BOARD OF STUDIES
NEW SOUTH WALES

2013 HSC Mathematics Extension 2
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	B
2	A
3	D
4	A
5	B
6	D
7	C
8	C
9	B
10	A

Section II

Question 11 (a) (i)

Criteria	Marks
• Correct answer	1

Sample answer:

$$\begin{aligned} z + \bar{w} &= \left(2 - i\sqrt{3}\right) + \left(1 - i\sqrt{3}\right) \\ &= 3 - 2\sqrt{3}i \end{aligned}$$

Question 11 (a) (ii)

Criteria	Marks
• Correct solution	2
• Finds modulus or argument	1

Sample answer:

$$|w| = \sqrt{1+3} = 2$$

$$\arg w = \tan^{-1}\frac{\sqrt{3}}{1} = \frac{\pi}{3}$$

Hence $w = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ in modulus–argument form.

Question 11 (a) (iii)

Criteria	Marks
• Correct solution	2
• Attempts to use de Moivre's theorem	1

Sample answer:

By de Moivre's theorem and part (ii)

$$\begin{aligned} w^{24} &= 2^{24}\left(\cos\left(24\cdot\frac{\pi}{3}\right) + i\sin\left(24\cdot\frac{\pi}{3}\right)\right) \\ &= 2^{24}\left(\cos 8\pi + i\sin 8\pi\right) \\ &= 2^{24} \end{aligned}$$

Question 11 (b)

Criteria	Marks
• Correct solution	2
• Finds value of one of A, B or C	1

Sample answer:

$$\frac{x^2+8x+11}{(x-3)(x^2+2)} = \frac{A}{x-3} + \frac{Bx+C}{x^2+2}$$
$$= \frac{A(x^2+2)+(Bx+C)(x-3)}{(x-3)(x^2+2)}$$

We need: $x^2+8x+11 = A(x^2+2)+(Bx+C)(x-3)$

Make particular choices for x and substitute:

$x=3:$ $\quad 44 = 11A$

$\quad A = 4$

$x=0:$ $\quad 11 = 2A - 3C$

$\quad = 8 - 3C$

$\quad C = -1$

$x=1:$ $\quad 20 = 3A - 2B - 2C$

$\quad 20 = 12 - 2B + 2$

$\quad B = -3$

Hence $A = 4$, $B = -3$ and $C = -1$

Question 11 (c)

Criteria	Marks
• Correct factorisation	2
• Attempts to use formula to solve a quadratic equation, or equivalent merit	1

Sample answer:

Find the zeros by using the quadratic formula:

$$z = \frac{1}{2}\left(-4i \pm \sqrt{(4i)^2 - 4.5.1}\right)$$
$$= \frac{1}{2}\left(-4i \pm \sqrt{-36}\right)$$
$$= -2i \pm 3i$$

The zeros are $-5i$ and i and therefore

$z^2 + 4iz + 5 = (z+5i)(z-i)$

Question 11 (d)

Criteria	Marks
• Correct solution	3
• Correct primitive, or equivalent merit	2
• Uses a suitable substitution to get an indefinite integral, or equivalent merit	1

Sample answer:

Use the substitution

$$x = \sin\theta$$

$$\frac{dx}{d\theta} = \cos\theta$$

When $x = 0,\ \sin\theta = 0,$ then $\theta = 0$

When $x = 1,\ \sin\theta = 1,$ then $\theta = \frac{\pi}{2}$

Hence

$$\int_0^1 x^3\sqrt{1-x^2}\,dx = \int_0^{\frac{\pi}{2}} \sin^3\theta\sqrt{1-\sin^2\theta}.\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \sin^3\theta\cos^2\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta\left(1-\cos^2\theta\right)\sin\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \cos^2\theta\sin\theta - \cos^4\theta\sin\theta\,d\theta$$

$$= \left[-\frac{\cos^3\theta}{3} + \frac{\cos^5\theta}{5}\right]_0^{\frac{\pi}{2}}$$

$$= 0 - \left[-\frac{1}{3} + \frac{1}{5}\right]$$

$$= \frac{2}{15}$$

Alternative: Use the substitution

$$u = 1 - x^2$$

$$\frac{du}{dx} = -2x$$

Where $x = 0,\ u = 1$

When $x = 1,\ u = 0$

Hence $\displaystyle\int_0^1 x^3\sqrt{1+x^2}\,dx = -\frac{1}{2}\int_1^0 (1-u)\sqrt{u}\,du$

$$= \frac{1}{2}\int_0^1 \left(u^{\frac{1}{2}} - u^{\frac{3}{2}}\right)du$$

$$= \frac{1}{2}\left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1$$

$$= \frac{2}{15}$$

Question 11 (e)

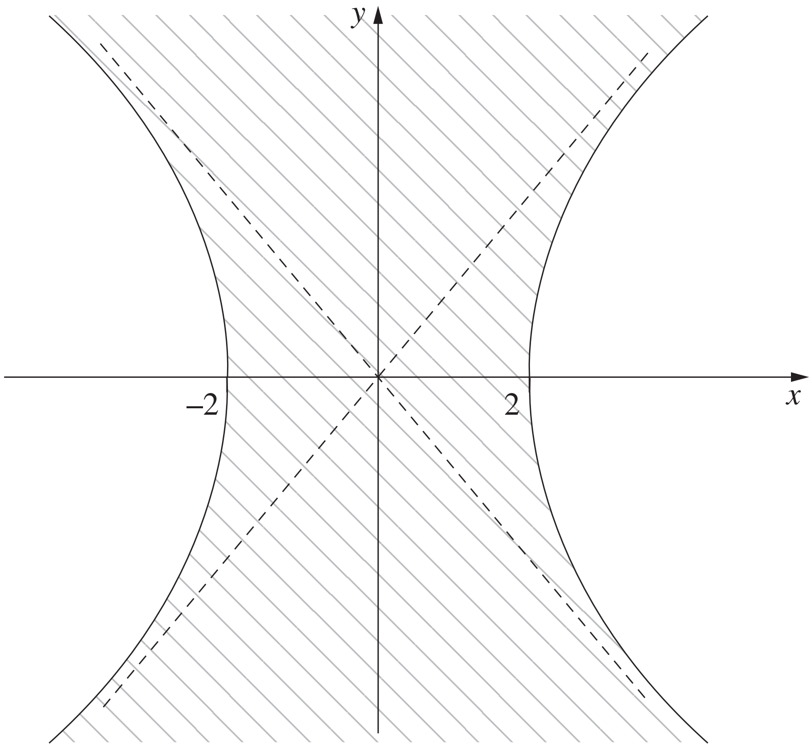
Criteria	Marks
• Correct sketch	3
• Sketches the correct boundary	2
• Obtains $x^2 - y^2 \le 4$, or equivalent merit	1

Sample answer:

$z = x + iy \quad (x, y \text{ real})$

$$\begin{aligned} z^2 + \overline{z}^2 &= (x+iy)^2 + (x-iy)^2 \\ &= x^2 + 2ixy - y^2 + x^2 - 2ixy - y^2 \\ &= 2x^2 - 2y^2 \\ &= 2(x^2 - y^2) \end{aligned}$$

Hence $2(x^2 - y^2) \le 8$, so $\dfrac{x^2}{4} - \dfrac{y^2}{4} \le 1$.



Question 12 (a)

Criteria	Marks
• Correct solution	4
• Makes significant progress towards finding a primitive in terms of *t*, or equivalent merit	3
• Makes complete substitution including the limits, or equivalent merit	2
• Correctly substitutes for $\cos x$ and attempts to write *dx* in terms of *t*, or equivalent merit	1

Sample answer:

First use the t-substition, $t = \tan\frac{x}{2}$

Then $dx = \frac{2}{1+t^2}dt$ and $\cos x = \frac{1-t^2}{1+t^2}$

When $x = 0$, $t = \tan 0 = 0$ and

when $x = \frac{\pi}{2}$, $t = \tan\frac{\pi}{4} = 1$

Hence $\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}dx$

$$= \int_0^1 \frac{1}{\left(4+5\frac{1-t^2}{1+t^2}\right)} \times \frac{2}{1+t^2}dt$$

$$= \int_0^1 \frac{2}{4\left(1+t^2\right)+5\left(1-t^2\right)}dt$$

$$= \int_0^1 \frac{2}{9-t^2}dt$$

$$= \int_0^1 \frac{2}{(3-t)(3+t)}dt$$

Next, do partial fraction:

$$\frac{2}{(3-t)(3+t)} = \frac{A}{3-t} + \frac{B}{3+t}$$

$$= \frac{A(3+t)+B(3-t)}{(3-t)(3+t)}$$

$$\therefore 2 = A\left(3+t\right)+B\left(3-t\right)$$

Choose $t = 3$, so $2 = 6A$ and $A = \frac{1}{3}$

Choose $t = -3$, so $2 = 6B$ and $B = \frac{1}{3}$

Hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{4+5\cos x}dx = \frac{1}{3}\int_0^1 \left(\frac{1}{3-t}+\frac{1}{3+t}\right)dt$$

$$= \frac{1}{3}\left[-\ln(3-t)+\ln(3+t)\right]_0^1$$

$$= \frac{1}{3}\left[\ln\frac{3+t}{3-t}\right]_0^1$$

$$= \frac{1}{3}\ln 2 - \frac{1}{3}\ln 1$$

$$= \frac{1}{3}\ln 2$$

Question 12 (b)

Criteria	Marks
• Correct solution	2
• Correct implicit differentiation, or equivalent merit	1

Sample answer:

Differentiate both sides of the given identity implicitly with respect to x:

$$\frac{1}{y}\frac{dy}{dx} + \frac{1}{1000-y}\frac{dy}{dx} = \frac{1}{50}$$

$$\left(\frac{1}{y} + \frac{1}{1000-y}\right)\frac{dy}{dx} = \frac{1}{50}$$

$$\frac{1000 - \cancel{y} + \cancel{y}}{y(1000-y)}\frac{dy}{dx} = \frac{1}{50}$$

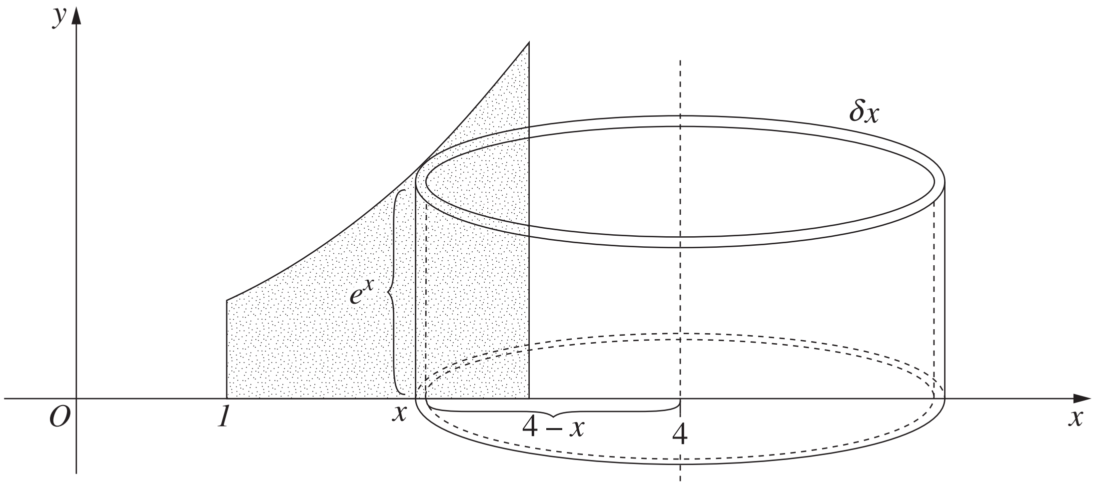
$$\frac{dy}{dx} = \frac{1}{50}y\left(\frac{1000-y}{1000}\right)$$

$$= \frac{1}{50}y\left(1 - \frac{y}{1000}\right)$$

Question 12 (c)

Criteria	Marks
• Correct solution	4
• Makes substantial progress towards evaluating the integral	3
• Correct integral for volume, or equivalent merit	2
• Attempts to use cylindrical shells, or equivalent merit	1

Sample answer:



Volume of thin cylindrical shell: $2\pi(4-x)e^x\delta x$.

Hence the volume V of the solid formed is

$$V = \int_1^3 2\pi(4-x)e^x\,dx$$

$$= 2\pi\int_1^3 \left(4e^x - xe^x\right)dx$$

$$= 2\pi\left[4e^x\right]_1^3 - 2\pi\int_1^3 xe^x\,dx$$

$$= 8\pi\left(e^3 - e\right) - 2\pi\int_1^3 xe^x\,dx$$

Using integration by parts:

$$\int_1^3 xe^x\,dx = \left[xe^x\right]_1^3 - \int_1^3 e^x\,dx$$

$$= \left[xe^x - e^x\right]_1^3$$

$$= \left(3e^3 - e^3\right) - (e - e)$$

$$= 3e^3 - e^3$$

$$= 2e^3$$

Hence $V = 8\pi\left(e^3 - e\right) - 2\pi\left(2e^3\right)$

$$= 8\pi\left(e^3 - e\right) - 4\pi e^3$$

$$= 4\pi e^3 - 8\pi e$$

$\therefore$ Volume is $4\pi e\left(e^2 - 2\right)\text{units}^3$

Question 12 (d) (i)

Criteria	Marks
• Correct solution	2
• Finds correct gradient, or equivalent merit	1

Sample answer:

$xy = c^2$, so $y = \dfrac{c^2}{x}$

Now $\dfrac{dy}{dx} = -\dfrac{c^2}{x^2}$

Hence the slope of the tangent at $\left(cp, \dfrac{c}{p}\right)$ is

$-\dfrac{c^2}{c^2 p^2} = -\dfrac{1}{p^2}$

The equation of the tangent is

$$y - \frac{c}{p} = -\frac{1}{p^2}\left(x - cp\right)$$
$$p^2 y - cp = -x + cp$$
$$x + p^2 y = 2cp$$

Question 12 (d) (ii)

Criteria	Marks
• Correct solution	2
• Finds the coordinates of *A* and *B*, or equivalent merit	1

Sample answer:

Find coordinates of *A* and *B* using part (i).

$A: y = 0,\ \text{so } x = 2cp \Rightarrow A(cp, 0)$

$B: x = 0,\ \text{so } p^2 y = 2cp$

$$y = \frac{2c}{p} \Rightarrow B\left(0, \frac{2c}{p}\right)$$

Now find distances from *P*:

$$PO: \sqrt{(cp)^2 + \left(\frac{c}{p}\right)^2} = c\sqrt{p^2 + \frac{1}{p^2}}$$

$$PA: \sqrt{(2cp - cp)^2 + \left(\frac{c}{p}\right)^2} = c\sqrt{p^2 + \frac{1}{p^2}}$$

$$PB: \sqrt{(cp)^2 + \left(\frac{2c}{p} - \frac{c}{p}\right)^2} = c\sqrt{p^2 + \frac{1}{p^2}}$$

All distances are equal, so *A*, *B*, *O* are on a circle with centre *O*.

Question 12 (d) (iii)

Criteria	Marks
• Correct proof	1

Sample answer:

The slope of PQ is

$$m_0 = \frac{\frac{c}{q} - \frac{c}{p}}{cq - cp}$$

$$= \frac{\frac{1}{q} - \frac{1}{p}}{q - p}$$

$$= \frac{p - q}{pq(q - p)}$$

$$= -\frac{1}{pq}$$

Using part (i), the tangent to the hyperbola at Q is $x + q^2 y = 2cq$.
At C, $y = 0$ and $x = 2cq$ ie $C\ (2cq, 0)$.

Hence the slope of BC is

$$m_1 = \frac{\frac{2c}{p} - 0}{0 - 2cq}$$

$$= -\frac{2c}{2cpq}$$

$$= -\frac{1}{pq}$$

As $m_0 = m_1$, the lines PQ and BC are parallel.

Question 13 (a) (i)

Criteria	Marks
• Correct solution	3
• Obtains $I_n = n\int_0^1 x^2\left(1-x^2\right)^{\frac{n-2}{2}}dx$, or equivalent unit	2
• Attempts to use integration by parts, or equivalent merit	1

Sample answer:

Use integration by parts with

$$u = \left(1-x^2\right)^{\frac{n}{2}}, \qquad v' = 1$$

$$u' = -nx\left(1-x^2\right)^{\frac{n}{2}-1}, \quad v = x$$

Hence

$$I_n = \left[x\left(1-x^2\right)^{\frac{n}{2}}\right]_0^1 + n\int_0^1 x^2\left(1-x^2\right)^{\frac{n}{2}-1}dx$$

$$= 0 - n\int_0^1\left[\left(1-x^2\right)\left(1-x^2\right)^{\frac{n}{2}-1} - \left(1-x^2\right)^{\frac{n}{2}-1}\right]dx$$

$$= -n\int_0^1\left(1-x^2\right)^{\frac{n}{2}}dx + n\int_0^1\left(1-x^2\right)^{\frac{n-2}{2}}dx$$

$$= -nI_n + nI_{n-2}$$

Solving for I_n

$$(1+n)I_n = nI_{n-2}$$

$$I_n = \frac{n}{n+1}I_{n-2}$$

Question 13 (a) (ii)

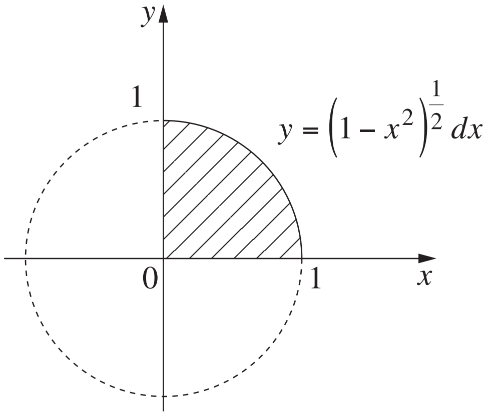
Criteria	Marks
• Correct solution	2
• Finds I_1, or equivalent merit	1

Sample answer:

$$I_1 = \int_0^1 \left(1 - x^2\right)^{\frac{1}{2}}\,dx$$

$=$ area of quarter circle with radius 1

$$= \frac{\pi}{4}$$



From the recursion relation in part (i),

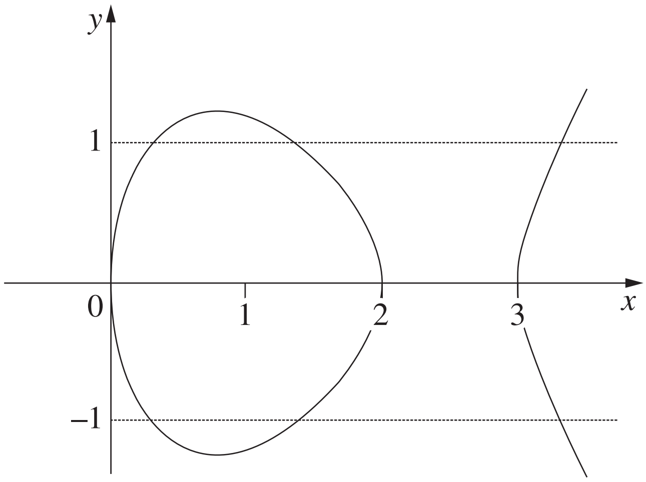
$$I_3 = \frac{3}{4} I_1$$

$$\begin{aligned} I_5 &= \frac{5}{6} I_3 \\ &= \frac{5}{6} \cdot \frac{3}{4} I_1 \\ &= \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{\pi}{4} \\ &= \frac{5\pi}{32} \end{aligned}$$

Question 13 (b) (i)

Criteria	Marks
• Correct sketch	2
• Sketches $y = \sqrt{f(x)}$, or equivalent merit	1

Sample answer:

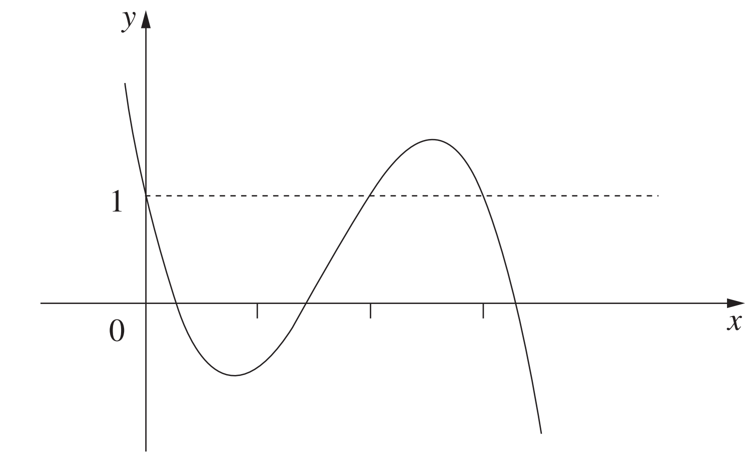


Question 13 (b) (ii)

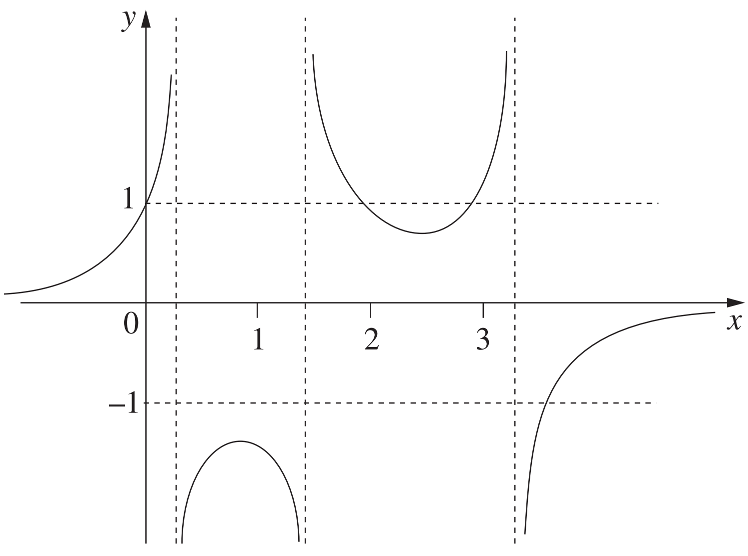
Criteria	Marks
• Correct sketch	3
• Sketches graph showing three correct vertical asymptotes, or equivalent merit	2
• Sketches graph of $y = -f(x)$, or equivalent merit	1

Sample answer:

First sketch $1 - f(x)$



Now sketch $\dfrac{1}{1 - f(x)}$



Question 13 (c) (i)

Criteria	Marks
• Correct solution	2
• Finds $\angle ABC = \alpha + \beta$, or equivalent merit	1

Sample answer:

$\angle ADC = 180° - (\alpha + \beta)$ (angle sum in triangle ACD)

$\angle ABC = 180° - \angle ADC$ (opposite angles of cyclic quadrilateral)

$= \alpha + \beta$

$\angle ACB = 90°$ (angle in semicircle, AB is diameter)

As $\triangle ACB$ is right-angled,

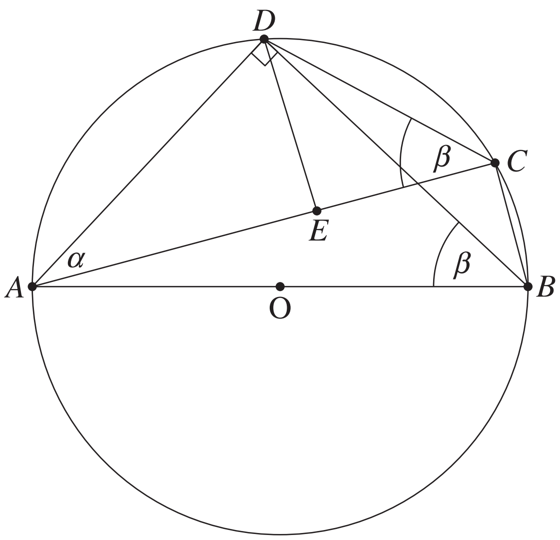
$$\sin(\alpha + \beta) = \frac{AC}{2r}$$

$$\therefore AC = 2r\sin(\alpha + \beta)$$

Question 13 (c) (ii)

Criteria	Marks
• Correct solution	2
• Observes that $\angle ADB$ is a right angle or $\angle ABD = \beta$, or equivalent merit	1

Sample answer:



Join BD.

$\angle ADB = 90°$ (angle in semicircle, AB is diameter)

$\angle ABD = \angle ACD = \beta$ (angles standing on the same arc AD)

As $\triangle ABD$ is right-angled,

$$\sin\beta = \frac{AD}{AB} = \frac{AD}{2r}$$

$$\therefore\ AD = 2r\sin\beta$$

As $\triangle AED$ is right-angled,

$$\cos\alpha = \frac{AE}{AD}$$

$$\therefore AE = AD\cos\alpha = 2r\cos\alpha\sin\beta$$

Question 13 (c) (iii)

Criteria	Marks
• Correct solution	1

Sample answer:

Similarly,
$CE = 2r\cos\beta\sin\alpha$

Now

$$\begin{aligned} AC &= AE + CE \\ &= 2r\cos\alpha\sin\beta + 2r\cos\beta\sin\alpha \\ &= 2r(\cos\alpha\sin\beta + \cos\beta\sin\alpha) \end{aligned}$$

From part (i) $AC = 2r\sin(\alpha+\beta)$, so
$\sin(\alpha+\beta) = \cos\alpha\sin\beta + \cos\beta\sin\alpha.$

Question 14 (a)

Criteria	Marks
• Correct solution	3
• Correctly finds both relevant areas, or equivalent merit	2
• Correctly finds one relevant area, or equivalent merit	1

Sample answer:

The area of the triangle with corners (1, 0), (t, 0) and (t, ln$t$) is given by $\frac{1}{2}(t-1)\ln t$.

The area under the graph $y = \ln x$ between $x = 1$ and $x = t$ is given by

$$\int_1^t \ln x\, dx = \int_1^t 1 \ln x\, dx$$

$$= \left[x \ln x\right]_1^t - \int_1^t x \frac{1}{x}\, dx \qquad \text{(integration by parts)}$$

$$= t \ln t - \left[x\right]_1^t$$

$$= t \ln t - (t-1).$$

As the graph $y = \ln x$ is concave down, the area under the graph is greater than the area of the triangle.

ie $t \ln t - (t-1) > \frac{1}{2}(t-1)\ln t$

$$t \ln t - \frac{1}{2}(t-1)\ln t > t - 1$$

$$\ln t\left[t - \frac{1}{2}(t-1)\right] > t - 1$$

$$\ln t \times \frac{1}{2}(t+1) > t - 1$$

$$\ln t > 2\left(\frac{t-1}{t+1}\right), \text{ for } t > 1$$

Question 14 (b)

Criteria	Marks
• Correct solution	3
• Uses the induction hypothesis, or equivalent merit	2
• Establishes initial case	1

Sample answer:

Show true for $n = 2$

$$\left|z_2\right| = \sqrt{1^2 + 1^2}$$
$$= \sqrt{2} \quad \text{ie true for } n = 2$$

Assume true that $\left|z_k\right| = \sqrt{k}$, for some $k \geq 2$

Show true for $n = k + 1$

ie $\left|z_k + 1\right| = \sqrt{k+1}$

$$\text{LHS} = \left|z_{k+1}\right| = \left|z_k\left(1 + \frac{i}{\left|z_k\right|}\right)\right|$$
$$= \left|z_k\right|\left|1 + \frac{i}{\left|z_k\right|}\right|$$
$$= \sqrt{k}\left|1 + \frac{i}{\sqrt{k}}\right|$$
$$= \sqrt{k}\sqrt{1 + \frac{1}{k}}$$
$$= \sqrt{k+1}$$
$$= \text{RHS}$$

$\therefore$ By the principle of Induction, $\left|z_n\right| = \sqrt{n}$ for all integers $n \geq 2$.

Question 14 (c) (i)

Criteria	Marks
• Correct solution	1

Sample answer:

As $\sec^2\theta = 1 + \tan^2\theta$ we apply the binomial theorem

$$\begin{aligned}\sec^{2n}\theta &= \left(\sec^2\theta\right)^n \\ &= \left(1+\tan^2\theta\right)^n \\ &= \sum_{k=0}^{n}\binom{n}{k}\tan^{2k}\theta\end{aligned}$$

Question 14 (c) (ii)

Criteria	Marks
• Correct solution	2
• Attempts to use part (c) (i), or equivalent merit	1

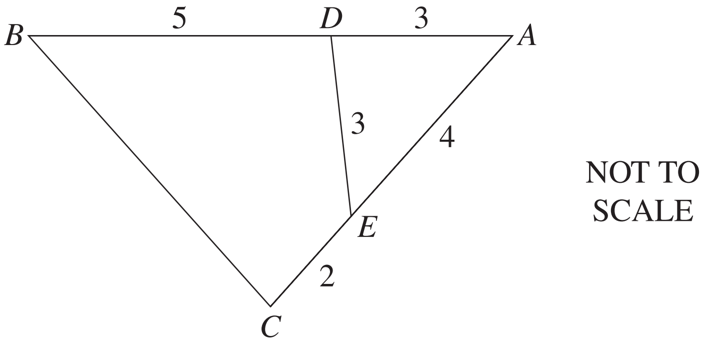
Sample answer:

$$\begin{aligned}\int \sec^8\theta\, d\theta &= \int \sec^6\theta \sec^2\theta\, d\theta \\ &= \int\left[\sum_{k=0}^{3}\binom{3}{k}\tan^{2k}\theta\right]\sec^2\theta\, d\theta \\ &= \sum_{k=0}^{3}\binom{3}{k}\int \tan^{2k}\theta\sec^2\theta\, d\theta \\ &= \sum_{k=0}^{3}\binom{3}{k}\frac{1}{2k+1}\tan^{2k+1}\theta + C \quad \left(\text{since } \frac{d}{d\theta}\tan\theta = \sec^2\theta\right)\end{aligned}$$

Question 14 (d) (i)

Criteria	Marks
• Correct solution	1

Sample answer:



$\angle DAE$ is common

$\frac{AB}{AE} = \frac{5+3}{4} = 2$ and

$\frac{AC}{AD} = \frac{2+4}{3} = 2$

So $\triangle ABC$ and $\triangle AED$ have pairs of sides in the same ratio, and the included angles equal. Hence $\triangle ABC$ and $\triangle AED$ are similar.

Question 14 (d) (ii)

Criteria	Marks
• Correct solution	1

Sample answer:

$\angle ADE = \angle BCE$ (corresponding angles in similar triangles $\triangle ABC$ and $\triangle AED$)

$\angle BDE = 180° - \angle ADE$

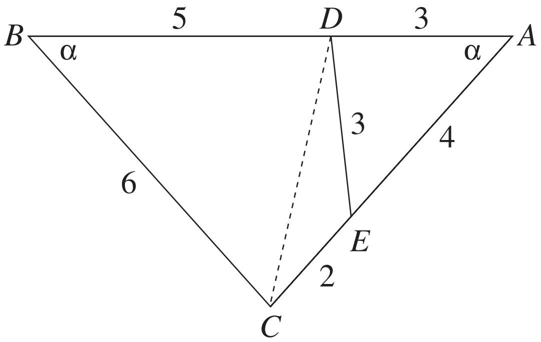
$= 180° - \angle BCE$

Hence $BCED$ is cyclic (opposite angles are supplementary).

Question 14 (d) (iii)

Criteria	Marks
• Correct solution	2
• Uses cosine rule in a suitable triangle, or equivalent merit	1

Sample answer:



$\triangle AED$ is isosceles, so by similarity with $\triangle ABC$, $\triangle ABC$ is also isosceles with

$BC = AC = 4 + 2 = 6$ and

$\angle ABC = \angle BAC = \alpha$

Applying the cosine rule in $\triangle ADC$

$CD^2 = 3^2 + 6^2 - 2 \times 3 \times 6 \cos\alpha$

Applying the cosine rule in $\triangle BDC$

$CD^2 = 5^2 + 6^2 - 2 \times 5 \times 6 \cos\alpha$

Eliminate $\cos\alpha$ from the two equations:

$$CD^2 = 45 - 36\cos\alpha \Rightarrow \cos\alpha = \frac{45 - CD^2}{36}$$

$$\begin{aligned} CD^2 &= 61 - 60\cos\alpha \\ &= 61 - 60\left(\frac{45 - CD^2}{36}\right) \\ &= 61 - 75 + \frac{5}{3}CD^2 \end{aligned}$$

$$\frac{2}{3}CD^2 = 14$$

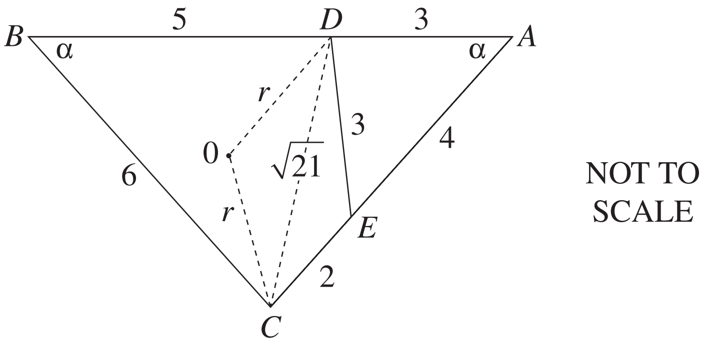
$$CD^2 = 21$$

$$CD = \sqrt{21}$$

Question 14 (d) (iv)

Criteria	Marks
• Correct solution	2
• Makes some progress	1

Sample answer:



Let O be the centre of the circle passing through B, C, E and D ($BC\,ED$ is a cyclic quadrilateral).

$\angle ABC = \alpha$, then $\angle COD = 2\alpha$ (angle at the centre is twice the angle on the circumference standing on the same arc).

From part (iii) $\cos\alpha = \dfrac{45 - CD^2}{36} = \dfrac{45-21}{36} = \dfrac{2}{3}$

Now $\cos 2\alpha = 2\cos^2\alpha - 1$

$$= 2\left(\frac{2}{3}\right)^2 - 1$$

$$= -\frac{1}{9}$$

From the cosine rule in $\triangle CDO$

$$CD^2 = r^2 + r^2 - 2r^2 \cos 2\alpha$$

$$\therefore 21 = 2r^2 + \frac{2}{9}r^2$$

$$= \frac{20}{9}r^2$$

Hence $r^2 = \dfrac{9 \times 21}{20}$ and $r = \dfrac{3\sqrt{21}}{2\sqrt{5}}$

Question 15 (a)

Criteria	Marks
• Correct solution	3
• Shows that $z\overline{w} - w\overline{z} = 2i\lvert w\rvert\lvert z\rvert(\sin\theta\cos\phi - \cos\theta\sin\phi)$, or equivalent merit	2
• Shows the fact that $z = \lvert z\rvert(\cos\theta + i\sin\theta)$, or equivalent merit	1

Sample answer:

Area $\triangle = \dfrac{1}{2}ab\sin C$

ie $A = \dfrac{1}{2}\lvert z\rvert\lvert w\rvert\sin(\theta - \phi)$

$2A = \lvert z\rvert\lvert w\rvert\sin(\theta - \phi)$

$$\begin{aligned}
z\overline{w} - w\overline{z} &= \lvert z\rvert(\cos\theta + i\sin\theta)\lvert w\rvert(\cos\phi - i\sin\phi) - \lvert w\rvert(\cos\phi + i\sin\phi)\lvert z\rvert(\cos\theta - i\sin\theta) \\
&= \lvert w\rvert\lvert z\rvert(2i\cos\phi\sin\theta - 2i\cos\theta\sin\phi) \\
&= 2i\lvert w\rvert\lvert z\rvert(\sin\theta\cos\phi - \cos\theta\sin\phi) \\
&= 2i\lvert w\rvert\lvert z\rvert(\sin(\theta - \phi)) \\
&= 2i2\,A \\
&= 4i\,A
\end{aligned}$$

Question 15 (b) (i)

Criteria	Marks
• Correct solution	2
• Correctly finds the value of b, or equivalent merit	1

Sample answer:

$P(1) = a + b + c + e = -3$ (remainder theorem)
$P(-1) = a - b + c + e = 0$ (as $x = -1$ is a root)

Subtracting

$2b = -3$, so $b = -\frac{3}{2}$

$$P'(x) = 4ax^3 + 3bx^2 + 2cx$$
$$P'(-1) = -4a + 3b - 2c$$
$$P'(-1) = 0 \text{ (as } x = -1 \text{ is a double root)}$$

Hence

$$\begin{aligned} 4a + 2c &= 3b \\ &= 3\left(-\frac{3}{2}\right) \\ &= -\frac{9}{2}. \end{aligned}$$

Question 15 (b) (ii)

Criteria	Marks
• Correct solution	1

Sample answer:

Slope of tangent at $x = 1$ is

$$\begin{aligned} P'(1) &= 4a + 3b + 2c \\ &= (4a + 2c) + 3b \\ &= -\frac{9}{2} + 3\left(-\frac{3}{2}\right) \\ &= -9. \end{aligned}$$

Question 15 (c) (i)

Criteria	Marks
• Correct solution	1

Sample answer:

The probability that a car completes all 4 days is

$$(0.7)^4 \approx 0.2401$$
$$= 0.24$$

Question 15 (c) (ii)

Criteria	Marks
• Correct solution	2
• Demonstrates some understanding of binomial probability using $n = 8$, or equivalent merit	1

Sample answer:

The probability that a car does not complete all 4 days is

$$1-(0.7)^4 \approx 0.76$$

Probability that no car completes all 4 days:

$$\binom{8}{0}(0.76)^8$$

Probability that precisely one completes all 4 days:

$$\binom{8}{1}(0.76)^7(0.24)^1$$

Probability that precisely two complete all 4 days:

$$\binom{8}{2}(0.76)^6(0.24)^2$$

Therefore the probability that 0, 1 or 2 complete all 4 days:

$$(0.76)^6\left((0.76)^2+8(0.76)(0.24)+28(0.24)^2\right) \approx 0.7$$

Hence the probability that at least 3 complete all 4 days is

$$1-0.7 = 0.3$$

Question 15 (d) (i)

Criteria	Marks
• Correct solution	1

Sample answer:

Terminal velocity occurs when acceleration is zero $\left(\dot{v} = 0\right)$,

ie $mg = kv^2$

$$v_T^2 = \frac{mg}{k}$$

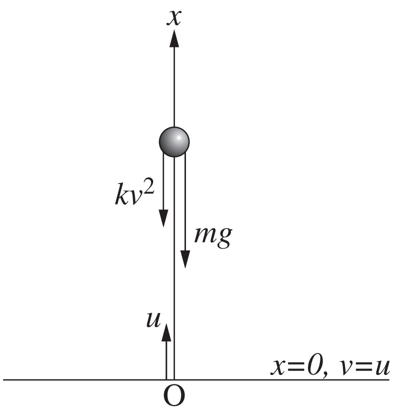
$$v_T = \pm\sqrt{\frac{mg}{k}}$$

ie $v_T = \sqrt{\frac{mg}{k}}$

Question 15 (d) (ii)

Criteria	Marks
• Correct solution	3
• Finds an expression for x in terms of v and an undetermined constant, or equivalent merit	2
• Obtains equation of motion, or equivalent merit	1

Sample answer:



Equation of motion for the ball going up:

$$m\dot{v} = -mg - kv^2$$

$$= -mg\left(1 + \frac{k}{mg}v^2\right)$$

$$= -mg\left(1 + \frac{v^2}{v_T^2}\right)$$

$$\therefore \dot{v} = \frac{-g}{v_T^2}\left(v_T^2 + v^2\right)$$

We know $\dot{v} = \frac{1}{2}\frac{dv^2}{dx}$, so

$$\frac{1}{2}\frac{dv^2}{dx} = -\frac{g}{v_T^2}\left(v_T^2 + v^2\right)$$

or $$\frac{dx}{dv^2} = -\frac{v_T^2}{2g}\left(\frac{1}{v_T^2 + v^2}\right)$$

Integrating we get

$$x = -\frac{v_T^2}{2g}\ln\left(v_T^2 + v^2\right) + C$$

When $t = 0$, $x = 0$ and $v = u$. Hence

$$0 = -\frac{v_T^2}{2g}\ln\left(v_T^2 + u^2\right) + C$$

so $$C = \frac{v_T^2}{2g}\ln\left(v_T^2 + u^2\right)$$

$$\therefore x = \frac{v_T^2}{2g}\left[\ln\left(v_T^2 + u^2\right) - \ln\left(v_T^2 + v^2\right)\right]$$

$$= \frac{v_T^2}{2g}\ln\left(\frac{v_T^2 + u^2}{v_T^2 + v^2}\right)$$

When the ball reaches its maximum height,

$v = 0$ and $x = H$. Hence

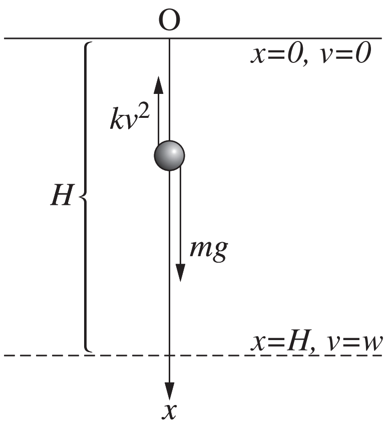
$$H = \frac{v_T^2}{2g}\ln\frac{v_T^2 + u^2}{v_T^2}$$

$$= \frac{v_T^2}{2g}\ln\left(1 + \frac{u^2}{v_T^2}\right)$$

Question 15 (d) (iii)

Criteria	Marks
• Correct solution	2
• Makes some progress	1

Sample answer:



Equation of motion for the ball coming down:

$$m\dot{v} = mg - kv^2$$
$$= mg\left[1 - \frac{k}{mg}v^2\right]$$
$$= mg\left(1 - \frac{v^2}{v_T^2}\right)$$
$$\therefore \dot{v} = \frac{g}{v_T^2}\left(v_T^2 - v^2\right)$$

We know $\dot{v} = \frac{1}{2}\frac{dv^2}{dx}$,

so $2\frac{dx}{dv^2} = \frac{v_T^2}{g}\left(\frac{1}{v_T^2 - v^2}\right)$

Hence $x = \frac{-v_T^2}{2g}\ln\left(v_T^2 - v^2\right) + C$

When $x = 0, v = 0,$

$$\therefore C = \frac{v_T^2}{2g}\ln v_T^2$$

$$x = \frac{v_T^2}{2g}\left[\ln v_T^2 - \ln\left(v_T^2 - v^2\right)\right]$$

ie $x = \frac{v_T^2}{2g}\ln\left(\frac{v_T^2}{v_T^2 - v^2}\right)$

When the ball hits the ground, $x = H, v = w$

$$H = \frac{v_T^2}{2g}\ln\left(\frac{v_T^2}{v_T^2 - w^2}\right)$$

but $H = \frac{v_T^2}{2g}\ln\left(1 + \frac{u^2}{v_T^2}\right)$ (from part (ii))

$$\therefore \frac{v_T^2}{2g}\ln\left(1 + \frac{u^2}{v_T^2}\right) = \frac{v_T^2}{2g}\ln\left(\frac{v_T^2}{v_T^2 - w^2}\right)$$

$$1 + \frac{u^2}{v_T^2} = \frac{v_T^2}{v_T^2 - w^2}$$

$$\frac{v_T^2 + u^2}{v_T^2} = \frac{v_T^2}{v_T^2 - w^2}$$

$$\left(v_T^2 + u^2\right)\left(v_T^2 - w^2\right) = v_T^2 \cdot v_T^2$$

$$\cancel{v_T^2 \cdot v_T^2} - v_T^2 w^2 + u^2 v_T^2 - u^2 w^2 = \cancel{v_T^2 \cdot v_T^2}$$

(dividing by $u^2 w^2 v_T^2$) $\frac{-1}{u^2} + \frac{1}{w^2} + \frac{1}{v_T^2} = 0$

ie $\frac{1}{w^2} = \frac{1}{u^2} + \frac{1}{v_T^2}$

Question 16 (a) (i)

Criteria	Marks
• Correct solution	2
• Solves $P'(x) = 0$, or equivalent merit	1

Sample answer:

$P(x) = 2x^3 - 15x^2 + 24x + 16$

$P'(x) = 6x^2 - 30x + 24 = 0$ (for max/mim)

ie $x^2 - 5x + 4 = 0$

$(x-4)(x-1) = 0$

$x = 4$ or 1

$P''(x) = 12x - 30$

$P''(4) = 48 - 30 > 0 \therefore$ min

$P''(1) = 12 - 30 < 0 \therefore$ max

$P(4) = 0 \therefore$ min at $(4,0)$

Test boundary: $x = 0$, $P(0) = 16$

$\therefore$ minimum value of $P(x)$ is 0 when $x = 4$

Question 16 (a) (ii)

Criteria	Marks
• Correct solution	1

Sample answer:

$(x+1)\left(x^2 + (x+4)^2\right) \geq 25x^2$

$\text{LHS} = (x+1)\left(x^2 + x^2 + 8x + 16\right)$

$= 2x^2 + 8x^2 + 16x + 2x^2 + 8x + 16$

$= 2x^3 + 10x^2 + 24x + 16$

Since $P(x) \geq 0$, ie $2x^3 - 15x^2 + 24x + 16 \geq 0$ (from part (i))

then $2x^3 + 10x^2 + 24x + 16 \geq 25x^2$

ie $(x+1)\left(x^2 + (x+4)^2\right) \geq 25x^2$.

Question 16 (a) (iii)

Criteria	Marks
• Correct solution	2
• Shows that $(m+n)^2 + (m+n+4)^2 \geq \dfrac{25(m+n)^2}{m+n+1}$, or equivalent merit	1

Sample answer:

$$(m+n)^2 + (m+n+4)^2 \geq \frac{100mn}{m+n+1}$$

From part (ii)

$$(x+1)\left(x^2 + (x+4)^2\right) \geq 25x^2$$

Since $x \geq 0$ then

$$x^2 + (x+4)^2 \geq \frac{25x^2}{x+1}$$

Let $x = m + n$

$$(m+n)^2 + (m+n+4)^2 \geq \frac{25(m+n)^2}{m+n+1}$$

Now $\quad (m-n)^2 \geq 0$

$$m^2 - 2mn + n^2 \geq 0$$

$$m^2 + n^2 \geq 2mn$$

and $m^2 + 2mn + n^2 \geq 2mn + 2mn$

$$(m+n)^2 \geq 4mn$$

$$\therefore \ (m+n)^2 + (m+n+4)^2 \geq \frac{25(4mn)}{m+n+1}$$

$$\geq \frac{100mn}{m+n+1}.$$

Question 16 (b) (i)

Criteria	Marks
• Correct answer	1

Sample answer:

The string has length $2a$, so $PS' + PS = 2a$.
This characterises an ellipse with focal length a and focal points S and S'.
Also, for an ellipse, $SS' = 2ae$, which is given.

Question 16 (b) (ii)

Criteria	Marks
• Correct solution	1

Sample answer:

For an ellipse, the distance from the centre O to the directrix is $\dfrac{a}{e}$.

Hence the distance from P to the directrix is
$\dfrac{a}{e} - OQ = \dfrac{a}{e} - a\cos\theta$.

By the focus-directrix definition of an ellipse, $SP = ePM$ (where M is on the directrix)

$\therefore SP = e\left(\dfrac{a}{e} - a\cos\theta\right)$
So, $SP = a - ae\cos\theta$

Question 16 (b) (iii)

Criteria	Marks
• Correct solution	2
• Finds the length of $S'P$, or equivalent merit	1

Sample answer:

$$
\begin{aligned}
\sin\beta &= \frac{QS'}{PS'} \\
&= \frac{ae + OQ}{2a - SP} \\
&= \frac{ae + a\cos\theta}{2a - a(1 - e\cos\theta)} \\
&= \frac{a(e + \cos\theta)}{2a - a + ae\cos\theta} \\
&= \frac{a(e + \cos\theta)}{a(1 + e\cos\theta)} \\
&= \frac{e + \cos\theta}{1 + e\cos\theta}.
\end{aligned}
$$

Question 16 (b) (iv)

Criteria	Marks
• Correct solution	2
• Finds the component of one tension force in the vertical direction, or equivalent merit	1

Sample answer:

The vertical component of the force along $S'P$ is

$$\text{T}\sin\beta = T\frac{e+\cos\theta}{1+e\cos\theta}$$

Now

$$\begin{aligned}\sin(\angle QPS) &= \frac{QS}{PS}\\ &= \frac{ae-OQ}{a(1-e\cos\theta)}\\ &= \frac{ae-a\cos\theta}{a(1-e\cos\theta)}\\ &= \frac{e-\cos\theta}{1-e\cos\theta}\end{aligned}$$

The vertical component of the force along SP is therefore

$$T\sin(\angle QPS) = T\frac{e-\cos\theta}{1-e\cos\theta}$$

Therefore

$$\begin{aligned}mg &= T\left(\frac{e+\cos\theta}{1+e\cos\theta}-\frac{e-\cos\theta}{1-e\cos\theta}\right)\\ &= T\frac{(e+\cos\theta)(1-e\cos\theta)-(e-\cos\theta)(1+e\cos\theta)}{1-e^2\cos^2\theta}\\ &= \frac{T}{1-e^2\cos^2\theta}\left[e+\cos\theta-e^2\cos\theta-e\cos^2\theta-\left(e-\cos\theta+e^2\cos\theta-e\cos^2\theta\right)\right]\\ &= \frac{T}{1-e^2\cos^2\theta}\left[2\cos\theta-2e^2\cos\theta\right]\\ &= \frac{2T(1-e^2)\cos\theta}{1-e^2\cos^2\theta}.\end{aligned}$$

Question 16 (b) (v)

Criteria	Marks
• Correct solution	3
• Makes substantial progress	2
• Finds QP, or equivalent merit	1

Sample answer:

As P lies on an ellipse of eccentricity e

$r = b = QP = a\sqrt{1-e^2}\sin\theta$

Since ellipse has $b^2 = a^2\left(1-e^2\right)$

$$\cos(\angle QPS) = \frac{r}{SP} = \frac{\not{a}\sqrt{1-e^2}\sin\theta}{\not{a}(1-e\cos\theta)} = \frac{\sqrt{1-e^2}\sin\theta}{1-e\cos\theta}$$

$$\cos(\beta) = \frac{r}{S'P} = \frac{\not{a}\sqrt{1-e^2}\sin\theta}{\not{a}(1+e\cos\theta)} = \frac{\sqrt{1-e^2}\sin\theta}{1+e\cos\theta}$$

Hence the horizontal force is

$$mrw^2 = \frac{T\sqrt{1-e^2}\sin\theta}{1-e\cos\theta} + \frac{T\sqrt{1-e^2}\sin\theta}{1+e\cos\theta} = \frac{2T\sqrt{1-e^2}\sin\theta}{1-e^2\cos\theta}.$$

Question 16 (b) (vi)

Criteria	Marks
• Correct solution	1

Sample answer:

$$\frac{mrw^2}{mg} = \frac{\dfrac{\cancel{2T}\sqrt{1-e^2}\sin\theta}{\cancel{\left(1-e^2\cos^2\theta\right)}}}{\dfrac{\cancel{2T}\left(1-e^2\right)\cos\theta}{\cancel{\left(1-e^2\cos^2\theta\right)}}}$$

$$= \frac{\sqrt{1-e^2}\sin\theta}{\left(1-e^2\right)\cos\theta}$$

$$\frac{rw^2}{g} = \frac{1}{\sqrt{1-e^2}}\tan\theta$$

$$\tan\theta = \frac{rw^2}{g}\sqrt{1-e^2}.$$

Mathematics Extension 2

2013 HSC Examination Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	8.0	H5, E8
2	1	3.2	E4
3	1	2.2	E3
4	1	7.5	E4
5	1	2.5	E3
6	1	4.1	E8
7	1	6.3	E5
8	1	5.1	E7
9	1	1.5	E6
10	1	8.0	HE3, E2

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	1	2.1	E3
11 (a) (ii)	2	2.2	E3
11 (a) (iii)	2	2.4	E3
11 (b)	2	7.6	E4
11 (c)	2	2.1	E3
11 (d)	3	4.1	E8
11 (e)	3	2.5	E3
12 (a)	4	4.1	E8
12 (b)	2	1.8	E4, E6
12 (c)	4	4.1, 5.1	E7, E8
12 (d) (i)	2	3.3	E4
12 (d) (ii)	2	3.3, 8.1	PE3, E3, E4
12 (d) (iii)	1	3.3	E3, E4
13 (a) (i)	3	4.1	E8
13 (a) (ii)	2	4.1	E8
13 (b) (i)	2	1.7	E6
13 (b) (ii)	3	1.8	E6
13 (c) (i)	2	8.0, 8.1	E2
13 (c) (ii)	2	8.0, 8.1	E2
13 (c) (iii)	1	8.0, 8.1	E2
14 (a)	3	4.1, 8.3	PE3, E2
14 (b)	3	2.1, 2.2, 8.2	HE2, E2. E3

Question	Marks	Content	Syllabus outcomes
14 (c) (i)	1	8.0	HE3, E2
14 (c) (ii)	2	4.1	E8
14 (d) (i)	1	8.1	P4, H5
14 (d) (ii)	1	8.1	PE3, E2
14 (d) (iii)	2	8.1	H5, E2
14 (d) (iv)	2	8.1	PE3, E2
15 (a)	3	2.2	E3
15 (b) (i)	2	7.2, 8.0	PE3,E4
15 (b) (ii)	1	7.2, 8.0	PE3,E4
15 (c) (i)	1	8.0	HE3, E2
15 (c) (ii)	2	8.0	HE3, E2
15 (d) (i)	1	6.2	E5
15 (d) (ii)	3	6.2	E5
15 (d) (iii)	2	6.2	E5
16 (a) (i)	2	8.0	H6, E6
16 (a) (ii)	1	8.3	PE3, E4
16 (a) (iii)	2	8.3	PE3, E4
16 (b) (i)	1	3.1	E3, E4
16 (b) (ii)	1	3.1	E3, E4
16 (b) (iii)	2	8.0	H5
16 (b) (iv)	2	6.3	E5
16 (b) (v)	3	6.3	E5
16 (b) (vi)	1	6.3	E5