BOSTES
BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2014 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–17

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 What are the values of a, b and c for which the following identity is true?

$$\frac{5x^2 - x + 1}{x\left(x^2 + 1\right)} = \frac{a}{x} + \frac{bx + c}{x^2 + 1}$$

(A) $a = 1,\ b = 6,\ c = 1$

(B) $a = 1,\ b = 4,\ c = 1$

(C) $a = 1,\ b = 6,\ c = -1$

(D) $a = 1,\ b = 4,\ c = -1$

2 The polynomial $P(z)$ has real coefficients, and $z = 2 - i$ is a root of $P(z)$.

Which quadratic polynomial must be a factor of $P(z)$?

(A) $z^2 - 4z + 5$

(B) $z^2 + 4z + 5$

(C) $z^2 - 4z + 3$

(D) $z^2 + 4z + 3$

3 What is the eccentricity of the ellipse $9x^2 + 16y^2 = 25$?

(A) $\frac{7}{16}$

(B) $\frac{\sqrt{7}}{4}$

(C) $\frac{\sqrt{15}}{4}$

(D) $\frac{5}{4}$

4 Given $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, which expression is equal to $\left(\overline{z}\right)^{-1}$?

(A) $\frac{1}{2}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

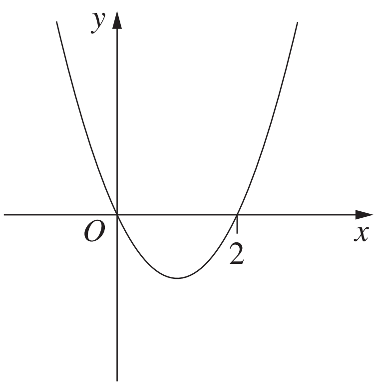
(B) $2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

(C) $\frac{1}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

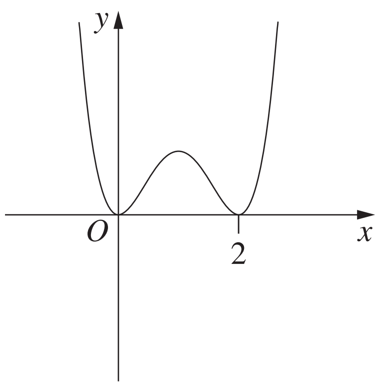
(D) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

5 Which graph best represents the curve $y^2 = x^2 - 2x$?

(A)



(B)



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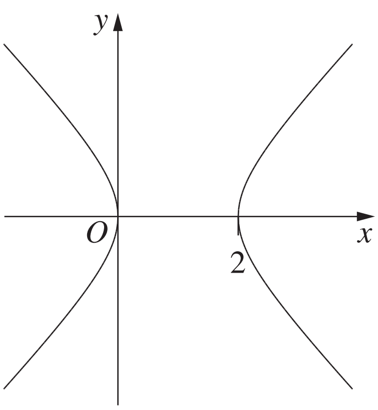

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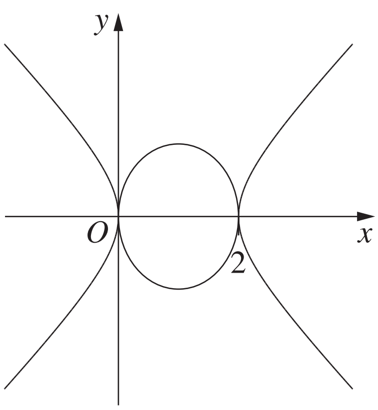
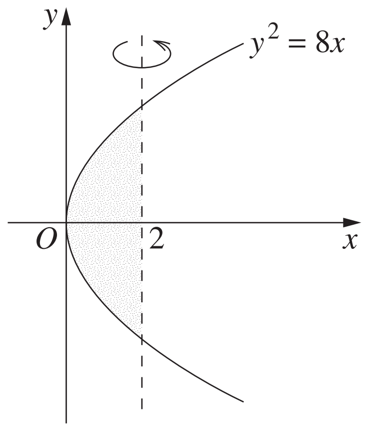


6 The region bounded by the curve $y^2 = 8x$ and the line $x = 2$ is rotated about the line $x = 2$ to form a solid.



Which expression represents the volume of the solid?

(A) $\pi\displaystyle\int_0^4 2^2 - \left(\frac{y^2}{8}\right)^2 dy$

(B) $2\pi\displaystyle\int_0^4 2^2 - \left(\frac{y^2}{8}\right)^2 dy$

(C) $\pi\displaystyle\int_0^4 \left(2 - \frac{y^2}{8}\right)^2 dy$

(D) $2\pi\displaystyle\int_0^4 \left(2 - \frac{y^2}{8}\right)^2 dy$

7 Which expression is equal to $\displaystyle\int \frac{1}{1-\sin x}\,dx$?

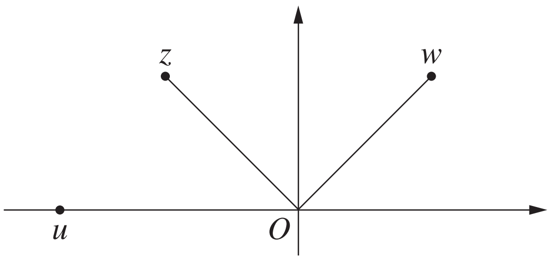
(A) $\tan x - \sec x + c$

(B) $\tan x + \sec x + c$

(C) $\log_e(1-\sin x) + c$

(D) $\dfrac{\log_e(1-\sin x)}{-\cos x} + c$

8 The Argand diagram shows the complex numbers w, z and u, where w lies in the first quadrant, z lies in the second quadrant and u lies on the negative real axis.



Which statement could be true?

(A) $u = zw$ and $u = z + w$

(B) $u = zw$ and $u = z - w$

(C) $z = uw$ and $u = z + w$

(D) $z = uw$ and $u = z - w$

9 A particle is moving along a straight line so that initially its displacement is $x = 1$, its velocity is $v = 2$, and its acceleration is $a = 4$.

Which is a possible equation describing the motion of the particle?

(A) $v = 2\sin(x-1) + 2$

(B) $v = 2 + 4\log_e x$

(C) $v^2 = 4(x^2 - 2)$

(D) $v = x^2 + 2x + 4$

10 Which integral is necessarily equal to $\int_{-a}^{a} f(x)\,dx$?

(A) $\int_{0}^{a} f(x) - f(-x)\,dx$

(B) $\int_{0}^{a} f(x) - f(a-x)\,dx$

(C) $\int_{0}^{a} f(x-a) + f(-x)\,dx$

(D) $\int_{0}^{a} f(x-a) + f(a-x)\,dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the complex numbers $z = -2 - 2i$ and $w = 3 + i$.

(i) Express $z + w$ in modulus–argument form. **2**

(ii) Express $\dfrac{z}{w}$ in the form $x + iy$, where x and y are real numbers. **2**

(b) Evaluate $\displaystyle\int_0^{\frac{1}{2}} (3x - 1)\cos(\pi x)\, dx$. **3**

(c) Sketch the region in the Argand diagram where $|z| \le |z - 2|$ and $-\dfrac{\pi}{4} \le \arg z \le \dfrac{\pi}{4}$. **3**

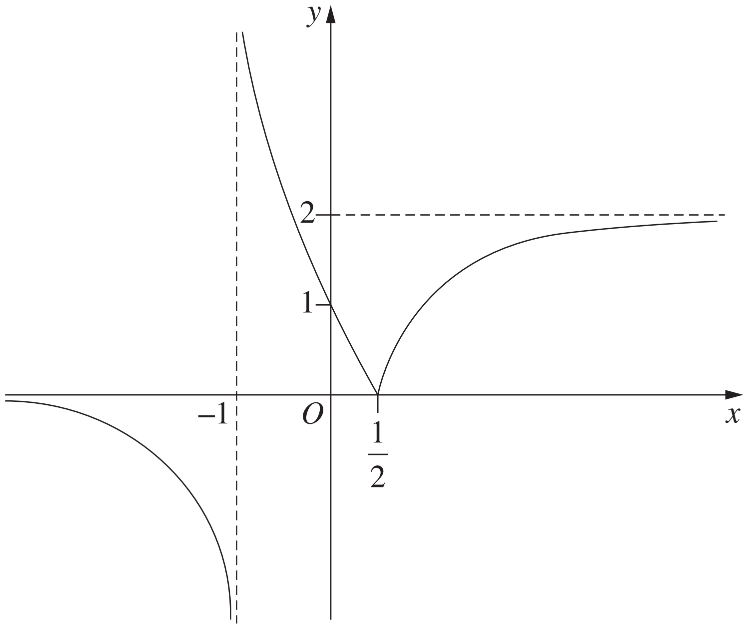
(d) Without the use of calculus, sketch the graph $y = x^2 - \dfrac{1}{x^2}$, showing all intercepts. **2**

(e) The region enclosed by the curve $x = y(6 - y)$ and the y-axis is rotated about the x-axis to form a solid. **3**

Using the method of cylindrical shells, or otherwise, find the volume of the solid.

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of a function $f(x)$.



Draw a separate half-page graph for each of the following functions, showing all asymptotes and intercepts.

(i) $y = f(|x|)$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(b) It can be shown that $4\cos^3\theta - 3\cos\theta = \cos 3\theta$. (Do NOT prove this.)

Assume that $x = 2\cos\theta$ is a solution of $x^3 - 3x = \sqrt{3}$.

(i) Show that $\cos 3\theta = \dfrac{\sqrt{3}}{2}$. **1**

(ii) Hence, or otherwise, find the three real solutions of $x^3 - 3x = \sqrt{3}$. **2**

Question 12 continues on page 9

Question 12 (continued)

(c) The point $P(x_0, y_0)$ lies on the curves $x^2 - y^2 = 5$ and $xy = 6$. **3**

Prove that the tangents to these curves at P are perpendicular to one another.

(d) Let $I_n = \displaystyle\int_0^1 \frac{x^{2n}}{x^2+1}\,dx$, where n is an integer and $n \geq 0$.

(i) Show that $I_0 = \dfrac{\pi}{4}$. **1**

(ii) Show that $I_n + I_{n-1} = \dfrac{1}{2n-1}$. **2**

(iii) Hence, or otherwise, find $\displaystyle\int_0^1 \frac{x^4}{x^2+1}\,dx$. **2**

End of Question 12

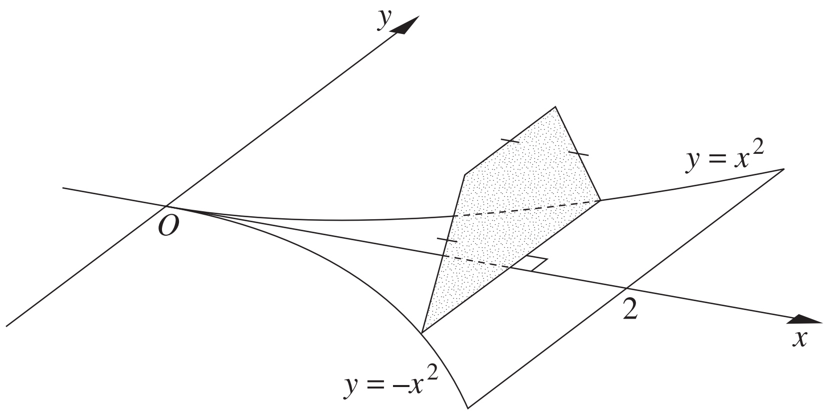
Please turn over

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Using the substitution $t = \tan\dfrac{x}{2}$, or otherwise, evaluate **3**

$$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \frac{1}{3\sin x - 4\cos x + 5}\,dx\,.$$

(b) The base of a solid is the region bounded by $y = x^2$, $y = -x^2$ and $x = 2$. Each cross-section perpendicular to the x-axis is a trapezium, as shown in the diagram. The trapezium has three equal sides and its base is twice the length of any one of the equal sides. **4**



Find the volume of the solid.

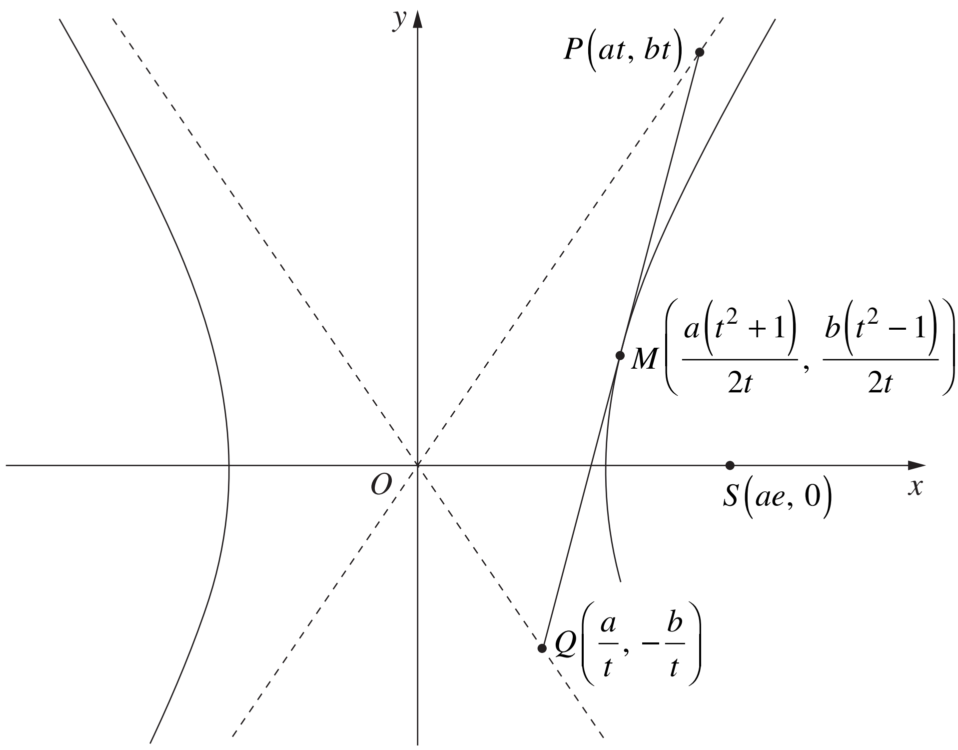
Question 13 continues on page 11

Question 13 (continued)

(c) The point $S(ae, 0)$ is the focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ on the positive x-axis.

The points $P(at, bt)$ and $Q\left(\frac{a}{t}, -\frac{b}{t}\right)$ lie on the asymptotes of the hyperbola, where $t > 0$.

The point $M\left(\frac{a(t^2+1)}{2t}, \frac{b(t^2-1)}{2t}\right)$ is the midpoint of PQ.



(i) Show that M lies on the hyperbola. **1**

(ii) Prove that the line through P and Q is a tangent to the hyperbola at M. **3**

(iii) Show that $OP \times OQ = OS^2$. **2**

(iv) If P and S have the same x-coordinate, show that MS is parallel to one of the asymptotes of the hyperbola. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

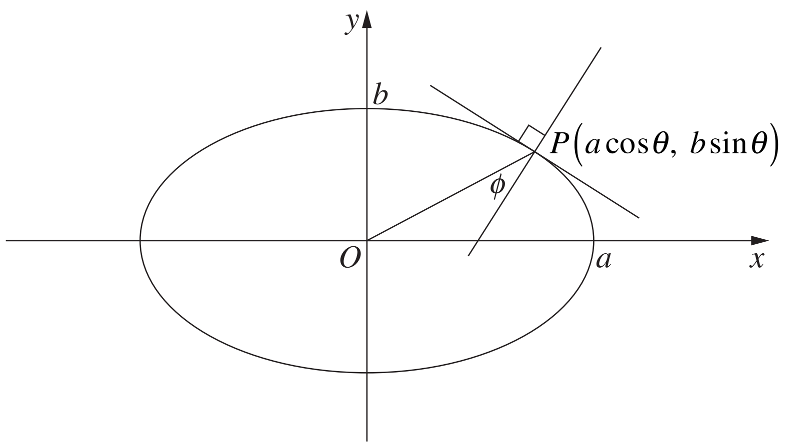
(a) Let $P(x) = x^5 - 10x^2 + 15x - 6$.

(i) Show that $x = 1$ is a root of $P(x)$ of multiplicity three. **2**

(ii) Hence, or otherwise, find the two complex roots of $P(x)$. **2**

(b) The point $P(a\cos\theta,\ b\sin\theta)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$.

The acute angle between OP and the normal to the ellipse at P is ϕ.



(i) Show that $\tan\phi = \left(\frac{a^2 - b^2}{ab}\right)\sin\theta\cos\theta$. **3**

(ii) Find a value of θ for which ϕ is a maximum. **2**

Question 14 continues on page 13

Question 14 (continued)

(c) A high speed train of mass m starts from rest and moves along a straight track. At time t hours, the distance travelled by the train from its starting point is x km, and its velocity is v km/h.

The train is driven by a constant force F in the forward direction. The resistive force in the opposite direction is Kv^2, where K is a positive constant. The terminal velocity of the train is 300 km/h.

(i) Show that the equation of motion for the train is **2**

$$m\ddot{x} = F\left[1 - \left(\frac{v}{300}\right)^2\right].$$

(ii) Find, in terms of F and m, the time it takes the train to reach a velocity of 200 km/h. **4**

End of Question 14

Please turn over

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) Three positive real numbers a, b and c are such that $a + b + c = 1$ and $a \le b \le c$. **2**

By considering the expansion of $(a + b + c)^2$, or otherwise, show that

$$5a^2 + 3b^2 + c^2 \le 1.$$

(b) (i) Using de Moivre's theorem, or otherwise, show that for every positive integer n, **2**

$$(1 + i)^n + (1 - i)^n = 2\left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}.$$

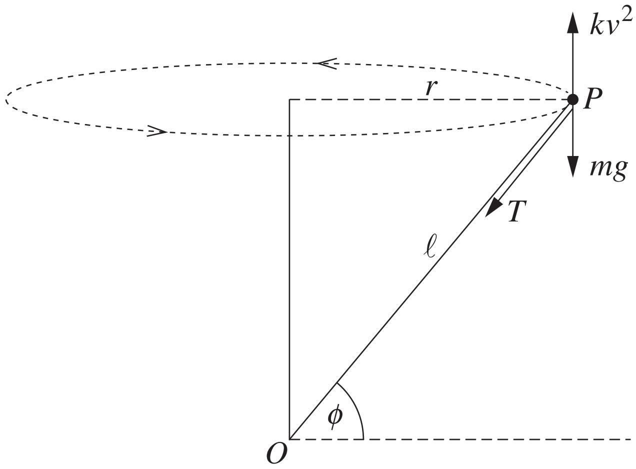
(ii) Hence, or otherwise, show that for every positive integer n divisible by 4, **3**

$$\binom{n}{0} - \binom{n}{2} + \binom{n}{4} - \binom{n}{6} + \cdots + \binom{n}{n} = (-1)^{\frac{n}{4}}\left(\sqrt{2}\right)^n.$$

Question 15 continues on page 15

Question 15 (continued)

(c) A toy aeroplane P of mass m is attached to a fixed point O by a string of length ℓ. The string makes an angle ϕ with the horizontal. The aeroplane moves in uniform circular motion with velocity v in a circle of radius r in a horizontal plane.



The forces acting on the aeroplane are the gravitational force mg, the tension force T in the string and a vertical lifting force kv^2, where k is a positive constant.

(i) By resolving the forces on the aeroplane in the horizontal and the vertical directions, show that $\dfrac{\sin\phi}{\cos^2\phi} = \dfrac{\ell k}{m} - \dfrac{\ell g}{v^2}$. **3**

(ii) Part (i) implies that $\dfrac{\sin\phi}{\cos^2\phi} < \dfrac{\ell k}{m}$. (Do NOT prove this.) **2**

Use this to show that

$$\sin\phi < \frac{\sqrt{m^2 + 4\ell^2k^2} - m}{2\ell k}.$$

(iii) Show that $\dfrac{\sin\phi}{\cos^2\phi}$ is an increasing function of ϕ for $-\dfrac{\pi}{2} < \phi < \dfrac{\pi}{2}$. **2**

(iv) Explain why ϕ increases as v increases. **1**

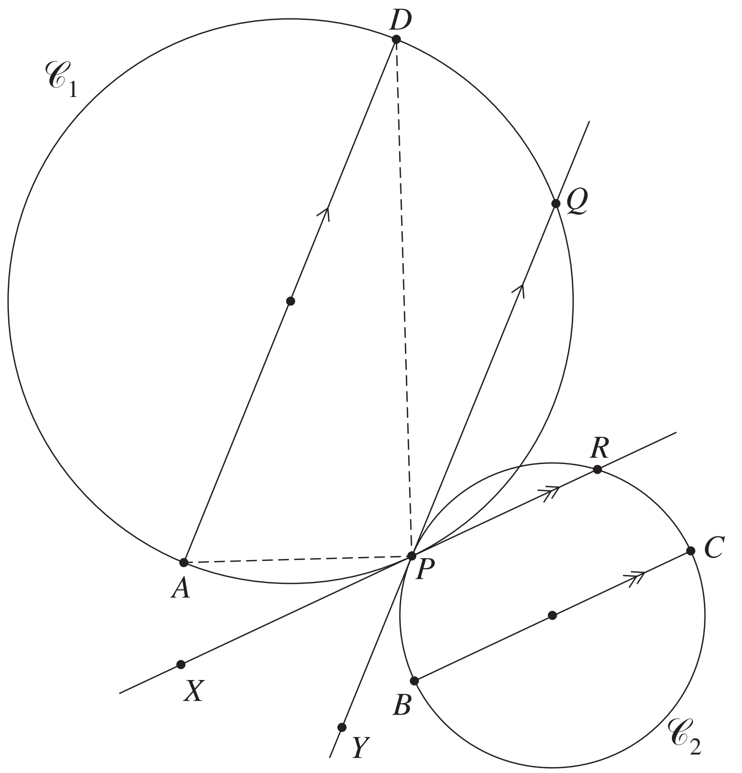
End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows two circles $\mathscr{C}_1$ and $\mathscr{C}_2$. The point P is one of their points of intersection. The tangent to $\mathscr{C}_2$ at P meets $\mathscr{C}_1$ at Q, and the tangent to $\mathscr{C}_1$ at P meets $\mathscr{C}_2$ at R.

The points A and D are chosen on $\mathscr{C}_1$ so that AD is a diameter of $\mathscr{C}_1$ and parallel to PQ. Likewise, points B and C are chosen on $\mathscr{C}_2$ so that BC is a diameter of $\mathscr{C}_2$ and parallel to PR.

The points X and Y lie on the tangents PR and PQ, respectively, as shown in the diagram.



Copy or trace the diagram into your writing booklet.

(i) Show that $\angle APX = \angle DPQ$. **2**

(ii) Show that A, P and C are collinear. **3**

(iii) Show that $ABCD$ is a cyclic quadrilateral. **1**

Question 16 continues on page 17

Question 16 (continued)

(b) Suppose n is a positive integer.

(i) Show that **3**

$$-x^{2n} \le \frac{1}{1+x^2} - \left(1 - x^2 + x^4 - x^6 + \cdots + (-1)^{n-1} x^{2n-2}\right) \le x^{2n}.$$

(ii) Use integration to deduce that **2**

$$-\frac{1}{2n+1} \le \frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + (-1)^{n-1} \frac{1}{2n-1}\right) \le \frac{1}{2n+1}.$$

(iii) Explain why $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$. **1**

(c) Find $\int \frac{\ln x}{(1+\ln x)^2}\, dx$. **3**

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$