counting is often more common in some cultures than in others. Teachers should be aware of this and be careful to find out what the child might be doing before using fingers themselves to illustrate finger patterns. The
visual images and subitising skills of students are important aspects of students' number knowledge on entering school.

There is evidence that five-year olds are building on their grasp of part-whole relationships. Sophian and Vong (1995) conclude from their study of four- and five-year olds that during the preschool period the ability to take into account part-whole relations among sets in arithmetic story problems develops. The study involved students in knowing the solution to engaging story problems involving adding or subtracting one given either the beginning or end number of the problem. For example, the Bear may have a basket of pears and give his friend one so he now has 4. How many did he have to start with. Knowing whether the answer was smaller or larger than the given number was one aspect of the study as well as knowing the answer. During the beginning school year, students were able to coordinate a conceptual understanding of part-whole relations to achieve correct solutions in both addition and subtraction. Their conclusion is reasonable although the study had too few students for adequately carrying out the multi-variate analysis selected.

Teachers need to appreciate how children perceive numbers and to use numbers in as many natural contexts as possible. However, unlike literacy, mathematics may not be naturally fun. Nevertheless, stories, rhymes, and pretend play can encourage the use of numbers in enjoyable ways. Young children need to see numbers being used incidentally (Munn, 1994; Owens, 1994). Awareness on the part of teachers could increase this usage. For example, Gifford (1995) tells of how play around an "aeroplane" simulation at the pre-school had many numbers involved from the name of the plane to the dials. Pretend activities such as making appointments, and shopping, and games (Gifford, 1995), buses, phones, cards, and, to a lesser extent, coins (Ewers-Roger & Cowan, 1996) were also quite important contexts. Many of the uses of number is as labels such as the car number plate. Some students at this age enjoy big numbers and effectively deal with half (Gifford, 1995; Hunting & Davis, 1991).

Young students also develop some form of representation of number, especially if they are encouraged in play or tasks like writing birthday invitations, preparing a list of requirements for a party (Ewers-Roger & Cowan, 1996). The students may have used quite idiosyncratic, pictographic, iconic, or symbolic representations which were often indecipherable for others. Gifford (1995) illustrated how valuable it was to get children to record or tally as they played games or took turns in an activity. Zero seems to create interesting recordings as children engaged in its meaning. Calculators also seemed to interest children who recognised symbols whether or not they differentiated them from other numbers or letters. Nevertheless, when using rhymes and telling stories, children seem to be able to participate in problem solving in which subtractions or divisions might occur (Gifford, 1995).

Gifford (1995) noted that preschool carers differed in how much they valued and saw their role in early numeracy for children. In addition, some children seemed to be quite anxious about numbers, especially about their recognition of numerals.

**Counting**

There are a number of studies that have considered the developmental levels of students. In general, these have fitted specific frameworks. For example, the *Count Me In* project provided stages of development for different aspects of number. These included Counting Forward and Backwards from a specified number, subitising or recognising small numbers of dots without counting, and Numeral Recognition. While these are seen as developmental in terms of number size, the Early Arithmetic Strategies are much more pronounced as skills and concepts. These include
counting all, counting on from the larger number when objects are present, imagining
the numbers, and then counting on figuratively. Finally students develop facile use of
numbers in which they remember various combinations of numbers. For example,
they may use the fact that 13 is one ten and three ones. Recognising ten ones as a
composite unit of one ten is also a significant developmental point and manifest in
the way students might use number facts that add up to ten, or they may count by
tens.

A Victorian project headed by Doug Clarke has developed a number of tasks that
they use to establish students' progress through a number of significant growth
points. To illustrate they give the addition and subtraction growth points as

1. Count all (two collections)
2. Count on
3. Count back. Count down to. Count up from
4. Basic strategies (doubles, commutativity, adding 10, ten facts, other known
   facts)
5. Derived strategies (near doubles, adding 9, build to next ten, fact families,
   intuitive strategies)
6. Extending and applying addition and subtraction using basic, derived and
   intuitive strategies.

For example in the last stage, given a range of tasks (including multi-digit numbers),
students can solve them mentally, using the appropriate strategies and a clear
understanding of key concepts. (Clarke, 2000, p. 16).

Again with multiplication and division, the concepts are built up perceptually through
equal grouping and counting. Verbal aids such as rhythmic, skip or double counting
are used. Students will then develop composite groupings without visible or sensory
models, with a further development in being able to keep track of the number of
repeated groups. At the stage when multiplication and division are viewed as
operations, they are ready to relate the concepts to fraction knowledge. Tasks
include both the comparison of groups as well as the hidden, missing or taken away
contexts used for subtraction. Both sharing and grouping tasks are given for division
problems (Clarke, 2000; Gould, 2000).

Young-Loveridge (1991) also provided items that could indicate students
development Her set of items covered counting items, numeral identification, pattern
recognition, and forward counting. She included both the concrete and imaginary
counting of objects.

The *Count Me In Too* kit (Department of Education and Training, 1999) provides a list
several pages long with summaries of selected articles that support the Count Me In
Too approach to early number learning. However, the notion of distinct stages of
development still needs to be questioned.

An interesting longitudinal study was carried out by Bruniges (1999). This study used
individual assessments of 389 students aged six to ten on eight occasions spanning
two years. Some of the assessment task items were similar to those used by other
researchers such as Mulligan (1992). The results show that there is an apparent
growth spurt, as indicated by degree of variance of the scores of the cohort, in term 3
of Year 1. At this point it can be expected that students will begin to think
operationally in terms of the concrete materials that they may have been
manipulating.

Mulligan and Thomas (1998) found that the processing of images plays an important
role in the development of young children's understandings of numeration and
number operations. Children’s external representations (symbols, drawings and models), as opposed to their internal representations, do not always correspond to conventional mathematics, nevertheless, Mulligan and Thomas argued that teaching and learning situations that allow students to describe, draw and discuss their personal mental images should be encouraged. The images children used to describe number concepts were often idiosyncratic but helped generate powerful understandings through both static and dynamic imagery. Mulligan, Mitchelmore, Outhred, and Bobis (1996) and Mulligan, Mitchelmore, Outhred, and Russell, 1997, for example, developed task-based interviews that encouraged Year 2 children to represent their solutions by modelling, drawing and symbolising their visual images.

Teachers need to challenge students once basic ideas are handled with some degree of confidence. If textbooks (K–3) are any guide, teachers are more likely to concentrate on small number (1–5) and addition knowledge than questions with higher numbers (6–10), and operations with 0, and 1. This limited range of questions is also likely to reduce students' development of different arithmetic strategies. Ashcraft and Christy (1995) showed that this lack of experience was linked to slow response times in answering number fact questions.

There has been a debate for some years about the use of number lines in early primary school, and a recognition of some of the difficulties of taking a ruler as a number line when it is so readily available. One difficulty of the use of the number line is that length is representing the size of the number but only the order of the numbers is transparent with the distance to zero or other point is not obvious. Another problem is knowing how numbers less than one fit onto the number line. The abacus is a good visual representation to assist with place value and groups of ten. It also assists with the numbers between 5 and 10 (Cotter, 2000). Visual assistance was given by using pictures of base-10 blocks for larger numbers. One point is made that in Japan, children are discouraged from counting by one and are encouraged to consider number combinations. Using counting to add and subtract develops a unitary concept of number and this interferes with their understanding of place-value concepts.

**Ordinal Numbers**

Students in Years 5 and 6 in Belgium generally failed to gain the correct answers to word problems that relied on a sense of ordinal numbers. Their solutions were generally one too many or too few for these kinds of word problems due to a reliance on formal algorithms for addition and subtraction. When number differences were small, informal methods were more likely to result in correct solutions. The main problems were postulated as being a limited understanding of ordinal numbers, lack of heuristic methods such as making a diagram, or metacognitive awareness about possible fallacious patterns of thinking and a distrust of "weak" informal solutions (Verschaffel, De Corte, & Vierstraete, 1999).

**Mental Computation**

Morgan (1999) summarises many of the issues raised about teaching mental computation. Mental computation is not to be viewed as just a first estimate, nor as a series of rules, nor as speed calculating. It must be seen as an integral part of learning number operations from the beginning. The intention is to encourage children to develop flexible, idiosyncratic mental strategies, emphasising the mental processes involved. Such a focus would allow mental computation to be used as a tool to facilitate the meaningful development of mathematical concepts and skills to promote thinking, conjecturing and generalising based upon conceptual understanding (Reys & Barger, 1994). Hence, mental computation is closely linked to the development of number sense, and as a bridge between learning and using
school mathematics and that used outside the classroom (Carroll, 1996; Willis, 1990).

Interestingly, the test items frequently used for mental computation could have been those found on tests from the past in which students follow rules (e.g. McIntosh, Nohda, Reys, & Reys, 1995). For example, they include questions such as 165 +99, 79 + 26. However, in more recent studies it is expected that students may use a range of strategies besides those prompted by the question. A study by O’Rode (1999) indicates that students in reform schools were not only more successful on the tasks but they also used a variety of strategies, often quite efficiently. Mentally using a standard algorithm was more likely in traditional school, and these students were less able to suggest an alternative solution strategy for the problems (a factor that limits checking methods). It was also found that high achieving students in both the reform and traditional groups were more inclined to not use the standard algorithm, two thirds of the students in the traditional middle group used it whereas only a third of the reform middle group used standard algorithms.

With an emphasis on mental computation and estimation, Carroll (1996) found students performed much better than the baseline students, especially on the story problems and items for which chaining from left to right was an advantage. However, the students still had difficulties with questions like 265 minus 98 and 25 x 28 when presented orally. Carroll also pointed out that division needs to be taken as an operation in its own right so that students have alternative strategies (e.g. subtraction and halving as well as reversal of multiplication). The students preferred pencil and paper but not for a standard written algorithm which was not taught, but to assist them with their own methods, especially the lattice and partial products methods for multiplication. It was clear that the students who had invented their algorithms in the primary years were still using these in the high school. The students all talked of adding 60 and 30 and not 6 and 3 as they worked from left to right in a 2-digit addition. Flexibility was most apparent in the different ways students tackled different questions. Generally doubling was done as an addition, and a subtraction as an addition with a missing addend and solved by at least one student counting-on in tens. Students also developed the important ability to know if an answer was reasonable but Carroll suggested that not all students monitored the meaningfulness of answers nor used a clear, efficient strategy. Some students also did not do so well on standardised tests that were designed to test written computation skills.

In the program presented by Carroll (1996), curricula and instruction emphasised exploration of numbers and operations, of which mental computation was a central on-going, integrated part. Mental computation was not a series of tricks, one for addition, another for certain numbers nor was it taught as a separate topic.

*Changed classroom approaches.* Effective teachers attempt to develop a rich network of connections between different mathematical ideas (Askew, Brown, Rhodes, Willaim, & Johnson, 1997). In order to develop teachers’ proficiency in using new approaches to mental calculation, it is necessary to inform them of the purpose. In addition, teachers currently recognise that children who are proficient at mentally calculating exact answers use personal adaptations of written algorithms and idiosyncratic mental strategies (Morgan, 1999). This notion can also be built on. There needs to be a corresponding reduced emphasis on the standard written algorithms accompanied by an increased emphasis on non-standard mental, calculator and paper-and-pencil strategies. For some children, the spontaneous invention of mental strategies may be difficult (Carroll, 1996; Reys, Reys, Nohda, & Emori, 1995). Low proficiency with mental computation may benefit from a teacher-
directed focusing on selected mental strategies (Cooper, Heirdsfeld, & Irons, 1996),
modelling by the teacher and other students, and the valuing of easily grasped
approaches. The easier strategies are those counting and heuristic strategies
considered accessible for most children at a particular stage (Morgan, 1999). They
are the strategies that contribute to and draw upon a student’s sense of number.
Mental computation should not be considered as a discrete topic. The wrong
approach is the one-step questions marked as correct or incorrect by answer only
which is perpetuated by the nature of some textbooks, especially those called
Mentals.

Encouraging students to develop their own solution strategies for solving problems
values students’ ability. We are learning how capable students are at thinking through
challenging tasks and taking ownership of their computational strategies as they work
to make sense of numbers (Trafton & Hartman, 1997).

Allowing children to calculate using self-generated written strategies requires
procedures for organising mathematics learning and conferencing techniques that
promotes individualised learning and thinking creatively. Such procedures will
encourage a focus on concepts rather than calculations (NCTM, 2000; Queensland
School Curriculum Council, 2000b, Thompson, Thompson, & Boyd, 1994, cited in
Morgan, 1999). Such an orientation is in harmony with the current view of mental
computation and how it should be taught. Thompson et al. (1994) contend that
"conceptually oriented teachers focus children toward a rich conception of situations,
ideas and relationships among ideas” (p. 86), which are key factors in developing
and being able to apply flexible mental strategies.

Curriculum writers, researchers, and teachers need to give thought to

- The way in which self-generated mental strategies mesh with formally taught
  algorithms.
- The level of place value understanding required for particular strategies.
- The demands of particular strategies on working memory.
- The effects arising from the need to regroup or carry, and the size, and type of
  numbers being manipulated.
- The consequences arising from using visual and oral stimuli, and examples
  presented in and without context.
- The appropriateness of various models which may be used to represent the
  calculative situation.
- The effectiveness of discussion as a means for encouraging strategy growth,
  and how it should be managed within the classroom. (Morgan, 1999, p. 378)

McIntosh, Reys, and Reys and their colleagues have written a number of articles on
mental computation. In 1997, they summarised some key issues. First they define
mental computation as computing an exact answer in the head, estimation as gaining
a close enough approximate answer to make a decision, and "number sense as a
person's general understanding of numbers and operations along with the ability and
inclination to use this understanding in flexible ways to make mathematical
judgments and to develop useful and efficient strategies for dealing with numbers
and operations" (McIntosh et al, 1997, p. 322). It includes both estimation and mental
computation. Thinking strategies are closely linked to the user's conceptual
understanding of numbers and flexibility in decomposing and recomposing numbers.
For upper primary school students McIntosh et al emphasise that mental computation
is important but that different students view it quite differently. These students can
and do invent thinking strategies for computing but this varies greatly and is often
quite creative. The highest levels and use of mental computation are found in
students who are confident in their thinking, value alternative-thinking strategies, and
see the invention of techniques as a powerful result of their understanding of mathematical relationships. The presentation format of tasks, visual or oral, and context stimulates different approaches and different performance levels. These authors emphasise that teachers need to show they value and model mental computation, and ask students frequently to explain how they performed a mental calculation. In particular, teachers should encourage students to use strategies different from written algorithms. Much less time needs to be spent on written algorithms with more time allocated to mental computation. They list the following computational skills are needed by upper primary school students:

- The intelligent use of calculators with a clear understanding of their shortcomings;
- A good operational knowledge of basic facts—addition, subtraction, multiplication, and division of single-digit numbers;
- A clear, practical understanding of place value and the system of numeration;
- An understanding of simple fractions, decimals, and percents, including their interrelatedness;
- Self-confidence in, and ability to use, a range of techniques for computing;
- The inclination and ability to use thinking strategies first when checking calculations, estimating results, or performing various calculations;
- The instinct and willingness to think about numbers in natural, comfortable, and flexible ways when computational results—whether exact answers or estimates—are produced; and
- the instinct and willingness to reflect on numerical results so as to judge their reasonableness. (McIntosh et al, 1997, p. 326)

Bobis (2001) provides another comparison of different mental strategies explaining that worthwhile strategies are those that no longer use counting by ones but some form of number relations and landmark numbers. She provides a range of activities for encouraging these practices in K–3.

**Using Calculators**

Ruthven (1998) points out that official UK ambivalent comments have not strengthened balanced use of calculators in schools. Calculators are seen as a fast and efficient means of calculation, and mental methods as a first resort. However, the Secretary of Education still questioned the exclusion of pencil and paper methods for long division and multiplication resulting in non-calculator methods being statutory and calculator methods as non-statutory.

Students in the CAN project schools used calculators throughout primary school, under the guidance of teachers encouraging a much more rounded view of calculation. They still performed better on some areas of the national numeracy test (Ruthven, 1997).

However, Ruthven (1998) points out the effect of a National Curriculum on both CAN project and non-project schools. To some extent they became similar in the last two years of the project. However, in non-project schools, a closer look showed that the teacher's use of calculators was much more restricted than in the project schools, and mental computation was seen as quick exercises like tables, and finding out answers to realistic problems like the change from the supermarket. In project schools, mental work was seen as mental strategies, discussing and investigating number relationships, valuing individual work and efficiency. When the testing began, teachers dropped investigative time to ensure students could do written algorithms for the test. Self-regulation of calculator use was still better in project schools but less developed as they began more work on mental computation.
When students’ work was examined, it was found that students used mental methods, mental methods with recording, tallying, written methods, written methods that implied mental methods, calculators, or deferred calculators implying mental methods. Mental methods, followed by much less frequent calculator methods, were used. Calculating the cost of 5 stamps at 19 p each, showed a relatively low success rate for splitting the number mentally whereas $10 + 10 – 1$ and $20 – 1$ methods were much more effective. Written column multiplication was also effective but so was calculator use once a student selected to use it. Students from the calculator project schools were more likely to use mental strategies based on splitting the number and compensation. Interestingly the type of computational strategy used did not relate much to number-concept attainment. Lower attaining students did use mental jump methods and less written column multiplication methods and were as likely to use split or distribution and compensation methods ($20–1$) as their higher-attaining peers.

An emphasis on place value and written column methods in the UK from an early age leads to striking differences between the Netherlands Realistic Mathematics and the United Kingdom's National Curriculum. Computation is embedded in real problems in both the Netherlands and in calculator schools. However, in the Realistic Mathematics calculator use is introduced much later as the gradual process of structuring and curtailing strategies is developed. The Netherlands have swung away from the point that it was more important to discover number facts than know them.

Tasks that matched the assessment level of students in number, and those that were more difficult showed that students were able, with less success as the tasks became more difficult, to estimate results of number problems. Indeed the zone of proximal development or ability of students to apply what they know to novel situations is fairly broad (Dowker, 1997).

Groves and her co-researchers have shown that calculators can be used from kindergarten as scratch pads, counting by ones or larger numbers, establishing notions of size and place value, exploring numbers, playing games like guess my number, putting numbers in order and many more (Groves, 1996). The overwhelming support for the use of calculators by teachers who used them and the children's ability to handle bigger numbers than expected supports the use of calculators as a tool for learning and not just as a means of checking or carrying out calculations. Nevertheless, Groves mentions the need for concrete materials and illustrates the need for paper-and-pencil for recording.

Despite the extensive literature (see summaries of studies, often large ones, in Groves, 1996; 1997; Groves & Stacey, 1998; Hembree & Dessart, 1986; Shuard, 1992, Stacey & Groves, 1996) on the value of using calculators from early schooling, their use is still not as frequent as it should be for effective teaching (e.g., Anderson, 1997; Sparrow & Swan, 1997a, 1997b). Sparrow and Swan suggest that an emphasis on standard written algorithms and the generally reserved and negative attitude and belief of teachers obstructs the use of calculators. This means that curriculum need to integrate how calculators are used for teaching and learning mathematics. White (1998) was more hopeful that teachers might soon implement their intention to use calculators for what they see as valuable aids in problem solving and for fast calculation but currently some Principals' support and parental support was needed.

Marley, Skinner and Kenny (1998) also emphasise how conversations were a key to why calculators were valued by students in their first year at school.
Development of Students’ Own Written Algorithms

The availability of calculators and the greater use of the decimal systems has meant less need for paper-and-pencil algorithms for whole numbers and fractions. Narode, Board and Davenport (1993), among others, have shown in their case study that traditional drill of standard algorithms has not been very successful. Several researchers show how mental computation can lead to sound written calculations. For example, Hédren (1999) illustrates the strategies that students generate by developing their own written algorithms based on mental computations. In most cases, students work from left to right. NCTM’s (2000) Standards recommends that children in primary years are encouraged to develop, record, explain, and critique one another's strategies for solving computational problems, and a number of important kinds of learning can occur (see, e.g. Hiebert, 1999). Efficiency and generalisability can be discussed.

And experience suggests that, in classes focused on the development and discussion of strategies, various ‘standard’ algorithms either arise naturally or can be introduced by the teacher as appropriate. The point is that students must become fluent in arithmetic computation—they must have efficient and accurate methods that are supported by an understanding of numbers and operations. ‘Standard’ algorithms for arithmetic computation are one means of achieving this fluency (NCTM, 2000, p. 35).

Hedrén (1999) lists the following advantages of the use of students’ own algorithms: ownership, similarity to mental computation and estimation, sense of magnitude as a result of working from the left, more natural working from the left, involvement of number sense, easier to understand and to avoid misunderstood systematic errors, and suitability for social constructivist development.

Nevertheless, Hédren (1999) lists the following advantages of the traditional standard algorithms. They are invented and refined through centuries and are part of our mathematical cultural treasures, and they can be used no matter how complicated the numbers.

Interestingly, students who have been taught formal written algorithms can still be encouraged to consider alternative strategies and to justify their solutions (Sáenz-Ludlow, in press). Sáenz-Ludlow provides case studies to show how students changed to using left to right additions of 3 digit numbers (with trading or renaming) and how these methods were grounded in developing a strong sense of place value.

The use of mental computations along with visual models for establishing algorithms is a worldwide area of research. For example, Hedrén is in Sweden; Kamii, Cobb and co-workers, Sáenz-Ludlow, and Narode et al are in United States classrooms; Duffin in the United Kingdom; Murray, Olivier, and co-workers are in South Africa; Beishuizen and many others in The Netherlands, and Leidtke in Canada to name just a few mentioned in this literature review.

Development of Place Value and the Use of Concrete Materials

Development of understanding of the place value system has not necessarily been assisted by the algorithmic use of base-10 arithmetic blocks (Thomas & Mulligan, 1999; NCTM, 2000; Price, 1999). A range of difficulties arise regarding the focus of the students, the lack of congruence between different shapes and the place values associated with the blocks (especially the use of a block for 1 000 rather than a linear representation as used for ones and tens). Concrete materials do not of themselves provide transparent abstract mathematical concepts (Cobb, Yackel, & Wood, 1992; Howard & Perry, 1999; Owens, 1994; Perry & Howard, 1994). While these blocks can be used well as a good representation to lead into the standard written
algorithms, their use in a drilled, algorithmic way is not seen as necessarily promoting understanding of the algorithms (NCTM, 2000). Price (1997, 1999) found that students were more able to focus on the meaning of place value when they could manipulate computer screens than when they manipulated real materials. Counting, and keeping track were not distracting from the main purpose of calculating and manipulating the virtual blocks.

Nevertheless, the importance of making numbers and number relations with concrete materials (NCTM, 2000) and matching by orally counting and counting by tens is a sound beginning for establishing place value concepts. The four constructs of counting, partitioning, grouping, and number relationships were seen as central to the development of multidigit number sense in the classroom research developed by Jones et al (1996). These teacher-researchers actively developed these constructs. "Students also develop understanding of place value through the strategies they invent to compute" (NCTM, 2000, p. 82). The selection of concrete materials is important for establishing both language and visualisation. Besides other materials that grouped by five, Cotter (2000) found the abacus with light and dark beads on each strand and two strands for each of the place values was good at establishing the visual imagery similar to using multiple 10-frame cards. Language and visual imagery plus the opportunity to work with the numbers on the material was important. Actively working with smaller numbers and gradually larger numbers will be necessary for sound place value understanding. This is not necessarily progressed in a strict single-digit to two digit to three digit number order. It is the constructs that need emphasis rather than the size of numbers. Indeed, the Count Me In studies and the Victorian calculator studies (Groves, 1996) show that students in Kindergarten are gaining a more holistic view of number if they are allowed to count and make numbers beyond a restricted range in the early years of schooling. When operational concepts and numerical relationships are being established, the main deterrent in using larger numbers is their impact on focussing attention on the concept. Interestingly, Legrande (2001) discussed emphasising building on the aptitude to see small collections and avoid counting and the oral words for the tens from thirty, forty etc to encourage a closer English oral link to groups of tens and ones before returning to 10 to 29.

It is quite clear that the way in which concrete materials are used might need to be different. While they provide opportunities for discussion between teachers and students and a means of establishing visual imagery and mathematical relationships, they are not always useful as a regular calculating aid especially for multidigit numbers (Threlfall, 1996). The students’ use in this way loses its value as students may not notice what teachers want them to notice. "To use the apparatus to get answers is to obscure its purpose and prevent it from having its true value, which is to bring meaning into arithmetic" (Threlfall, 1996, p. 10). Their use can in fact be intrusive and delay paper and pencil skills. After meaning is established, the materials may shadow the calculation and be a safety net or check after students have tried to calculate without them.

We can couple these comments with the results of a study by Moyer and Jones (1998). They found that teachers who first provided access to a variety of concrete materials and initially controlled their use generally allowed the students to continue to use them in free access. Control included illustrating how materials can be used in a specific situation. Students may have also been controlled initially in being allowed to use them. However, at times, teachers intervened and restricted an efficient use of materials. By contrast, some teachers who provided greater autonomy had withdrawn the basket of materials from students’ desks in order to restore classroom
order. Students were not always as successful in selecting materials when they were not first introduced in a controlled manner. Although it is not mentioned in their study, it is likely that the early controlled use was a time when imagery and meaning was being established. Related to this issue, is a greater awareness of teachers, especially in the Years 5 to 8 of the mathematics that students are learning (NCTM 2000). Such knowledge would assist teachers to see potential in equipment and use it wisely.

Owens' major concern was for students to use materials in problem solving activities (Owens, 1994, Owens & Clements, 1998). In this way, students were engaged more, noticed features of the materials, and employed visual imagery, conceptual knowledge, and various checking and heuristic approaches to solving the problems. The materials needed to be fairly generic and the problems suitably open-ended.

Thomas and Mulligan (1999) further summarised research and emphasised how important it was for students to recognise the underlying structures (see also Bove, 1995; Nagel & Swingen, 1998; Thomas & Mulligan, 1995; Thornton, Jones & Neal, 1995). Bove (1995) and Groves (1996) illustrate how a continuous vertical list of numbers can assist students to gain a good sense of size of numbers. Bove used a matching place-value mat and blocks while Groves used a calculator. Thomas and Mulligan used students' descriptions of 100 to illustrate how students may have adequate or inadequate views of a hundred. One student had tried to use classroom tasks such as an array of 100 probably learnt from a hundreds chart but failed to have 10 rows of 10 whereas others managed to illustrate counting by 10, a hundreds chart, and numbers increasing by 10 as they are recorded. Establishing this structure, as an abstraction is important just as developing other structures in mathematics e.g. in algebra, have proved to be essential for further development.

Children need to better understand the importance of ten in mental operations and so develop thinking strategies, such as bridging tens, for addition and subtraction. Mental imagery and verbalisation of the structures within the number sequence will help develop an appreciation of the importance of ten and act as a template for extending the system to incorporate the powers of tens. Far too many children take with them into secondary schools a reliance on unitary counting as a strategy for basic addition. (Thomas and Mulligan, 1999, p.9)

Thinking Strategies for Arithmetic Problems

One of the clearest messages coming from all the research in number has been the importance of having students to think about strategies for solving problems. This includes the simplest addition and subtraction problems. Year 2, 3 and 4 students find difficulties in understanding the situation but teaching by key words or operation helps little. (Christou & Philippou, 1998).

Many research articles deal with the efficiency of certain mental strategies and the flexibility of use. One summary article by Beishuizen, van Putten, and Mulken (1997) gives some of the approaches. They group the strategies as (a) the split method in which tens and units are split off and handled separately and (b) the sequential counting or jump method in which students work with the whole number and add on. It seems that this second method is more common in Europe than in the United States. Nevertheless the European students for whom there has been emphasis on mental computation by the jump method still use the other method because it is probably less effort cognitively. However, there are more mistakes in the split method, especially with the subtraction of taking the smallest from the largest, when students start from the unit digits following a traditional written algorithm for
subtraction. In Japan one study had shown a very narrow range of strategies with the most popular being the learned paper/pencil strategy used mentally (Reys, et al., 1995). In Beishuizen et al.'s 1997 study, they found that students used a range of strategies including addition and subtraction strategies of splitting and jumping. The students were more successful in working out 87 – 39 if they used the number jump methods and less successful if they used one of the splitting methods. However, a contextual shoe problem, though the numbers were closer, was not well done if students used the split number method. However, if the problem required restructuring, e.g. 27 + ... = 67 they found that students who used splitting strategies and generally did well on mental computations, were able to switch to the jump method whereas those who did not do well on mental computations did not make the switch. The students who did use jump methods, especially adding-on methods were more successful on this kind of problem. The main problem with splitting into tens and units was the recombining of the results since a compensation procedure was often required. Students did not seem to have an overall picture of the problem. It should also be noted that the jump method was favoured by the Dutch curriculum but the same problem of ability had been found by Fuson (1992) in the USA. In conclusion, Beishuizen et al. (1997) felt that with Realistic Mathematics Education the counting by tens using the empty number line as a sequential model stimulates not only mental modelling but also invites flexibility in the use of procedures and strategies. Thereafter the splitting method could be introduced as a transition to place-value based procedures (Klein, Beishuizen & Treffers, 1995).

Sáenz-Ludlow (in press) illustrates how mathematical interactions between the teacher and the students and among the students both creates and reflects their mental acts and therefore their personal mathematical functioning. This mediating by language, gesture, diagrams, mathematical notations, idiosyncratic and conventional symbolisms, and the task format renders actions to involve the self (Vygotsky, 1986). Students and teacher were expected to listen carefully to other solutions and to express agreement or disagreement with justifications. This was coupled with a task format that encouraged students to estimate, to show a strategy to find the exact answer and then to verify and ask the question does your answer make sense. The actions and interactions promoted a better understanding of the algorithms for addition and subtraction and ground the basis for a conceptual understanding of the algorithms for multiplication and division of whole numbers.

**Can All Students Develop their Own Satisfactory Algorithms?**

The results of Bashash and Outhred's (1996) study of 10 students with moderate intellectual disabilities (aged 12 to 18 years) suggested that the students were capable of developing rule-based number skills based on comparing and deciding whether one number was larger than another. The authors conclude that children with moderate intellectual disabilities should benefit from an emphasis on cognitive techniques of instruction rather than on rote learning.

Assisting students with learning difficulties in mathematics requires knowing whether there are reading difficulties as well. A variety of problems such as compare and find the difference may be more difficult than change story problems when reading difficulties are influencing performance (Jordan, 1998).

Liedtke (1996) posed the problem of why a student had been unable to use the fact that if 5 x 8 is 40 to give 6 x 8. The student gave the answer as 46. Liedtke questioned the goal of speed answering which emphasised learning by recitation only. She also felt copy, cover and recall methods did not lead to retention. Her emphasis was on expecting and teaching number sense relationships, even if it took time. If a deficit
model is to be avoided, then we should expect that conceptually focused instruction and thinking strategies will be more powerful than direct instruction. However, she did advocate modelling specific ideas like doubling and/or using ten times for a number to get nine times and even eight times or to halve for five times. Relationships need to be visualised and based on skip counting with objects. Clearly the idea of having strategies rather than rote recall are essential if students with learning difficulties are to repair when they have forgotten. Nevertheless, the initial problem of using $5 \times 8$ to get $6 \times 8$ raises again the issue of whether $4 \times 8$ is 4 groups of 8 or as the Europeans and Asian communities seem to indicate it is 4 being operated upon (8 times).

Based on variations of Piaget’s theory and Steffe’s theoretical understanding of cognition, there have been a number of research studies and level approaches. With the proviso that students are not held back by only remaining at one level, these approaches for students with learning difficulties are well tried. Wright (Stewart, Wright, & Gould, 1998; Wright, Stanger, Cowper, & Dyson, 1996), Bobis and Gould (1999), Pearn (1998) and many others have shown that the use of developmental levels in counting, place value knowledge, and more advanced arithmetic strategies, such as knowledge of numbers that add up to ten, assist students to develop their number knowledge. This is a far cry from direct teaching that drills number facts. Rather the approach is to ensure that students experience activities in which they are actively engaged in making number combinations, visualising these combinations, and working with these number combinations. Initially the approach is to encourage perceptual and then mental counting-on until other strategies have been developed by the students. The use of other combinations is assisted by both physically making and mentally imagining combinations (Bobis, 1996a, 1996b; 1998). While three-quarters of students who participated in the Count Me In Too project used more sophisticated strategies to solve simple addition and subtraction problems and 90% had progressed on numerical categories, the lower ability students progressed at a slower rate. Another large evaluation study also indicated that Kindergarten students enter with relatively high levels of knowledge but that there is much diversity in levels. However, participation in the project showed improvements, often exceeding expectations. This large cohort included nearly 50 out of the total of 866 students who were Aboriginal or Torres Strait Islander (Stewart, et al., 1998).

For students with learning difficulties, learning to count up to 100 in different ways was important. This counting needed to be associated with the counting materials until the concepts and mechanical counting approaches were developed (Menne, 2000). At the structured calculating stage, skills concerning number concepts are: counting from any number, locating numbers in relation to each other, and identifying jumps towards a number. Skills concerning number operations were known, complements in 10, splitting numbers up to ten, and jumps of ten. The jumps are treated as units, and the empty number line was often used for the jumps. This was the reproductive practice that concentrated on fundamental skills. The next stage concentrated on own productions and was called productive practice. This Netherlands approach for assisting students in Year 2 with learning difficulties and language of instruction as a second language was particularly successful.

One study by Wilson and Robinson (1997) is somewhat inconclusive in explaining which of three methods employed by parents was most effective: (a) encouraging skip count, or (b) reciting multiplication facts, or (c) reducing the speed by which a teacher-researcher gives the answer to a multiplication. While the last method was apparently more effective, it was hard to be conclusive because parental involvement per se was highly influential.

Carroll (1996) indicated that some students would have difficulty developing their
own mental strategies. However, she found that there were approaches to help students develop mental strategies based on their existing counting techniques and number facility.

These students will resort to writing information on paper to help them keep track of how they are thinking. This has been indicated by having younger children attempt novel calculations (Baek, 1998).

The issue of cognitive load in solving problems was raised by English. For example, she noted the problems of using base 10 blocks (English & Halford, 1995). In her later studies (1996, 1998, 1999) she has shown that reasoning by analogy or finding a similar problem does improve solutions if problems are in everyday contexts but this reasoning is still difficult in formal written problems, which the students have trouble even generating.

Simplifying the resources used by students with special needs was emphasised by Vacc and her colleagues (Vacc, 1995; Vacc, Ervin, & Travis, 1995). Although she used a column presentation starting on the right to form a hundreds chart to be in line with working right to left with algorithms, there may have been other confusions not addressed such as columns for place value, especially the third column that is used for the twenties (not hundreds) or the lack of link to number lines. Her main suggestions of importance were (a) the use of zero on the chart and beginning each column with a new decade 10, 20 etc. and (b) a number of simple uses of the hundreds chart that helped to bridge the link between concrete and abstract, the only manipulatives for students being the see-through counters. For example, she illustrated how counters can be placed on the numbers to 8, then to subtract 3 remove the counters on 8, 7, and 6 leaving counters up to 5. Reading the equivalent statements in different ways was also emphasised for each operation.

Thorpe (1995) provides a salutory reminder that students with special needs may tend to play alone with concrete materials and be more dependent on adults than communicate with their peers, possibly because the adult-child relationship is higher. In this case, student talk may need to be promoted.

**Practice**

Meaningful practice is necessary to develop fluency with basic number combinations and strategies with multidigit numbers. ... Practice needs to be motivating and systematic if students are to develop computational fluency, whether mentally, with manipulative materials, or with paper and pencil. Practice can be conducted in the context of other activities, including games that require computation as part of score keeping, questions that emerge from children's literature, situations in the classroom, or focused activities that are part of another mathematical investigation. Practice should be purposeful and should focus on developing thinking strategies and knowledge of number relationships rather than drill isolated facts ... Students can learn to compute accurately and efficiently through regular experience with meaningful procedures. They benefit from instruction that blends procedural fluency and conceptual understanding (Ginsburg, Klein, & Starkey, 1998; Hiebert, 1999). This is true for all students, including those with special educational needs. ...Comprehensive instruction uses the child's abilities to offset weaknesses and provides better long-term result (Baroody, 1996). ...More cumbersome computations should be done by calculators (NCTM, 2000, p. 87).

When practice is given on drill facts that students are ready to learn as ascertained by assessing their current knowledge, the practice can be beneficial. A project on connecting common fractions to decimal notation in a controlled experiment indicated that there was success when the repetition involved the establishment of the visual and symbolic connections (Lynch & Cuvo, 1995). However, the study involved a
small sample and a technology-based situation that was not relevant to many classrooms.

Addition and subtraction. When students are unable to recall number facts, they will resort to various approaches. Average or above average children reduce a problem so they can use known facts (Gray & Tall, 1994). These strategies are not so common with below average students who may resort to counting long numbers and need to know the strategies of counting-on from the larger number or other non-count-by-one strategies. A particularly useful strategy is that of counting from five and knowing the up-to-ten relationships (Cobb, Boufi, McClain, & Whitenack, 1997; Hatano, 1982). These strategies need to be clearly valued by teachers and modelled by other students in order for the below average students to develop these strategies.

Counting and Place Value
The English counting sequence does not transparently show that the teens are ten and units. Counting sequence is more transparent for students with average or above IQs who have an East Asian language. These students are more able to add 10 and a single-digit number than 4 and a number. For English speaking students, adding by counting was the main method for solving this kind of problem. (Ho & Fuson, 1998). Skemp as far back as 1986 provided a clever activity in which the tens and ones were clearly articulated along with the everyday words for the number. Although artificial for the kindergarten child, nevertheless it seems that teachers will need to emphasise the ten hidden in the counting sequences for number if the place value idea is to be well established. In addition, emphasis must be placed on the other relations associated with counting-up-to and beyond 10.

Recital of the counting words needs to be matched to digital codes emphasising the place value relations (Giroux & Lemoyne, 1998). This became apparent when students had difficulty with what to add to 17 to get 19 even though they can do 17 + 2. Even when a student can do a question like this or "What is the difference between the numbers 19 and 21 and between 39 and 41", they may still struggle with a question like "What is the difference between the numbers 19 and 28 and the numbers 30 and 39". It is likely that the part-whole relationships that are not directly apparent in the digits is not so easy to establish and is not part of school practice. However, this is an important issue for coordination of number knowledge.

On other issues, Giroux and Lemoyne point out that the joining or adding process communicates a moving direction in a number sequence whereas other questions that use "more" or "less" suggest a more static comparison. Counting backwards may be a useful assessment task (but only one) to check if students have developed the part-whole relationship for numbers and can work out the numbers before in a long sequence. It would be inappropriate for teachers to drill counting back for its own sake. Further, for students with learning difficulties it can be too difficult and demoralising when alternatives for subtraction are available to them in the long run (e.g., using addition facts).

On number facts and basics, the Dialogues (Lott, 1999) clearly pointed out that it is not recital of number facts. Included in the issues were the importance of number sense or using number structure, problem solving strategies, how numbers work, what operations mean, and preparation for tomorrow which includes technology. Another point was that children, not content, are the basics (Walker in Dialogues). For Year 5–10 students, proportional thinking was seen as an emerging concept that should be fostered and used to solve a wide range of practical problems. It was basic for many careers. It was also important for students to work with indirect measures.
and dynamic events. Teachers should not draw away from exploring interesting mathematics because students do not recall isolated number facts that were previously presented in fragments and not in a holistic way. Basic operations is not stuff you do but a means for observing and organising data, increasingly assisted by technology (Burnside in *Dialogues*). Coverage of facts and formulas that are merely memorised is not mathematics. Mathematics is to analyse, visualise and find patterns, make and test conjectures, experiment and guess, and describe mathematics orally and in writing. Interesting problems are needed for this. For example, after using Geometer's Sketchpad to see Pythagorean relations, students can try to find out if the relationship holds for quarters of circles (Snyder in *Dialogues*) or for other regular shapes and what happens to non-right triangles.

Students are less likely to select subtraction as a process for solving a problem requiring it, nor to carry out the subtraction calculation satisfactorily when compared with their competence with the other operations that have been taught. Perry and Howard (2000) found that only approximately 33% of the Year 7 *Counting On* students in NSW were able to do two digit subtractions by mental computation. Adetula (1996) also found that the problems requiring joining an unknown number to gain a result or separating an unknown number to gain a result were not as well done as the straight joining (addition) and separating (subtraction) problems. Without training in arithmetic strategies, the public school students in his Nigerian town did not do well on subtraction problems. These students had traditional lessons with using concrete materials to represent and carry out addition and subtraction problems. With training that encouraged students to know the various strategies for solving the problems, the students fared better. He noted, however, that while learning to count down to the number in a "separate change-unknown problem" (see Carpenter & Moser, 1984), which he previously found students did not succeed on, was this time done with moderate success. This is consistent with earlier studies by Clements and colleagues (Del Campo & Clements, 1987; Lean, Clements, & Del Campo, 1990) who felt that students generally were inclined to use the addition equivalent rather than counting backwards for some of the problems. Atedula and other studies show that this is not a strategy that students tend to use naturally. Interestingly, Atedula (1996) found that when recall of basic number facts failed students, only those with strategies were able to have successful attempts at problems. He also noted that students who overlearned and repeated the counting sequence and counting concrete materials without being taught arithmetic thinking strategies did not fair too well.

**Proportional Reasoning**

In this section, we will look at rational numbers, fraction notation, decimal notation, percent, ratio, money and measurement. We will look more closely at multiplication and division. Equivalence between different representations for a number is also a key issue. These aspects of mathematics are all closely related and rely on multiplicative reasoning. Money is included because of its decimal notation but also because it is a context in which students might first encounter proportional reasoning, for example, 10 ten-cent coins for a dollar. Alternative scales is also raised under this topic.

*Multiplying and Dividing*

It is clear that greater emphasis on the meaning of the concepts for operations has been established in projects such as the Count Me in Too program. In this project, the use of repeated addition for multiplication, the link between arrays and
multiplication, and between sharing or grouping for division is strengthened. The rhythmic counting, skip counting and use of words is closely linked, moving students on quickly from using one-to-one counting of equal groups which was used in establishing the concepts of multiplication and division as equal groups. If students continue to count one-to-one, their sense of composite units is underdeveloped and they will be held back (see, e.g., Wheatley & Reynolds, 1996).

Mulligan (1998) presented a framework that begins with initial grouping and counting and then expecting students to use repeated addition and subtraction and then to use multiplication and division as operations. Students progress in their ability to interpret word problems, in recognising equal-sized group structures in problem situations, and in retrieving and using more number facts (Mulligan & Mitchelmore, 1996; 1997).

There are several ways of interpreting the symbolic product. For example, 3 x 6 can be considered 3 groups of 6. Not so in Japan. This remains groups of 3, and we have 6 of them. The sense of product or of being operated upon by the 6 is similar to having 3 + 6 as 3 is divided by 6 and 3 + 6 or 3 - 6, all have 3 operated on. The commutative properties are taught in Year 3 as a useful way to solve a problem rather than as just a way of reversing the symbols for equivalent addition or multiplication expressions. This means there is more consistency across the approaches (Mousley, 2000).

**Multiplicative Thinking**

Doubling is taught in Japan in the first year of school and subsequently the two times table is taught along with multiplying by 5.

Students seem to have a natural tendency to enlarge by adding (Hunting & Davis, 1991; Lamon, 1999; Owens, 1992). Nevertheless, students are fascinated by doubling. For this reason, students will frequently begin to grasp a sense of products by making two-row arrays and later larger arrays for grasping number concepts. While skip counting assists with this introduction to number, it is also important for students to grasp the linear (rather than area) increase in size associated with multiplication, doubling and higher ratios. This is an important aspect of the jumps along an open number line as used in the Netherlands (Heuvel-Panhuizen, 1999). Dole (1999) used these ideas in more advanced work on percentages.

Similarly Pepper and Hunting (1998) emphasise that students should be encouraged to share as a precursor to division, even though their counting skills are not strong. The sense of equality in sharing will also need to be recognised but different strategies (not one by one) are automatically used in the sharing process (Hunting & Davis, 1991).

One of the concepts that assists in uniting the mathematics curriculum is that of splitting. Confrey and Smith (1995) elaborate on this idea and show that the world may be seen in terms of counting and splitting. Splitting involves division in its simplest form and so is a primitive idea for multiplication, ratio, and rate. For example, students in early primary school were able to select similar shapes indicating that similarity and its related concept of splitting is an early understanding like counting. Splitting provides an alternative to repeated addition for multiplication as seen in a doubling situation. Its origin or identity element is one rather than zero and splitting and counting can be matched for a range of properties. For example, composite units are made by counted units whereas raising units to the next power produces splitting units. Exponentials are created by repeated multiplication as a characteristic of splitting, and multiplication (additive) is constructed as repeated
addition based on counting. It is helpful to consider non-counting index systems such as pH concentration, Richter scale, decibels and musical scales when establishing proportional reasoning.

In relation to the development of rational number sense, Moss and Case (1999) base their work on two primitive psychological units (a) global structure for proportional evaluation and (b) a numerical structure for splitting or doubling, both of which appear to be in place by the age of 9 to 10 years. As students grow older they learn about different forms of splits and the relationship between different types of fractions. The order is more arbitrary for sequencing experiences but using one form of representation of fractions seems worthwhile. A curriculum based on these ideas was given to Year 4 students with a comparison group who were also given the rational numbers test. The program was similar to other progressive programs such as Confrey and Smith's (1995) but differed in not using pizza or round shapes for fractions. Instead it used a beaker which was split e.g. halved to show proportion. Percents were also the springboard to which number proportions e.g. 25% is half of half was developed. Once proportions were in place they could be added or subtracted and divisions by 10 achieved. Two-place decimals were then introduced.

In order to evaluate the program, percentages who achieved on each item were given and responses such as interchangeability of representations were discussed. The experimental group tended to do better when novel questions were asked or visual miscues and when the whole number - rational number confusion was prevalent. The experimental group of students developed a qualitatively better understanding of rational number. Students in both groups did not reach a product or mixed number subtraction involving thirds.

Percent teaching frequently lead to misconceptions (Szymanski, 1998). A percent sign is frequently not explained, calculators do calculations but do not put in the percent sign. Students need to make links to other procedures if they are to use an algebraic rather than a visual approach like Dole used. Students may not appreciate how decimals and percentages interlink. If they realise that multiplying by a hundred then assigning a percent sign is like then dividing by 100, they may follow. However, the pattern of percents and decimals is probably easier to manage.

Post, Cramer, Behr, Lesh, and Harel (1993) emphasise that multiplicative relationships must be introduced into early primary school in order that whole number and additive ideas do not dominate students thinking about number relationships. In the past, students may have found continuous interpretations of rational number more difficult than discrete because they were not so easily able to regress to counting strategies and, with the wrong emphasis in learning, they had not yet developed flexible partitioning strategies.

They conclude that general curriculum implications from research are:

- Extend interpretations of rational numbers and develop connections among them. Instruction should build on previous learning and understanding should be expected to evolve over a several-year period of time.
- Instruction should emphasize the interrelationships within the rational number domain (part-whole, decimal, ratio, measure, and operator).
- Delay procedures and operations until an understanding of quantities is established. Understanding of quantities should include an emphasis on order and equivalence ideas.
- Develop understandings via instructional models that reinforce links between concepts and procedures as well as translations within and between modes or representations. (p. 344)
Such approaches were especially supported by research with Year 7 students (Heller, Post, Behr, & Lesh, 1990). They also go on to illustrate the kind of assessment items that would reflect students ability to show understanding by their mode of representation (Lesh, 1979). For example, draw a picture to show the situation and explain how the picture can be used to find the answer to: Alice owned _ of a hectare of land. She planted corn on one half of this. What fraction of a hectare did she plant in corn? (Post et al, 1993). Kieren (1993) emphasises that students need a mental visual representation to allow them to fold back and work out multiplicative thinking problems. Warrington & Kamii (1998) illustrate this with a series of simple problems that can be represented by fraction circular "pizzas". They carefully planned an order of questions that are easy to handle (e.g. halves and thirds) in order for students to construct the meaning of finding a fraction of another number. They emphasise that taught phrases such as "of means multiply" and algorithms do not allow for this intuitive development (see also Kamii & Warrington, 1997). Nowlin (1998) also mentions how formally taught algorithms can hinder understanding of the division of fractions. He illustrates how concrete examples can be used to establish the meaning of the division of fractions by students drawing on their understanding of how many for division and applying it to fractions. Warrington and Kamii (1998) illustrate the variety of ways that Year 5 and 6 students will find half of _ or 2/3 of 3 using diagrams to help their rational number thinking. They began with fractions of whole numbers. The students were already good at making equivalent fractions in dealing with concrete examples of addition and subtraction problems. Thirds, sixths and ninths are more difficult than halves, fourths, and eightths. "Children ... could simply look at 4/5 and know that half of it was 2/5" (1998, p. 343). Non-unit fractions were more difficult. Warrington and Kamii emphasise that without the exploration of problems, "by telling children that 'of means to multiply,' ... [and] by giving them the algorithm of multiplying the numerators and the denominators, we impose a rule that does not make sense to students" (1998, p. 343).

The usual algorithm for division of fractions is "invert and multiply". When this is followed as a quantitative procedure and practised without referential meaning, it can lead to the following situation. Half a pizza divided by a quarter pizza is 2. When students were asked to explain the answer, they drew 2 pizzas. The idea that they were now multiplying made it seem okay for the answer to be bigger than either of the two original numbers. The notion that in reference to the problem, the solution might be 2 quarter pizzas was not easily understood. This is just one example of the importance of learning mathematics with real problems where elements of the problem structure, checking of solution accuracy and evaluation of a solution's reasonableness are all part of mathematical reasoning. "Teaching computational procedures stripped of referents to which they apply isolates mathematical meaning from, rather than disclosing it within, the context of their use. In short, mathematics is reasoning about things as well as about their relationships." (Piel & Green, 1994, p. 49)

**Thinking about Common Fractions**

Pitkethly and Hunting (1996) identify five sub-constructs associated with initial fraction concepts: part-whole, quotient, ratio, operator and measure and outline the foundation of rational number knowledge, addressing such things as fraction language, intuitive fraction knowledge, constructive mechanisms, whole number schemes, partitioning schemes, measuring schemes, and so on. Whole number schemes can serve both to inhibit or advance rational number development (see also Lamon, 1999). After reviewing a number of studies, Pitkethley and Hunting suggest
that two avenues are suggested for the emergence of initial fraction concepts, from application of intuitive mechanisms or from ideas of ratio and proportion. Although it is possible to build experiences for young children which reflect individual sub-con structs, it is also important to know how both the part-whole and ratio sub-constructs relate to the growth of rational number understanding. The work of Moss and Case (1999) and of Confrey and Smith (1995) illustrate this position.

If students are "to conceive of fractions as objects embodying numerical relationships that transcend particular instances and contexts" (Hunting, 1996, p. 123) then students need opportunities to engage in problems, tasks, and explorations to promote rich conceptual understanding. Hunting (1996) provides some classroom activities that build on children's intuitive division, sharing, and fraction knowledge.

The major difficulty with common fractions is that they require seeing numbers in different ways (Lamon, 1999). For example, taking two thirds of 15 requires 15 to be seen as a whole of which two thirds is a part, that is 15 is made up of 5-unit parts. Watanabe (1995) illustrated the different ways in which numbers had to be considered. He also showed the value of counting by fractions, renaming wholes as, for example, three thirds. There is a significant jump to recognising equality of fractions that are represented by repeated addition of different fractional parts. For example, three red rods may equal two green rods in length representing different collections of parts of a whole. Lamon (1999) points out too that fractions can be considered a specific way of writing rather than equivalent to rational number. She also provides 12 examples of meaning that may be attached to three quarters. The point she is making is that students from K to 4 need considerable talk about fractions in different contexts. Kindergarten children can partition and talk of part of a whole, especially if the package is already marked, e.g. a sheet of stamps. Students can usually make halves, quarters, eighths etc but can also manage thirds with assistance at an early age. By investigating students may record and understand some numeral combination such as 3 sixths being a half. Besides the division of a whole into parts, students need to experience parts of multiple wholes, for example, how many quarters in 2_. Some uses of the word fraction could also misguide. For example, a fraction of the population usually means a small fraction. It does not refer to any possible fraction. Students also need to discuss the comparative situations where you might say the package has more. This may happen when the boxes being compared actually start with different amounts so that more means relatively more. Finally it should be remembered that probability is expressed as a fraction.

A key issue in the teaching of fractions is whether to use discrete objects or continuous objects. Singaporean curriculum recommends the division of areas initially. Behr, Wachsmuth, and Post (1998) recommend the division of continuous objects as early representation in Year 4. Their study illustrated the many different ways in which students thought about the tasks they gave. In particular, the tasks looked at both the egg carton with discrete holes and the representation of the fractions by covering with L-shaped cardboard so that the egg-carton was considered as continuous. They showed that the latter method was a good approach and that when discrete objects were later introduced this did not interfere with the fraction concept. They also commented on the introduction of different manipulative aids. For example, if folding a circle was the first aid for investigating simple fraction situations (e.g., a third compared to a half; half of a half), then they moved to another manipulative such as rods when students became too comfortable with the first aid. They may model with one of the aids, asked students to explain and then take the other aid (usually the new one) and explain the fraction in terms of it. This was important in being able to generalise the concept. They could do this too with moving
from the continuous to the egg carton as discrete holes. The continuous partitioning seemed to establish a sounder understanding of the fraction concept. Further down the track it could assist with the operations, even if there were discrete parts as part of the fraction (see Lubinski, Fox, & Thomason, 1998).

A thoroughly reported study by Empson (1999) with a Year 1 class illustrates not only the initial fraction concepts that students had (e.g., about half and half again) but also the development of students’ fraction thinking. By the end of a series of problem-solving lessons on equal sharing of, for example, cookies, half the students were able to illustrate how 14 cookies could be equally shared between 5 children. Students showed that all but 3 children could predict how to share equally, could discuss part-whole relationships with fractions and mixed numbers, could order unit fractions, and half the class could apply their learning to new situations such as operator on a set (\(_\frac{1}{2}\) of 12) and add and subtract simple cases such as \(_\frac{1}{2} + \_\) and \(_6 - \_\), and about a third of the students could give equivalence by comparing or giving a missing value. Some even thought of this in terms of a ratio. The report gives details of the problems used and how students negotiated the results of the problems, thus developing their various fraction concepts. Except for the operator items, the sharing dealt with parts of continuous wholes like pancakes.

The discussion on overgeneralisation of symbolic representations and understandings between fraction notation and the whole number system has led some people to suggest that fraction notation should be left until later. However, there is some good reason for introducing fraction symbolic notation building on informal knowledge of fractions at an early stage so it is built up with the whole number system. Mack (1995) does not provide a definitive answer to this but she showed that students will also overgeneralise the fraction notions to whole numbers and vice versa when the fraction symbolic notation is introduced. She suggests that this overgeneralisation may be a way of learning new concepts, no matter when the symbols are introduced. However, a realisation of different forms of symbolic notation (not just whole numbers) may be valuable. She reiterates the value of the symbolic system being a shared communication (Pimm, 1987).

There seems to be no doubt that younger children in Year 1 can begin to think about fractions through problem situations in which the teacher selects appropriate problems, the students work on representative objects, and the class negotiates the solutions and share alternative solutions. It would be based on these early concepts that the later lessons in Years 3–5 would take students much further from practical operations on fractions.

Making fractions, especially drawing representations, seems to be a key for working with them adequately (Busatto, 2000). This is particularly useful when working with fractions with the denominators that are easily factored by which some meaning of addition of fractions can be established. Area and number line diagrams have both been successfully used for this and makes the addition of fractions meaningful for a younger age group. Diagrams can help to establish products and divisions when whole numbers are concerned. It may be for some time that students work out fraction questions informally so that algorithmic approaches do not interfere with rational number sense (Kamii & Warrington, 1999). Riddle and Rodzwell (2000) illustrated Year 3 and 4 students working with adding mixed numbers successfully used the strategy of filling up wholes, using drawing pies to illustrate and keep track. The few Year 5 students who applied the traditional equivalent fraction method tended to make errors. Kamii and Warrington (1999) tackled products successfully beginning with halving and then taking thirds. Nevertheless, developing the sense of
division of fractions as how many of that fraction in the first fraction is critical but not as easy as it might first seem. Lubinski et al.(1998) illustrates the struggle that a teacher education student went through to construct this meaning. Nevertheless, the diagrams were a key to thinking. But so too was the ability to reunite the parts of the diagram.

Reunitising numbers and chunking them in different ways is a key to being able to work with fractions. This should be done with manipulatives and drawings with talk about the context of a problem. Teachers may need inserviceing on the different kinds of fraction situations with examples of questions that might arise. Flexibility between different uses of the word fraction and reunitising will be keys to early fraction work. These experiences must be quite extensive before any numerical manipulation is undertaken if good fraction sense or relative sense is to be developed.

The partitioning strategies that students use can be quite varied. Year 3 students have shown up to 12 different kinds with circular, rectangular and length models but some efficient strategies do not necessarily assist students' part-whole knowledge of fractions (Charles & Nason, 1999).

Further to the debate on what form of analogue or manipulatives should be used for developing fraction knowledge, Charles and Nason (1999) evaluated different circular (e.g., pizzas and pikelets), rectangular, and length models for ecological validity, abstraction ability, and ease of partitioning for Year 3 students. Teachers need to take account of the positive and negative features of each analog for fraction understanding.

The issue of building on whole-number facts or developing intuitive knowledge from physical materials was raised by Pearn (1996) in her comprehensive study of Year 3 students on partitioning tasks, fraction tasks, and a ratio task. It is clear that one difficulty is that students do not understand the meaning of most fractional terms beyond _, and lacked general conceptual fraction knowledge. Descriptions of the tasks and students' responses are provided in some detail in her chapter.

"A firm foundation of whole number strategies and relationships established in early years of schooling" is important for tapping into whole number knowledge through appropriate tasks to build fraction knowledge (Hunting, Davis, & Pearn, 1996, p. 375). In a two-year teaching experiment with two students (aged 8 and 9), they used activities with the computer program CopyCat. In a follow-up study (Hunting, Davis, & Pearn, 1997), they show how two other students' limited whole-number knowledge frequently prevented them from successfully operating in the fraction domain. Over time, "development of fraction knowledge in this setting [paired learning with the computer program] was stimulating the creation of whole number knowledge which in turn enabled success with the fraction tasks" (p. 246). However, there can be difficulties that face teachers when students are at risk. Both conceptual and procedural difficulties may be faced, and multiple solution strategies are possible. Some of the children revealed faults with basic skills, but others had more deep-seated difficulties. One child lacked the metacognitive skills to keep track of his solution; another had developed a slow, cumbersome but meaningful method.

Not only with fractions but with operations on whole numbers, students are required to construct the abstraction of units as composite units (Wheatley & Reynolds, 1996). This notion would seem to be important in other areas such as measurement too. Image development can assist with this. Even a drill and practice computer program on fractions showed that the success comes from the graphical presentations as well as oral feedback (Abercrombie & King, 1996).
Students show a tendency to use economic partitioning strategies when sharing and thus develop composite units for fractions. The commodity being shared influences performance but not as much as the numerical proportions used. "Partitioning has not been fully exploited as a didactic device for helping children to develop rational number ideas." (Lamon, 1996, p. 190). Partitioning has a role beyond Year 3 as the curriculum provides for greater depth. For example, partitioning may be used in an increasing variety of problems, in moving to composite wholes, the notion of how much rather than how many (a rate idea), and flexibility in unitising.

The idea of partitioning a rectangle to represent fractions has been encouraged by numerous authors. Armstrong and Larson (1995) also show that students will use first direct comparisons and then part-whole relationships in deciding on fraction comparisons. Students tended to use part-whole relationships more as they reached higher years and after fraction symbolic notation was introduced. One concern was that students should continue to experience fractional situations after Year 8. They concluded that teachers should:

1. recognise the developmental sequence that students follow in their acquisition of rational number reasoning abilities: direct comparison strategies precede part-whole comparison strategies;
2. give students opportunities to experience the meaning of fractions, other than one-half;
3. give students opportunities to connect fractional symbols to fractional terms and models form third grade through middle school [to Year 8]
4. recognize the developmental and experiential aspects of constructing those connections. (Armstrong & Larson, 1995, p. 18)

Watson, Collis, and Campbell, (1995) illustrate how important it is to associate both fraction and decimal representations of rational numbers. If this is not done students do not gain meaning. Despite the stilted and complicating use of the SOLO taxonomy to describe development, it is clear that each representation has both its visual and overgeneralisable influences and necessary underlying constructs (such as place value for decimal fractions) that need to be mastered if students are to understand the meaning behind the notations.

Reasoning with Decimals

The major body of work on decimals has been carried out by Stacey and colleagues. Their website <http://dsme.edfac.unimelb.edu.au:8000/> at the University of Melbourne provides an excellent summary of work on decimals. The actual decimal notation is difficult with many considering a decimal number like 0.2 as being less than zero. This seems to arise from students associating the decimal number with a number line. Another visual problem arises from the symmetry around the decimal point rather than around the ones place, leading to students thinking about onesth. Some students have a poorly developed sense of the columns being related by multiplying, or dividing, by ten. Another misconception can arise with the use of money in which the decimal part is considered in terms of tens e.g $2.50 is considered 2 fifty and operated upon as 52 or 250. There is also a poorly established relationship between tenths and the decimal digit. For example, many consider 0.2 to be half. Finally many students have a poorly developed sense of how big a decimal number is when they see the written form. So, for example, they consider 0.8 as smaller then 0.75 because 8 is smaller than 75.

Students' incorrect responses to items on comparing decimal numbers generally result from application of three common erroneous rules: (a) whole-number rules in which 4.125 is thought to be bigger than 4.7 because 125 is bigger than 7, (b)
fraction rules in which fewer decimal places is thought smaller, (c) zero rules in which a zero after the decimal point makes it smaller but then students resort to whole-number rule for other numbers, or (d) unclassified (Moloney & Stacey, 1997). In their study the students showed little change in understanding of decimals from Year 7 to Year 9 when they were retested. Results also suggested that students' misconceptions about decimals are a significant problem, and that in a follow-up cross-sectional study of 379 students in Years 4–10 over 50% of Year 8 students, and more in lower years, could not compare decimals correctly. By Year 10, 73% could compare decimals. Whole number misconceptions were prominent in the early years, but they tended to disappear over time; that the fraction rule misconceptions persisted, and were retained by approximately 20% of Year 10 students; that the zero rule was uncommon; and that most movement between categories was evident in Years 4–7. In 1999, Stacey and Steinle (1999) also showed the whole-number rule or longer is larger, and fraction rule or shorter is larger were prevalent. Results indicated that in the long term, there is a general trend towards expertise, but that 20% of students with misconceptions remain in that category.

Basic misconceptions associated with fractions and decimals were prevalent, and persisted even among 14-year olds in four countries Australia, United Stated, Sweden and Taiwan according to multiple-choice test data analysed by Bana, Farrell, and McIntosh (1997) The authors raise the issue of requiring students to perform operations with fractions and decimals as set in school mathematics curricula when clearly many students have limited and limiting conceptual understanding of these topics.

Of all the misconceptions about decimal numbers, Putt (1995) found that the rule to select as smaller the number that has more digits in its decimal portion (e.g. 12.94 is smaller than 12.7) is the most persistent even into adulthood.

According to Baturo (1997), there are three levels of knowledge that enable functioning within the decimal number system, to varying degrees, but that Level 3 knowledge enables expert functioning within the system. In Baturo's model, broadly speaking, Level 1 is baseline knowledge and Level 2 is connection knowledge. At Level 3, structural knowledge comprises additive structures, multiplicative structures and reunitisation, and is the "superstructure for integrating all levels of knowledge" (p. 88). From testing and interviewing some Year 6 students, Baturo determined that multiplicative structure was a determining factor in differentiating high performance; that only students in the high proficiency with tenths and hundredths category could process tenths and hundredths with understanding. This research has implications for teaching particularly when students are introduced to the next decimal place of thousandths.

Baturo (1998) explored the interaction between available and accessible place-value and regrouping knowledge for Year 5 students. Available knowledge can be directly probed, and accessible knowledge becomes evident from situations where it is applied. Students' performance was better on place-value than regrouping items and that there appeared to be a strong direct relationship between available and accessible place-value knowledge. There was evidence also of students using over-practised rules (e.g., insert a right-most zero) and impoverished prerequisite whole number and fraction concepts and processes in some classes and "that correct individual performance is possible without understanding and that availability of knowledge does not mean it will be used (accessed)" (p. 96). Baturo raises the issue of the place of pen-and-paper computation in schools, and suggests that the place for such computations is as a vehicle for promoting structural principles of
mathematics (e.g., set inclusion, commutative and distributive laws) that apply across whole numbers, fractions and algebra domains.

Irwin (1996) reported on the development of 11- and 12-year old students' decimal number understanding through their engagement in pair discussion on contextualised (real-world) and decontextualised (numerical) problems over a week. Greatest improvement was made and retained by students who had worked on contextualised problems. Analysis of interactions that occurred between student pairs revealed differences in interaction, thinking and discussion between students according to ability ranking. Irwin notes that errors made by students were not random and that contextualised situations enabled students to draw on their experiential knowledge to assist with problem interpretation and understanding. As a result of this study, Irwin concluded that to understand decimal numbers, students need to reconstruct their ideas of whole numbers to include decimal fractions. The tasks were intended to cause some degree of cognitive conflict but this led to a greater increase in decimal knowledge when the students already had some initial knowledge, usually in higher ability pairs (Irwin, 1997).

Putt (1995) recommends that the concrete, oral/written and symbolic representations of decimal fractions and the various equivalent forms (e.g. 7 tenths and 70 hundredths) need to be well established. Concrete representations are particularly useful in encouraging students' mental models or images from which they can reason. In using words, he supported Resnick's earlier work that it was more valuable to use the additive nature of the base 10 decimal system and refer to 0.463 as 4 tenths, 6 hundredths, and 3 thousandths than to refer to four hundred and sixty three thousandths. The implications for teacher-student discussion of problems brings to light conflicts in the students' thinking allowing students to overcome these misconceptions and not allow them to be reinforced by practice (Stacey & Steinle, 1999).

The diagnostic tests developed by Stacey and her colleagues (see also Steinle & Stacey, 1998) are useful for classroom teachers as they yield valuable information on students' understanding of such things as the relationship between place-value position, decimal to fraction conversions, equivalent fractions, identification of decimals on a number line, naming decimals between two decimals, and stating the value of certain digits in decimal representations.

The use of a 100 for establishing the links between decimals and fractions is supported by a study involving individual interviews with 49 Year 6 students' (Hunting, Oppenheimer, Pearn, & Nugent, 1998). Using 100 to explain fraction equivalents and comparisons was the most common of the "range of different relational connections upon which students' knowledge of fractions and decimals are based" (p. 277). Numerous misconceptions were also evident and students could not see or explain the inconsistencies. Nevertheless, the researchers point out the value of providing students with opportunities to reflect and explain their thinking processes.

However, counting by decimals, placing decimals on number lines with increasing magnification are highly critical aspects of developing decimal fraction number sense. Moreover, linear modelling of decimals seems to be far more valuable than area models although both are used extensively (Helme & Stacey, 2000). Nevertheless, a linear model does not directly link back to the density of decimals as expressed on a number line. In particular, it is important to keep to one visual image where possible (Irwin, 2000). Emphasis on the relative size of decimals in terms of number line position seems to be successful. The links to fractions, that is the part of a whole aspect of a decimal, must be clear.
While the decimal format provides an easy application of written formal algorithms to numbers with decimals, nevertheless the initial meaning of decimals, the sense of size, and a well understood algorithm seem to be priorities.

Money
Money is introduced in the *Years 1 to 10 mathematics syllabus-in-development* (Queensland School Curriculum Council, 2000b) under number concepts with recognising coins and notes by the numbers displayed and other features, and with explaining the use of money and that change may be given. It continues in level 2 with relationships between number knowledge and the value of coins, being able to make a certain value with different coins and being able to write in conventional ways.

By level 3 students are making comparative judgements, using calculators to add and subtract monies and to interpret the result, and to know the amount to tender by using the ideas of rounding. There is little further development in level 4.

*Investigations* from USA deal with money in Year 2. In particular, money becomes a way of skip or jump counting. For example, making 64 is 10 + 10 + 10 + 10 +10 + 10 + 4.

Our current coins and rounding, can be useful for encouraging counting in 5s and 10s and 20s. Discussion of rounding reinforces numbers that are near each other but it may lead to other discussions about other ways in which decisions are made about numbers and money when the issue arises (i.e., sometimes rounding to the nearest number and sometimes rounding down only), probably in a higher year.

Saul (1997) provides a number of suggestions for lessons on money for early to later primary school. They include activities and games for recognition of coins, noting value of money, a coin a day and equivalents, shops and ticket booths, planning a store or advertising, thinking of science explorations such as metal, weight, shape, and what happens to paper money when it gets wet. There are a range of art and language lessons, and a discussion on how money is earned. Another teacher devised an alternative to counting up the amount tendered because children would get lost and frustrated. Instead she went through a chart that started at .01 and in columns went up to .99. Next to it students helped the teacher to complete the amount of change that would be given for each number in the columns. Students saw a pattern developing so that when they had to give change for .56, they knew they should get something in the forties. The procedure allowed students to go into change from larger amounts and to see how to subtract, for example from 2000 (that is cents equivalent to $20).

Several studies have shown how students familiar with selling goods in a market place are very able to calculate monetary amounts in their heads but struggle with simple school problems. However, these would only represent a small proportion of Australian students. The main recent research article on money is by Brenner (1998). Although her study covered Hawaiian students with different opportunities for using money than may be found in Australian society, nevertheless her findings are very pertinent. She indicates that there is a clear disjunction between out of school use of money and the school lessons on money as observed and recorded at interview with a wide cross-section of students and ages.

Students would confuse cents with coins in the younger years so asking how many cents were needed was misunderstood. Shop keepers dealing with children tended to avoid using "pennies" but Australian stores already round off. Learning that a dollar is 100c or a particular coin was so many cents was not as common out of school as
learning that four quarters equalled a dollar. In Australian terms, students would be learning that two big coins (fifty cents) made a dollar or that five ten-cent coins made the big coin. Children could read numbers on receipts and on brochures although they had not really developed a sense of part of the dollar. It is this aspect of money which needs to be clearly articulated.

Ratio

Carpenter et al (1999), Lamon (1999), Conroy & Perry (1997) would suggest that ratio must be intuitively developed by middle primary school. It is otherwise hard to teach in high school. In particular, Dole (Dole, Cooper, Baturo, & Conoplia, 1997) has shown that proportion and work with percents has been difficult. She selected to teach percents by using a vertical linear model with numbers of comparison on one side and percents on the other. In this way, students readily developed equations or ratios that could be compared (Dole, 1999).

An example for early primary school comes from Lamon (1999). What comparative understanding is conjured up by the following. Think of 5 people in a 2-seated car, 5 people in a football stadium, 5 people on an 8-person elevator. A ratio is a comparative index. This relative thinking is linked to fractions or rational number comparisons and are one kind of ratio. If the ratio compares measure of different types, it is called a rate. Multiplicative thinking also needs to be developed. How many times would you have to stack up Joe's biscuits to get a pile as high as Cindy's? (Lamon, 1999). The next idea is to think about the unit as discussed under rational numbers. For example there are 6 cakes but 4 are a unit e.g. a pack of 4 cakes. So we have 1_ units. As experience allows, students can reunite and make comparisons. This links to fractions and equivalent fraction ideas. These key ideas of part-whole relationships include different-sized partitions. Students tend to gain the idea of equivalence if the larger parts are divided up and then shared. This occurs, for example if you ask how 3 people might share 4 pizzas. Developing flexibility in thinking is important as the relative reasoning develops. Ratios differ from rational numbers unless they are part-whole, operator, measure or quotient fractions. For example, the comparison or ratio of circumference of a circle to its diameter is not a rational number. Discussion of change situations are important so that students learn to distinguish proportional and non-proportional change. Lamon recommends an arrow diagram for showing the jumps that are made in thinking about a ratio.

- Integrating areas of mathematics seems to be an important issue. Area has been a clear way of presenting fractions in the Years 5–8. The idea of seeing relative sizes is significant (e.g. Brahier & Speer, 1997; Lamon, 1999; NCTM, 2000, p. 219).

A particularly useful real-world example of proportion was discussed by Mittag and van Reusen (1999). The problem dealt with the capture-recapture procedure for finding out how many fish are in a lake. A number of fish are captured and tagged and the number of a specific kind out of the total recorded. The fish are returned unharmed to the lake. On the next recapture, the number of tagged fish is noted out of the total caught, and the proportion of the total obtained. Since a bag could represent the lake with fish of different kinds (pretzels) they could simulate the problem. The ratio of the number of marked fish in the population compared to total number of fish in population equals the ratio of the number of marked fish in sample compared to total number of fish in sample. The total number of fish in the population is to be found. The lesson organiser for this idea which included statistics also emphasised word problems and equations, fractions, ratios and proportions and estimates.
Later developments towards the end of compulsory high school revolve around the use of graphs to represent the scale factor of rates or ratios. The idea is for students to become proportional thinkers. They can think in terms of a 3-unit or 10-unit or composite unit.

Carpenter et al (1999) report on the observation of a Year 4/5 classroom using a social constructivist model for teaching ratio over a two week period. Lessons consisted of blocks of story problems in which one term of a proportion, two equal ratios was missing. They carefully selected stories and numbers for students to try and discuss. All problems were discussed without any written approaches (the symbolic representations below were not used but it aids in reporting their study). The first set was missing the fourth term.

For example

\[
\frac{3}{10} = \frac{6}{X}
\]

They observed that students may develop several methods for solving the problems. For example, they may use the multiplicative relation of the two terms within the same ratio or of two terms from different ratios. Students generally selected the method that provided an easy solution strategy but they had to explain it. Some students also developed a level of thinking where they could explain the relationship between the procedures used. One approach was to reduce a term to one or the unit ratio but this was not a necessary procedure. Some students used the ratio as a unit, a complex composite unit (that the researchers described as a vector upon which scalar multiplication could be performed). These students may have transferred the two terms together making a new equivalent ratio. For example, this is represented in symbolic form although students would be expected to do this with words orally.

\[
\frac{2}{5} \times \frac{X}{40} \quad \frac{14}{35} \quad \frac{2}{5} \cdot \frac{14}{35} \cdot \frac{16}{40} \quad \text{or} \quad \frac{2}{5} \times \frac{X}{40} \quad \frac{2}{5} \cdot \frac{X}{40} \quad \text{multiply by } 40
\]

*Figure 1. Symbolic description of young student's ratio ideas*

Sometimes they then added this ratio to the first to get the target ratio. This was called a build up strategy. At a higher level, the students were able to work with non integer multiples of the ratio, often using a unit ratio but dealing with it as 1:1 which is multiplied by, say, 10. Data and transcripts indicated how students were able to discuss the ideas of ratio within the story contexts with the members of the class.
achieving these levels. Students began to explain the associations between the use of between (second approach shown in the above figure) and within methods (first approach in the above figure) as described earlier.

In summarising these ideas on proportional reasoning, we can look at Langrall’s (1999) summary of instructional strategies and examples of questions for stimulus of discussion of relative or proportional reasoning. Langrall gives the critical components for formal proportional reasoning as

- Understanding the difference between relative and absolute change
- Recognising situations in which the use of a ratio is reasonable or appropriate
- Understanding that the quantities composing a ratio may change together in such a way that the relationship between them remains unchanged
- Building increasingly complex unit structures or unitizing. (p. 1).

Langrall suggests that students first reason informally using pictures or manipulatives and can make qualitative comparisons. Although it may be discussed earlier, it is mainly a Year 5 and 6 activity. Later they unitise or use composite units, find and use unit rates and identify or use scalar factors. They use equivalent fractions and build up both measures. This usually begins in Years 6 and 7 with formal solutions beginning in Year 8. Too early introduction of formal methods for solving proportions can lead to misunderstandings that are difficult for students to reconcile with real problems. Establishing the proportional reasoning slowly over the years is much needed. Later they set up proportion using variables and solve using equivalent fraction or other methods, fully understanding the relationships. Proportion needs to be integrated with other aspects of the Years 5–8 curriculum. It should be noted that these ideas are covered earlier in Singapore and some studies detailed above suggest the ideas can develop earlier.

Lamon (1999) provides a comprehensive text for teachers based on research and allowing teachers to explore the concepts as students might develop them. She emphasises the same points as those raised above. Students need to partition physically and develop mental images. They also need the long development of understanding of rational numbers if they are to think relatively and apply proportional reasoning.

One way to define proportional reasoning is to say that it is the ability to recognise, to explain, to think about, to make conjectures about, to graph, to transform, to compare, to make judgments about, to represent or to symbolize relationships of two simple types. The first type of relationship, a direct proportion, occurs when two quantities change in the same direction, that is as one quantity gets larger (or smaller), the other also gets larger (or smaller), but the size of one quantity relative to the size of the other always stays the same. (p. 8)

In the second case, the quantities move in opposite directions.

One key idea is that of being able to explain a unit and later to be able to give different units. These units are composite units. Ratio is a part of the whole where the whole may be a composite unit of 3 parts.

**Measurement**

The research on measurement is suggesting that we may need to reconsider how we have been teaching it. For one thing, the issues of doubling, and multiplying, the ideas of composite units are closely linked to early measurement sense and relative size. There is some agreement from researchers There is much less research available on the learning and teaching of measurement than there is for number.
However, it is clear that much of the number research, particularly that dealing with the concept of fractions and the notion of iterable and composite units, is particularly pertinent to measurement (McClain, Cobb, Gravemeijer & Estes, 1999; Stephan, 2000).

Length

that using informal units in early measurement lessons may leave the activity as one of counting with little concept of what is being measured and why counting is giving a measure (Clements, 1999, Owens & Outhred, 1998). However, the intuitive ideas about number seem to have resulted from the uses of rulers (Boulton; Lewis, Wilss, & Mutch, 1996; Clements, 1999). Children preferred to measure by using rulers. Clements suggests that we may need to revisit the traditional "instructional sequence: Traditionally, measurement of length has been taught through a sequence of activities described by Clements (1999b, p. 5) as: "gross comparisons of length, measurement with nonstandard units such as paper clips, measurement with manipulative standard units, and finally measurement with standard instruments such as rulers." This sequence is often repeated with other measurement constructs such as area, volume, and mass. However, this may need to be reconsidered in the light of research which has gone beyond that of Piaget and his colleagues (see, for example, Piaget, Inhelder, & Szeminska, 1960).

On the one hand, there is some evidence to suggest that using informal units in early measurement lessons may leave the activity as one of counting, with little concept of what is being measured and why counting results in a measure rather than a number (Bragg & Outhred, 2000; Clements, 1999a; Owens & Outhred, 1998). As well, there is evidence (Boulton-Lewis, Wilss, & Mutch, 1996; Clements, 1999a that the use of rulers may facilitate the develop of length measurement ideas and may be preferred by many students. Boulton-Lewis et al. (1996) found that Kindergarten students could select strings of the same length, by Year 1 they could measure a rope, and by Year 2, students had difficulty doing a task that is often expected by using nonstandard units. The task required them to compare 5 matches in a row, 10 half matches, and 5 in a zigzag and 10 halves in a zigzag. Using tasks with nonstandard units does not necessarily precede or show the need for standardised conventional units. The cognitive load in these tasks was exacerbated by the use of informal units and it was suggested that the introduction of such units is not helpful in students’ developing an understanding of the need for standardised units. Clements (1999a, p. 7) suggests that:

using non-standard units early so that students understand the need for standardization may not be the best way to teach. If introduced early, children often use unproductive and misleading strategies that may interfere with their development of measurement concepts.

On the other hand, Cobb, Stephan, McClain, & Gravemeijer, 1998; McClain et al., 1999; Stephan, 2000) have found that the introduction of an informal unit in an appropriate context can not only make the task of linear measurement more interesting for the children but can also strengthen the links between the number and measurement through the development of 'unitizing' in the measurement context. Students in a Year 1 teaching experiment not only used nonstandard units—both perceptual and conceptual—but also created from these composite iterable units which they could use to develop their measurement knowledge. In short, they created their own ‘rulers’ using these units and used them to measure and “to interpret their activity of measuring as the accumulation of distance” (Stephan, 2000, p. 4).
In a further departure from the guidance of Piaget and his colleagues, there is evidence to suggest that waiting for students to conserve length by the age 7 or understand transitivity as applied to the indirect comparison of lengths by age 9 simply has the effect of slowing down measurement development (Clements, 1999a). However, young students need to know that, to cover the same length, you need more of one iterable unit than you do of another which is of greater length. This has strong links to much of the recent work in the development of concepts of fractions which has been discussed earlier.

Teaching measurement in Year 1 through carefully sequenced instructional activities that encouraged students to explain and justify their thinking supported students' reasoning about space and distance, and not merely their ability to measure accurately (McClain, et al., 1999). By using 10 strips for measuring, it was possible to also support students' mental calculations to 100. Students reasoned quantitatively using jumps on an empty number line which seemed to develop from students' representations of measurement as an analogue to number.

In another study, students aged 6 to 8 were more able to use standard units and measurement instruments to reason about lengths than they were able to use informal units (Nunes, Light, & Mason, 1993). They even did better with a broken ruler! It seems that the rulers, especially if they make their own, build on cultural advantage and students develop new mental models in a meaningful sense in accord with Vygotskian perspectives.

Clements, Battista, Sarama, Swaminathan, & McMillen (1997) have shown the advantage of using computer Logo turtle graphics to develop measurement sense. By Year 3, students were drawing proportional line segments and rectangles with few needing to portion off units. They were able to reason about the lengths of all lines in a complex rectangular plan in which only some lengths were marked. Another suggestion is for students to use a marked but unnumbered ruler, thus developing their own links with number, especially some of the multiplicative aspects of number. The advantage is for the students to no longer rely on the number sequence per se but to consider the intervals and general concept of length.

The counting of units is an empirical approach that may not assist in the development of the measurement concept (Owens & Outhred, 1998). Kamii and Clark (1997) illustrate this using Wally's Stories (Paley, 1981) in which Kindergarten students try to decide which carpet is bigger. They suggest that students need to engage in problems that actually encourage them to compare, often using a third object (in Piagetian terms this is transitive reasoning). Kamii and Clark (1997) showed that 72% of Year 2 students in their sample could illustrate their transitive reasoning either with a strip of cardboard or with blocks. However, with their tasks, Year 4 was reached before students could explain unit iteration. This piece of research confirms that measurement is much more than an empirical procedure where the purpose is to find the answers to questions of How many? units fit into given length. Rather, it is a procedure which requires reasoning—in particular, transitive reasoning and unit iteration. Hence, they suggest that open-ended questions, which require indirect comparison of lengths using a third length (iterable unit) and not simply an empirical procedure, should be used to stimulate such reasoning. They suggest:

Instruction must encourage children to think hard and to modify their thinking, rather than teach empirical procedures that do not take their thinking into account. ...

Encouraging children to struggle with a problem and to debate among themselves is beneficial both to students who have no idea what to do and to
those who try to convince others of the value of their idea. (Kamii & Clark, 1997, p. 121)

Other Measurement Topics

The same pedagogical dilemma—do we introduce nonstandard units in order to give meaning to standard units or do we consider the use of standard measuring units and instruments—as was discussed with length above is pertinent to the other measurement topics of area, volume, and mass and less so with time and temperature.

The topic of area measurement has stimulated much research and discussion in recent years (Comiti & Moreira Baltar, 1997; Kidman, 1999; Kidman & Cooper; 1996; 1997; Outhred & McPhail, 2000; Outhred & Mitchelmore, 2000; Owens & Outhred, 1997). Much of this work has highlighted the fact that many students confuse perimeter and area measures and that this is related to whether or not they use additive or multiplicative judgement rules. Much of the research suggests that formulae may be introduced too early, making it more difficult for the students to distinguish between these two concepts. For example, Kidman (1999, p. 304) concludes her paper with the following comment.

The findings of the study suggest that these students, particularly those in Grades 6 and 8, may not have had sufficient opportunity to explore practically the spatial foundations of area and perimeter and the relationships between them. The author suggests structured classroom activities of a practical nature ... be more widely used to develop the notions of area and perimeter, and a delay in the presentation of area and perimeter formulae.

The premature introduction and reliance on the area formula for rectangles has been noted as a cause of many students' difficulties with the area concept by other researchers (Battista, Clements, Arnoff, Battista, & Borrow, 1998; Outhred & Mitchelmore, 2000).

In a major study designed to investigate a sequence of developmental levels for young children's strategies to find the number of unit squares that cover a rectangle, Outhred and Mitchelmore (2000) worked with children in Years 1 to 4 on a series of tasks which differed from each other in terms of whether the unit tile was movable and able to be placed onto the given rectangle (Task 1), or not (Task 2), and whether both the rectangle and the unit could be seen (Tasks 1 and 2 but not Task 3). Five strategy levels were found, ranging from incomplete coverings through more and more sophisticated physical arrays of iterated units to a final stage where the array was implied and the solution was found through calculation. The critical step seems to be the establishment of the importance of rows of iterated units. Next, the student needs to quantify the number of rows and then "a child is only a short step from being able to calculate the total number of units" (Outhred & Mitchelmore, 2000, p. 163). The study also highlighted

The importance of a good understanding of linear measurement, without which children are unlikely to learn the relation between unit size and rectangle dimensions. We have also shown that an understanding of multiplication is not necessary to an understanding of area measurement, although it is essential for the area formula. (Outhred & Mitchelmore, 2000, p. 165)

Outhred has used the results above, along with other research on linear and volume measurement to develop a framework for teaching early measurement (Outhred & McPhail, 2000). The framework consists of a theoretical summary, activities for measurement learning and example lesson plans. It was developed through an
intensive process of consultation and trialling with teachers. The framework is broken into three levels for each of length, area and volume: identification of the attribute; informal measurement and the structure of the iterated unit. This third step is seen a crucial, based on recent research (Battista et al., 1998; Bragg & Outhred, 2000; Outhred & Mitchelmore, 2000; Reynolds and Wheatley, 1996; 1997).

The importance of the row in the development of area measure is mirrored by the importance of the planar layer in the measurement of the volume of a rectangular prism (Battista, 1999; Battista & Clements, 1996; Nortvedt, 1999; Outhred & McPhail, 2000). For example, Nortvedt (1999) surveyed asked students in Grades 6 and 9 in Norway to write how they solved certain volume problems.

When calculating the number of cubes needed to build a three-dimensional array, the two most common approaches for successful children were to calculate the volume by a formulaic approach or building the volume in terms of layers. (Nortvedt, 1999, p. 305)

**Time**

The measurement of time is an interesting pedagogical area because of the relative abstraction of the time concept and the inability for direct physical comparisons to be made between the ‘length’ of time being measured and the units being used to measure it. Of course, other added difficulties might arise from the complexity of the system of time units. On the other hand, time is a pervasive part of most school students’ experience and time language is part of most of these students’ lives. Boulton-Lewis, Wilss, and Mutch (1997) have analysed primary school students’ abilities and strategies for reading and recording time using both common types of clocks—analogue and digital. For children in Years 1-3, they confirmed a sequence for the acquisition of these abilities. This sequence commences with time on the hour and progresses through half-hour, quarter hour, five minute and minute times. Times after the hour, particularly times such as 3:45, were found to be more difficult. Digital time was generally learned earlier as essentially it “is merely a matter of reading numerals separated by a colon. This ability does not necessarily indicate understanding of the relationships involved in digital notation” (Boulton-Lewis, Wilss, & Mutch, 1997, p. 137). The sequence confirmed for Years 1-3 was not confirmed for Years 4-6. The latter group of students showed many irregularities in their development due to greater individual differences in their abilities to read analogue times.

An interesting approach to the development of skills in reading analogue time is presented by Friederwitzer and Berman (1997). The authors describe

concrete time-teaching models that make connections to fractions and measurement. These models were used with second and third graders for several weeks. Previously taught concepts of whole, half, and quarter fractions are reinforced using fraction circles and relating them to the clock. Students literally “measure” time by using Cuisenaire rods on a form of number line to discover the meaning of the language used to describe the passing of time. (Friederwitzer & Berman, 1997, p. 255)

**Spatial and Geometric Thinking**

When discussing spatial concepts the terms space and geometry are often used. While space is often used in connection with primary-school mathematics and geometry with high-school mathematics, these two terms are inextricably linked, with the latter representing the more formal development and application of spatial
concepts. As such, mathematical content relating to understandings of two-dimensional (2D) and three-dimensional (3D) shapes and space (including angles); spatial concept development (including studies associated with the van Hiele theory); and understandings of formal geometry including proofs, are considered within the space strands of most national and/or state syllabus documents. Mathematical processes associated with visualisation, imagery and problem solving form strong relationships with the content areas of space and measurement. The role of imagery in the early stages of the problem-solving process and the extent to which diagrams and drawings assist in problem representation are key issues for all mathematics but are particularly important in space. They provide ways of integrating areas of mathematics and emphasising mathematical thinking and reasoning.

**Early Childhood**

Both Macmillan (1996) and Rogers (1999, 2000) have shown that young children develop mathematical knowledge through play. They will discuss qualitative and quantitative ideas in both space and number. They showed that early spatial and measurement understandings included (a) position language of degree, e.g., halfway, near; (b) shape and line names and classification characteristics e.g. "like a window", (c) enjoyment at seeing and making spatial patterns, and (d) turns and corners. In measurement, children’s actions illustrated informal measures and finding midpoints, and comments were on (a) strength and balance, size, comparisons decided by sight or direct matching (e.g., not equal, twice/half as big); (b) patterns of area; and (c) time and speed recognition.

Roger’s (1999) report included a diagrammatic representation of the interactions between problem solving, reasoning and communication and the mathematical processes of (a) visualisation in estimation, preconceived plan, projection and refinement; (b) experimentation in concrete problem solving, balanced, symmetrical, aesthetic structures, and (c) application in new structures, purpose, product—real and product—imagined. These interactions were illustrated by a vignette of the discussion between children on the building of a structure resembling a Greek temple. Not only was interaction between children important but adults’ modelling, acceptance, positive responses, and questioning helped students to feel confident, to cooperate, and to express mathematical ideas and purposes. Rogers concluded by claiming that block play, building and cleaning up activities prepare children for school mathematics. Further she says block play encourages children to learn through sharing of knowledge and skills with peers, experimenting, practising new discoveries or techniques, and applying what they had learned to different situations. Rogers also noted that, once the girls began to participate in block play, there was no gender differences in the themes for play given above, nor was there a difference in whether blocks were used in imaginative play or other play.

The analysis and presentation of the TIMSS results for Australia was carried out at the Australian Council for Educational Research (Lokan, Ford, & Greenwood, 1996a, 1996b). These studies indicated that 10% more Australian primary students (upper year was 74%, lower year was 65%) achieved correct answers to the geometry questions than the International average (upper year was 64%, lower year was 56%). Females did marginally, but not significantly, better than males in both years. On no item did Australian students do more poorly than the international average. There were some state differences with percentages correct ranging from 69% to 81%. The question on compass points showed a marked improvement between lower primary and upper primary. Only 40% of students in lower primary were able to give the correct number of edges of a drawn cube suggesting that students did it visually
rather than by knowing. Although space items were relatively well done compared to other items in middle primary school, they were a relative weakness at lower secondary level. Interestingly, a fairly low 59% of items were considered covered by the curriculum in lower primary. Most of the items that were not covered were in geometry as it was not emphasised adequately in curriculum documents (Lokan, et al., 1996a p. 136). Lower secondary school results showed that students were able to visualise a rotated 3D object without too much difficult. However, items on an unexpected 2D rotation, angles, angles and parallel lines, and congruent triangles were not well done. A study in the Philippines of Year 6 students in some relatively well-supplied schools across the country concluded that the geometry questions were relatively well done (Mecardo, 1999). The percentages correct for each item were generally at least 10% above the upper 25th international percentile except for three harder items. By contrast, items in other strands were lower than the international averages. This was explained in terms of the change of doing such items through spatial abilities rather than through mathematical development.

Visualisation

The use of imagery in space mathematics is now well documented as important. For example, Wheatley (1998) illustrates the development and use of imagery in space mathematics. Through imagery, it is known that different students will notice different things. "Imaging involves three activities, constructing an image, re-presenting the image, and transforming the image" (p. 66). The construction of the image may be a rigid mental picture which can limit reasoning. In future, the image may need to be re-presented when it is to be used. It is important that the imagery be dynamic if it is to assist with reasoning. Image schema are used in reasoning. Helping students to develop imagery can be achieved with such activities as asking them to recall what they see when they see a particular diagram and to share their various descriptions with a class. One such example is the square with a quarter square in a corner and an oblique line joining the further corner. The issue of assessing imaging is often bought up when it is not valued enough in mathematics. It is clearly observed in the way students react to manipulative tasks that are described or drawn as well as presented (Wheatley, 1998). In an analysis of results from NSW Years 3 and 6 students on Basic Skills items, Owens (1997) highlighted the diversity of visual imagery skills that students needed in answering the questions. Among these were the skills of transforming images, recognising shapes, analysing plans and recognising parts of shapes, and appreciating alternative perspectives. Some of the plan items were most difficult for students. Imagery seems to be affected by experiences, quality of images, interpreting of diagrams, remembering images, and assisting the understanding of images with verbal representations.

In a study on visual-spatial reasoning, Lowrie (1998) found that 7 and 9 year old children could construct spatial understandings associated with perspective and representation when using computer software that aimed to develop problem-solving skills. The activity required the children to construct a room on screen with the use of a series of furniture icons that could be moved into position in the virtual room. The children found it difficult to link three-dimensional space with images presented in simulation programs but discussion with adults seemed important if such programs are to encourage students to engage in mathematical thinking.

In a study designed to see whether preschoolers could think analytically about space, Feeney and Stiles (1996) have shown that by four and a half, children were able to distinguish wholes and parts of simple designs such as plus or cross signs. They could do this by construction (Tada & Stiles, 1996), by perception or just looking and
selecting from a picture (e.g., for a plus sign selecting from 2 long lines, a long and 2 short lines, and 4 short lines), and by drawing. There were correlations in results. Where inconsistencies occurred, students seemed to be in transition in mastering different media. For example, they may have been able to copy the designs as they were commonly used in early childhood but may not have worked with the design in the different format of the perceptual task. It was clear that students began this analytical thinking spatially earlier than four and a half years but by this age some students were able to select all possible drawings including that usually used by adults (2 long lines).

Young school children tend to focus on aspects of 3D objects that are not related to the 3D shape that they may represent (Everett, 1999). Nevertheless teaching programs are expected to show improvements in students’ 2D to 3D visualisation.

Symmetry difficulties were highlighted by Leikin, Berman, and Zaslavsky (2000) when student teachers prepared lessons for Year 8. The study showed how resistant the selection of a wrong line of symmetry was (students are confused between congruence and mirror image). The inclination of the line also caused a difficulty especially if a visual alternative was dominant, (e.g., part of a parallelogram in an oblique line) while most students did not consider a symmetry axis for a line. One particularly good idea was to define an isosceles triangle as a triangle with a line of symmetry. From that students can deduce many properties as they begin ideas on proof. Using technology to produce symmetric shapes can be valuable to engage students in thinking about symmetry (Seidel, 1998).

The research based on van Hiele and SOLO levels of development has indicated that the early levels of recognition and of early analysis needed to be modified (Pegg, 1997, 1998; Pegg & Davey, 1998). Basically, this recognised the importance of real world referents. When children first begin to reason geometrically, they use direct or indirect resemblance. Later they reason by attributes and then by properties (Fox, 2000; Lehrer, Fennema, Carpenter, & Ansell, 1994). Tasks of sorting and classifying are provided. In addition, it is clear that language and experiences of alternatives for a space concept were significant (Whitland & Pegg, 1999).

The earlier theories of Piaget and van Hiele have been critiqued by Battista and his colleagues (e.g., Clements, Swaminathan, Hannibal, & Sarama, 1999). Data from preschool and kindergarten children assist in suggesting that students show a prerecognitive level and early syncretic level in which they show some stronger imagistic prototypes and gradually gain verbal declarative knowledge. These two levels are identified by characteristics that are similar to those presented in the recent Count Me into Space project where the terms emerging strategies and perceptual strategies are used (Owens, 1999b). When students are using Emerging Strategies, they are beginning to attend purposefully to aspects of spatial experiences, to manipulate and explore shapes and space, to select shapes like ones shown or named, and to associate words with shapes and positions. Students using perceptual strategies are attending to spatial features and beginning to make comparisons, relying on what they can see or do.

Clements et al. (1999) used diagrams in individual assessments and asked students to select different shapes. The diagrams included shapes and near shapes. They started with a circle, with the rectangle being harder since there were more prototypical image possibilities. They also drew the shapes within shapes, not so much to see if students were disembedding but rather to provide a more complex item.
It is clear that early experiences with spatial concepts have been lacking in Australian schools (Currie & Pegg, 1997, 1998a, 1998b). That is, the predominance in naming shapes, often in standard form, is insufficient activity for students to explore the properties of shapes and to use them. Currie and Pegg classified students' cognitive differences. The issue of class inclusion is a logico-reasoning issue and needs to be addressed in different ways with younger students rather than considering it for the first time in high school. Furthermore, it is possible that primary school definitions are actually acting against this reasoning. For example, an isosceles triangle has two equal sides may encourage students to exclude it from the set of equilateral triangles. Often looking at an alternative definition, e.g. a line of symmetry definition may encourage better class inclusion.

A report to the Board of Studies, NSW by Pegg (1995) illustrates that a 1950s-1960s type syllabus would be ignoring the last 30 years' research on students learning geometry. In particular they point out that only 20% of Advanced students will be able to reason at a deductive level with 60% being able to deduce simple conclusions based on relationships from concrete or specific diagrammatic examples. Another 20% will only be able to provide different properties but without connections. The challenge is to prepare students so they can reach some of these goals but a curriculum from 1950s in less time and an outmoded approach to learning geometry will be doomed to fail. A set of detailed geometric statements that are to be covered by students will result in rote learning.

By taking results from 1985-1986 examinations when an older syllabus was used, Pegg (1995) showed that 70% or less of Advanced level students were unable to disembled angles and show two angels equal in triangles when they overlapped or overlapped other shapes (1985, Q. 6 in full response section and 1986, Q. 43). In the first question with a simple congruent triangle deduction (if the triangles could be visually isolated) less than 10% could do the question. In questions that linked to ratios of similar triangles, only 50 to 70% could get the correct solution and whenever letters were used, there was usually less than 50% correct for circular geometry questions and similar questions that would easily link to class exercises. It was not always the number of steps or the nature of the connections between properties that made these questions difficult for students but various visualisation problems. It is clear that establishing proportional reasoning, spatial thinking, and algebraic reasoning will assist students with geometry and geometry facts should not be seen as an end in themselves.

Since imagery and language seem to be two key aspects of spatial thinking, Owens (1999b; Owens, Leberne, & Harrison, 1999) has worked with teachers to develop a useful framework for teaching space mathematics in the early primary school. This work is based on Owens' earlier research (e.g., 1996b, 1998a) and the body of literature on visual imagery (e.g., Lowrie, 1997; Owens & Clements, 1998; Presmeg, 1986) and spatial abilities (from both a factor analytic approach and an information processing approach). The framework for early Space has now developed two broad strands: (a) Orientation and Motion; (b) Part-Whole Relationships; for which investigating and visualising strategies and describing and classifying are expected to develop. Initially students begin with Emerging Strategies as they begin to recognise shapes, orientation, and language. Then Perceptual Strategies develop in which physical representations are used to think spatially. Pictorial Imagery Strategies are then used by students who can work with an image when physical representations are not needed for decision-making. Pattern and Dynamic Imagery Strategies are expressed when students associate specific relationships between parts of shapes such as a grid and when students think dynamically by moving
images around or by internally changing shapes through extending, pulling, or folding parts. Efficient Strategies tend to develop as language assists recognition of properties of shapes, position relationships, and imagery associated with turning and folding experiences. Efficient Strategies involve reasoning and the beginnings of proof which develops through imagery as well as language. Significant developments occur as students move through the different kinds of strategies as a result of lessons that encourage investigative tactics such as turning, flipping and superimposing physical manipulatives, folding, and drawing. Language also intertwines to specify aspects of shape and position that are noticed and used by the learner in problem-solving situations (Owens, 1996a, 1996b). Students’ responses to the assessment tasks (Owens, 1999b) have been categorised according to strand and strategy. This framework extends notions of Space mathematics by considering conceptual development and imagery development in an analysis of content. The value of using manipulatives for investigating shapes for communication is well illustrated by Flores (1995) as he describes activities presented bilingually.

Studies of the development of angle concepts has been pursued by Mitchelmore and White after initial investigations using six pieces of apparatus which reflected contexts in which students are likely to become aware of angles (Mitchelmore, 1996; Mitchelmore & White, 1996). They developed a theoretical perspective in which angle contexts were the beginnings for students to abstract the concept of angles. Several tasks were given to Year 2 students from which it became clear that young children did have an initial contextual understanding of angles and that they were beginning to associate the different contexts into a general angle concept. (Mitchelmore, 1997). Children as young as Year 2 were able to say that a rotation of a small doll from forward to left was equal to that from right to forward but half of them did not interpret turning as an angle. The students could not match it with another angle representation, e.g. a tile corner. By Year 6, most students had developed a broad turning concept that included at least rotations and hinges like doors as subconcepts (Mitchelmore, 1997). Mitchelmore concluded that teaching about angles with clear lines, drawing these turns, and using static angles with two distinct lines will assist the early development of angles.

Mitchelmore and White (1998) noted that students frequently experienced numerous distinct angle situations prior to school. For example, students understood it was harder to walk up a steep hill than a slight hill. The researchers developed a theoretical position and argued that these situated angle concepts developed into contextual angle concepts such as hills and cranes both providing examples of slopes. In the next stage these angle contexts were linked to form an abstract angle concept. It remains to be seen whether young primary students can learn more about angles and whether teachers can build on their intuitive experiences. Owens’ (1996a, 1998a) study illustrates how students were learning in such contexts. The teaching focused on the context of angles on shapes and the forefinger and thumb were turned apart to mark the arms of the angle (this is not quite the same turn as Mitchelmore & White were using with the doll turning and these latter researchers would refer to this as an opening). The larger angle was marked by a larger turn or the overlaying of the angles exposing a larger region. Sticks were also used to represent angles as part of the outline of a shape. Not all students in the classroom were able to notice or disembed the angles but a number of Year 2 and 4 students were able to make these simple links between turning and fixed arm positions on a shape, especially when the teacher was able to hold conversations with groups of students. The focus of students’ attention was a key to learning about the angles and to distinguishing size of angles from the size of sides or the overall shape.
Similarity is a particularly interesting topic in space mathematics as it links with proportional reasoning. It is a particularly useful topic for students to investigate geometric relationships, especially from the environment. The notion of fixing a triangle will lead to numerous constructions of mathematics if they are given the opportunity (Slavit, 1998).

Much has already been said about geometry and proof in the section on mathematical reasoning. Glenda Lappan (1999) argues strongly for time to be spent on geometry:

All this wonderful mix of identifying and describing, moving and transforming geometric objects is a part of geometry right along with argument and proof. ...Activities such as examining enlargements ... will help students explore important geometric ideas. For instance, are two geometric objects congruent? Are they similar? If so, why? If not, why not? If we know certain properties of a geometric object, what else can we infer? What happens if we take a line or curve and rotate it in space? What sort of object does it sweep out as it rotates? Can we describe the curve and the object it sweeps out in mathematical terms? With questions like these, not only are we exploring geometry, we are connecting to other areas and levels of mathematics. ... We have a beautiful, important strand of mathematics that has too often not been given appropriate attention in the curriculum. (Lappan, 1999, p. 3)

Data Sense and Probability Sense

**Data sense**

In response to the critical role that data plays in our technological society, international pressure for reform in statistical education at all year levels has been mounting (Australian Education Council, 1994; Department for Education and Employment and Qualifications and Curriculum Authority, 1999; Lajoie & Romberg, 1998; NCTM, 1998, 2000; 1999). Scheaffer, 2000). These calls for reform have stimulated research on statistical thinking, especially in the primary school, where, to date, there has been a tendency to focus merely on graphing rather than on broader aspects of data handling and analysis (Lajoie & Romberg, 1998; Shaughnessy, 1992; Shaughnessy, Garfield, & Greer, 1996). Some elements of students' statistical learning have been investigated in areas such as data organisation (Mokros & Russell, 1995), data modelling (Lehrer & Romberg, 1996) and graph comprehension (Curcio, 1987; Friel, Bright, & Curcio, 1997). Jones and his international group based at Illinois State University (Jones, Thornton, Langrall, Mooney, Perry, & Putt, 2000; Perry, Putt, Jones, Thornton, Langrall, & Mooney, 2000; Putt, Perry, Jones, Thornton, Langrall, & Mooney, 2000) have developed a framework of students' statistical thinking that could be used to inform instruction. An extensive amount of work has been done by Watson and her team at the University of Tasmania which again highlights the value of introducing statistical thinking and data handling into the school mathematics curriculum at an early stage (Chick & Watson, 1998; Moritz, 1999; 2000; Watson & Moritz, 2000). Another approach to the question of students' statistical thinking is represented by the group from Vanderbilt University led by Paul Cobb and Kay McClain (McClain, Cobb, & Gravemeijer, 2000; McGatha, Cobb, & McClain, 1999). Working with middle school students, and using computer tools to assist in the students' analyses, they have helped the students develop deep understandings of important statistical concepts and to have them make data-based arguments.
It was for this reason that whole-class discussions throughout the classroom teaching experiment focused on the ways in which students organized data in order to develop arguments. In addition, students seemed to reconceptualize their understanding of what it means to know and do statistics as they compared and contrasted solutions. The crucial norm that became established was that of explaining and justifying solutions in the context of the problem being explored. This is a radically different approach to statistics than is typically introduced in middle schools. It highlights the importance of middle school curricula that allow students to engage in genuine problem solving that supports the development of central mathematical concepts. (McClain, et al., 2000, p. 186)

There seems to be general agreement in various national documents that students should be participants in the exploration of data, its representation, interpretation and analysis. For example, Shaughnessy, Garfield and Greer (1996, p. 208) make the point that

students should:
- collect, organise, and describe data;
- construct, read and interpret displays of data;
- explore chance and random phenomena;
- formulate and solve problems that involve collecting and analysing data;
- describe and interpret data;
- create visual and graphical representations of data; and
- develop a critical attitude towards data.

In their *Principles and Standards* document (NCTM, 2000), the National Council of Teachers of Mathematics stresses the growing importance of data analysis and the need for students at all levels of schooling to be actively involved in working with data.

The increased curricular emphasis on data analysis ... is intended to span the grades rather than to be reserved for the middle grades and secondary school ... To understand the fundamentals of statistical ideas, students must work directly with data. ... The data and statistics strand allows teachers and students to make a number of important connections among ideas and procedures from number, algebra, measurement, and geometry. Work in data analysis and probability offers a natural way for students to connect mathematics with other school subjects and with experiences in their daily lives. (NCTM, 2000, p. 48)

Students should be encouraged to collect, organise and display data which is relevant to them and should be encouraged to pose and answer questions which arise or might arise from the data and to make inferences, interpretations and predictions. Such activities incorporate what Curcio (1987, p. 387) coined as “reading between the data” and “reading beyond the data”. While there is ample evidence that many students may have conceptual misconceptions about some statistical ideas (Ben-Zvi, 1999; Cai, Moyer, & Grochowski, 1997; Mokros & Russell, 1995; Watson & Moritz, 2000), there is also a great deal of evidence that, under appropriate instructional approaches, these misconceptions can be avoided or, at least, alleviated (Garfield & Gal, 1999; Jones et al., 2000; Whitin & Whitin, 1999). Much of the Australian work in this area has relied upon analyses of student learning outcomes using the SOLO taxonomy (Biggs & Collis, 1982) and has categorised student responses to innovative questions according to this taxonomy. For example, Chick & Watson (1998) worked with their ‘data cards protocol’ in Years 5 and 6 to help student develop their skills in interpreting and representing data (see also, Watson, Collis, Callingham, & Moritz, 1995) while Moritz (1999, 2000) has explored students’ graphing and interpretation of statistical associations. In all of these studies, the SOLO taxonomy has proved a very useful tool for the analysis and has allowed
positive comment to be made about what the students could do and what they are likely to be able to do in future lessons.

Also working from the basis of the SOLO taxonomy as one of many possible explanatory tool, Jones and his team (Jones, et al., 1998; 2000; Perry, Jones, Thornton & Langrall, 1997; Perry et al., 2000; Putt et al., 2000) have devised, trialed, adjusted, and validated a statistical thinking framework which has then been used in teaching experiments to guide instruction and assess statistical thinking in primary school children. The results, with both US and Australian data available, suggest that this framework could prove very useful in guiding curriculum development and could possibly be extended to consider the statistical thinking of older students. The framework is described as follows.

The framework incorporates four key data handling processes: describing, organizing and reducing, representing, and analyzing and interpreting data. Describing data involves explicit reading of data contained in a visual display and recognition of graphical conventions. Organizing and reducing data incorporates mental actions such as ordering, grouping and summarizing data using measures of center and spread. Representing data involves the construction of visual displays that exhibit different organizations of data. Analyzing and interpreting data includes “reading between the data” and “reading beyond the data”. For each of these processes, four levels of thinking were hypothesized and validated. Level 1 is associated with idiosyncratic thinking, Level 2 is transitional between idiosyncratic and quantitative thinking about data, Level 3 involves the use of informal quantitative thinking, and Level 4 incorporates analytical and numerical thinking about data. These levels of thinking are consistent with neo-Piagetian theories that postulate the existence of levels of thinking that recycle during developmental stages. (Jones et al., 2000, p. 97)

The National Council of Teachers of Mathematics have made a number of suggestions as to what sorts of data experiences students might have as they pass through their compulsory schooling. For example,

By grades 3-5, students should be developing an understanding of aggregated data. … Statistics such as measures of center or location (e.g., mean, median, mode), measure of spread or dispersion (range, standard deviation), and attributes of the shape of the data become useful to students as descriptors. … Throughout the school years, students should learn what it means to make valid statistical comparisons. In the elementary grades, students might say that one group has more or less of some attribute than another. By the middle grades, students need to be quantifying these differences by comparing specific statistics. Beginning in grades 3-5 and continuing in the middle grades, the emphasis should shift from analyzing and describing one set of data to comparing two or more sets. As they move through the middle grades into high school, students will need new tools, including histograms, stem-and-leaf plots, box plots, and scatterplots, to identify similarities and differences among data sets. Students also need tools to investigate association and trends in bivariate data, including scatterplots and fitted lines in grades 6-8 and residuals and correlation in grades 9-12. (NCTM, 2000, p. 50)

Another example of the statistical and data experiences students might have during their school years is given by the Western Australian Curriculum Framework.

**Early childhood phase (typically K-3)**

Children should be assisted to classify consistently and to classify things in different ways. … They should record and represent data which arise from practical and problem-solving activities in all of the mathematics strands and all learning areas. Children should begin to construct graphs based mostly on one-to-one correspondence.
Middle childhood years (typically Years 3-7)
Children should collect, represent and interpret data in order to answer questions of interest to them. ... In conducting surveys, children should work collaboratively to clarify what type of information they need to collect ... and what people, things or events are to be surveyed. This should include drafting questions, testing different versions of them for clarity and bias, and redrafting to improve their usefulness. Children should be beginning to learn that data can be classified, organised, summarised and displayed in a variety of ways and that the choices depend upon the questions being asked of the data, the type of data and the audience. ... During these years, children should begin to consider whether it is reasonable to generalise from their data.

Early adolescence (typically Years 7-10)
Practical investigations should be undertaken which involve all of the facets of data handling. ... The activities should include careful consideration of procedures for choosing samples and designing and trialing questionnaires, the comparative advantages of different methods of organising and representing data ... and the difficulties which arise at the interpretation stage. ... Students should be learning to interpret various representations of data including means, measures of variability and association, line plots, histograms, stem-and-leaf plots, box plots, scatter plots, and lines of best fit; understand the conditions under which their use is appropriate; and compare and select from different possible representations of the same data. Calculators are a necessary tool in this. Students should be gaining experience which will, over time, enable them to distinguish between a population and a sample, informally draw inferences from data collected by themselves and others, construct convincing arguments based on such data, and evaluate arguments. (Western Australian Curriculum Council, 2000)

From yet another point of view, Garfield and Gal (1999, p. 218) list the following recommendations concerning the development of statistical reasoning.

1. Provide students with opportunities to work with real data, either solving interesting problems or posing problems of their own that involve going through the steps of a statistical investigation. Have students make decisions about data collection, coding and analysis. Students should justify their solutions.

2. Provide students with practice articulating their reasoning by including written or oral communication as a regular part of statistical problem solving. ...

3. Encourage students to become aware of their thinking and reasoning by having them discuss and compare different solutions to statistical problems and their interpretations, assumptions, and explanations of those problems.

4. Provide students with opportunities to use technology to manage and explore data, so that they can focus more on the reasoning and less on the calculations and constructions. ...

5. Introduce software that helps students develop and support their statistical reasoning. ...

6. Allow students to make predictions and to test them so that they may become aware of and confront misconceptions and faulty reasoning. ...

7. Build on students’ prior knowledge or real-world knowledge, so that they are able to construct appropriate relationships with this knowledge as they extend it and apply it to new situations in order to develop good statistical understanding.

Technology has a central place when students are working with data. At the primary level, computer technology has been used successfully to help introduce statistical ideas of mean, mode and spread (Perry et al, 1997, 1999; Putt et al, 1999) while, with older students, both computers and calculators have proven useful (Hancock, Kaput & Goldsmith, 1992; Kissane, 1998; McGatha, et al., 1999; Wilson & Jones,
1998). Of course, consideration must be given as to the most appropriate ways to use technology in the development of students’ statistical thinking. Citing Garfield (1990), Shaughnessy et al., 1996, p. 217) suggest that the attributes of technological environments that facilitate the learning of data handling would include:

- **Direct access**, which is the capability that allows students to view and explore data in different forms, whether subsets of data or different visual representations.
- **Flexibility**, which allows students to experiment with and alter displays of data, change intervals on a graph, and explore different models that may fit the data.
- **Connectedness**, which allows students to access networks, resources on the Internet, writing packages, and software or data used in the study of other disciplines.
- **Representations**, including dynamic ones from which students may choose between different graphs in order to select the best way to interpret and display a data set.

Technology which allows students to explore data at their own level of understanding and which helps show them how useful such exploration has the potential to facilitate their statistics understandings. However, as always, there is a danger that while the technology might reduce the drudgery involved with the physical creation of representations, it may exacerbate the difficulties students have in choosing appropriate representations, simply because it is so much easier than before to produce several representations. As always, students need to learn how to use the technology to meet their needs.

An important issue which is often overlooked in the development of data handling / statistical ideas lies in the affective domain. Shaughnessy et al. (1996, p. 206) have summarised this in the following way:

The beliefs and attitudes lying behind data are just as important to include in a treatment of data handling as are the methods of organising and analysing the data. Bias, attempts to disguise data, attempts to mislead with data, attempts to present the evidence from one point of view—all such misuses and abuses of statistics are an important part of the data-handling experience for students. If students are to interpret data presented to them in a myriad of different ways, they need to be aware of the possibilities of ‘filtering’ data according to certain biases or concerns. Students need to be able to evaluate the quality of the data gathering approaches, presentations and analyses in order to determine the relative validity of the data on which they are working.

The importance of data sense—being able to collect, describe, organise, represent, analyse, interpret, and infer from data in a flexible, efficient and accurate manner and to understand the uncertainty which surrounds particular types of data—has never been so great as it is today. Much of the information provided to society is provided in terms of data displays or interpretations. Inferences are made from data to ‘prove’ certain points of view. All of our citizens, including our students from K-10, need to be critically literate with data so that they can get the best from their lives without danger of missing opportunities because of misconceptions about statistical ideas.

**Probability sense**

> From an early age, children encounter the language of chance and begin to realise that there is an element of chance in many of the experiences they have in their lives. As they expand their experiences, children begin to quantify chance through comparison—rain is ‘likely’ or ‘unlikely’ today; it is ‘more likely’ that something will happen than not; I am ‘less likely’ to get a black Smartie than a red one from the
packet. Later still, children can order the likelihood of more than two events and, still later, attach numerical values to these likelihoods—I have a 50% chance of winning this race. All of this can—and often does—happen outside of mathematics classrooms. What can, or should happen inside these classrooms to help develop these ideas and to ensure that students are able to handle the probabilistic thinking demands of modern society and the other curriculum areas in which they are learning concurrently?

There is much evidence to suggest that students in secondary schools have difficulty in displaying probabilistic thinking—“an approach towards viewing reality in terms of uncertainty” (Borovcnik & Peard, 1996, p. 240)—at levels that might be expected (Linsell, 1997; Truran, 1995, 1998; Watson & Moritz, 1998, 2000). One possible reaction to this is to reduce the level of probabilistic thinking required of these students; another is to consider the introduction of structured elementary probability ideas into the primary school years. The former route denies the necessity for students, as they grow into adolescence, to understand issues of probability and chance and the consequences of intuitive decision making in their everyday lives, particularly in a country such as Australia. Perhaps because of this, the latter route is one which has been taken up in many other states of Australia and other countries of the world. “The kind of reasoning used in probability and statistics is not always intuitive, and so students will not necessarily develop it if it is not included in the [school] curriculum” (NCTM, 2000, p. 48). This is further reinforced by Bright and Hoeffner (1993, p. 87):

Unlike many other mathematics topics, probability instruction must compete with possibly strongly held intuitive beliefs and strategies that may be inconsistent with the instruction. ... students need to be asked to explain their reasoning so that intuitions are openly discussed.

Probability and data handling both provide marvellous opportunities for students and teachers to integrate mathematical ideas in other learning areas. Questions such as “Who is most likely to win the next election?” and “What coloured eyes are the most likely in the next student to walk into the classroom?” show how straightforward it is to use probability language and thinking in areas other than mathematics. Of course, there are numerous connections between probability thinking and knowledge and other areas of mathematics, especially number and measurement. It is difficult to talk about the quantification of probability without knowledge and understanding about fractions, decimals and percentages, although not all of these are needed all of the time. In fact, the discussion of probabilities has been shown to have some benefits in the development in these other mathematical areas.

So, what is it in probability that should be taught and when should it be taught during the K-10 Years? Jones and his colleagues (Jones, Langrall et al., 1999; Jones, Thornton, Langrall, & Tarr, 1999) have identified six key concepts through a series of teaching experiments with primary and middle school students. These concepts are: sample space; experimental probability of an event; theoretical probability of an event; probability comparisons; conditional probability and independence. Using these concepts, a framework for probabilistic thinking has been developed and validated (Jones, Langrall, Thornton, & Mogill, 1997; Tarr & Jones, 1997). This framework provides a structure around which teachers can plan, implement and assess their students’ probabilistic reasoning. It provides an outline for possible learning trajectories in probabilistic reasoning for students in the elementary and middle school and beyond.
We believe that probabilistic reasoning has a special place within the broader picture of mathematical reasoning because it involves reasoning that is associated with a context of uncertainty. Developing reasoning within such contexts presents special challenges to teachers because students’ probabilistic reasoning is diverse, idiosyncratic, and even subject to a certain mystique. However, within these variations there is a pattern of growth in probabilistic reasoning. … Moreover, we believe that the pattern of growth described and illustrated by our probabilistic reasoning framework can be used by teachers to construct appropriate probability tasks, to monitor and assess students’ reasoning, and to adapt instruction accordingly. (Jones, Thornton et al., 1999, p. 155)

The National Council of Teachers of Mathematics has suggested the following sequence of learnings and activities for students in the area of probability.

In prekindergarten through grade 2, the treatment of probability ideas should be informal. Teachers should build on children’s developing vocabulary to introduce and highlight probability notions. Young children can begin building an understanding of chance and randomness by doing experiments with concrete objects … In grades 3-5 students can consider ideas of chance through experiments … with known theoretical outcomes or through designating familiar events as impossible, unlikely, likely, or certain. Middle-grades students should learn and use appropriate terminology and should be able to compute probabilities for simple compound events … In high school, students should compute probabilities of compound events and understand conditional and independent events. Through the grades, students should be able to move from situations for which the probability of an event can readily be determined to situations in which sampling and simulations help them quantify the likelihood of an uncertain outcome. (NCTM, 2000, p. 51)

Another example of the probability experiences students might have during their school years is given by the Western Australian Curriculum Framework.

**Early childhood phase (typically K-3)**
Young children should investigate actions and events which involve unpredictability and refine their use of everyday language of chance such as “will” – “won’t” – “might” – “possible-impossible” – “certain” – “uncertain”. They should carry out experiments which involve chance processes … and examine the outcomes with discussions and predictions referring to the range of possibilities.

**Middle childhood years (typically Years 3-7)**
[Children] should plan and carry out investigations which involve chance processes … and discuss and compare the results of their experiments, noting that different results are likely in repeats. In doing so, they should be learning to order familiar and imaginary events informally from most likely to least likely.

**Early adolescence (typically Years 7-10)**
Students should learn to estimate probabilities experimentally and through the analysis of simple sample spaces. They should work collaboratively and with teacher support to set up and carry out simulations. (Western Australian Curriculum Council, 2000)

In the Northern Territory, the Curriculum Framework is in its pilot phase. However, it is instructive to see how this framework intends to provide scope and sequence for chance and probability ideas.

**Key Growth Point 3 (approximately Year 1)**
Learners use or respond to everyday language of chance such as ‘might’, ‘won’t’.

**Band 1 (approximately Year 2)**
Learners use and interpret everyday language of chance such as possible, impossible, maybe, likely, unlikely and classify familiar events as either likely or unlikely.

**Band 2 (approximately Year 4)**
Learners classify events as having an equal or more or less likelihood of happening.

**Band 3 (approximately Year 6)**
- Learners use quantitative data to make judgements about chance.
- Learners conduct investigations to produce data about chance events.

**Band 4 (approximately Year 8)**
- Learners explain that we describe events that cannot occur as having a probability of 0, events that are certain as having a probability of 1 and events that may happen as having a probability of between 0 and 1, depending on how likely they are to occur.
- Learners list all possible outcomes for the experiment.
- Learners use the results of a simple experiment to predict the results of a repetition of the experiment.
- Learners design a device to fit specified probabilities.
- Learners use fractions to assign probabilities.

**Band 5 (approximately Year 10)**
- Learners use published data to assign probabilities to events.
- Learners use census data to assign probabilities.
- Learners represent sample space using set notation, tree diagrams, Venn diagrams or lattice diagrams.
- Learners investigate independent events.
- Learners interpret ‘and’, ‘or’ and ‘not’ when used to describe events. (Northern Territory Department of Education, 2001)

It is clear from the above discussion that there is a place for the development of chance and probabilistic thinking in K-10. Not only does such a place recognise the fact that young children deal with chance language and activities from an early age in their everyday lives, it also recognises the importance of an understanding of probabilistic thinking in the development of many other skills in many other learning areas. For example, Johnson, Jones, Thornton, Langrall & Rous (1998) have shown how Year 5 students’ probabilistic thinking and writing levels both increased as a result of a program of writing instigated during probability lessons. The two activities of probabilistic thinking and writing seemed to develop a reflexive pairing in which development of one depended on and assisted in the development of the other.

While there is ample evidence of student misconceptions in the probability are, it would seem that these point to a need for further careful introduction of the probability ideas and reasoning rather than resulting in the deletion of probability from the curriculum, even in the early years. The links which are able to be made between this learning area, others areas within mathematics and other learning areas outside mathematics make it a prime contender for an integrating force in students’ development. Probabilistic thinking and ‘sense’ are, indeed major curriculum considerations for students who will be citizens of the twenty-first century.
Conclusion

This literature review has addressed the terms of reference supplied by the Board of Studies, NSW. It is not a complete review and the authors doubt that it could ever have been so, given the activity in mathematics education across the world. However, it is reasonably comprehensive and will, hopefully, be of value to the Board as it continues towards a new syllabus for mathematics K-10. The authors look forward to being a continuing part of that development and certainly look forward to its culmination. We commend the review and its recommendations to the Board.