## 2014 Notes from the Marking Centre – Mathematics Extension 2

### Introduction

This document has been produced for the teachers and candidates of the Stage 6 Mathematics Extension 2 course. It contains comments on candidate responses to particular parts of the 2014 Higher School Certificate examination, highlighting candidates' strengths and indicating where they need to improve.

This document should be read along with:

- the Mathematics Extension 2 Stage 6 syllabus
- the 2014 Higher School Certificate Mathematics Extension 2 examination
- the marking guidelines
- Advice for students attempting HSC Mathematics examinations
- Advice for HSC students about examinations
- other support documents developed by the Board of Studies, Teaching and Educational Standards NSW to assist in the teaching and learning of Mathematics in Stage 6.

#### Question 11

(a)(i) Most candidates correctly added the two complex numbers and found the modulus and the argument of the resultant simple expression.

A common problem was:

- determining the quadrant of $\theta$ when $\tan\theta$ was negative.

(a)(ii) Most candidates identified the need to multiply by the complex conjugate to realise the denominator.

A common problem was:

- carelessness in expanding and in collecting real and imaginary terms.

(b) Common problems were:

- splitting the integral into two separate integrals then applying integration by parts to one
- not selecting $u$ and $v'$ correctly
- incorrectly calculating $v$ when $v' = \cos \pi x$ (typical incorrect responses included $v = \pi \sin \pi x$, $-\frac{1}{\pi}\sin \pi x$ or $\pi \sin x$)

(c) Candidates had little difficulty sketching the region $-\frac{\pi}{4} \le \arg z \le \frac{\pi}{4}$.

Common problems were:

- presenting broken lines throughout the response
- showing correct boundaries but not indicating a region
- not understanding what was meant by the statement: $|z| \le |z-2|$ (many presented circles indicating a confusion with regions such as $|z| \le k$)

- making algebraic errors such as $(x-2)^2 = x^2 - 2x + 4$ when substituting $z = x + iy$.

(d) A wide variety of graphs was presented. The most appropriate responses showed a curve approaching the asymptote $y = x^2$.

Common problems were:

- not considering the significant features of a graph such as intercepts, asymptotes and symmetry
- failing to recognise an even function
- failing to recognise a discontinuity at $x = 0$.

(e) Many correct solutions were preceded by a correct diagram illustrating the axis of rotation and a typical slice or shell. The solutions that demonstrated understanding of the method of using cylindrical shells were generally successful in finding the volume.

A common problem was:

- presenting a rotation about the $y$-axis instead of the $x$-axis.

## Question 12

(a)(i) Those candidates who realised that $|x| = x \ \ for\ x \geq 0$ drew the conclusion that the resulting graph would have to be an even function symmetrical about the $y$-axis where the original portion of the graph of $f(x)$ found in the first quadrant is reflected through the $y$-axis.

A common problem was:

- reflecting that part of the graph below the $x$-axis through the $x$-axis thereby sketching $y = |f(x)|$ instead of $y = f|x|$ .

(a)(ii) Many candidates realised the horizontal asymptote given at $y = 2$ would become a horizontal asymptote at $y = \frac{1}{2}$ in their answer. Many candidates also realised there was an $x\text{-}intercept$ at $-1$ but indicated correctly that the point did not exist by using an open circle at $(-1, 0)$.

A common problem was:

- not clearly labelling the asymptotes nor the intercepts on the axes.

(b)(i) Well answered by most candidates.

(b)(ii) Most students realised that $x = 2\cos\theta$ substituted into the equation leads to $\cos 3\theta = \frac{\sqrt{3}}{2}$. They then proceeded to write $3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}$ and finally concluded that $\theta = \frac{\pi}{18}, \frac{11\pi}{18}$ and $\frac{13\pi}{18}$. Unfortunately they did not proceed to the next step, writing the solutions are $x = 2\ \cos\frac{\pi}{18}, 2\cos\frac{11\pi}{18}$ and $2\cos\frac{13\pi}{18}$.

Common problems were:

- not developing three distinct roots, often writing $x = 2\cos\frac{\pi}{18}$ and $2\cos\frac{-\pi}{18}$ or $2\cos\frac{\pi}{18}$ and $2\cos\frac{37\pi}{18}$ which in both cases are repetitions of the same answer
- making basic arithmetic errors when dividing by 3, to find $\theta$.

(c) Many candidates showed competence in implicit differentiation getting

$\frac{dy}{dx} = \frac{x}{y}$ for the hyperbola $x^2 - y^2 = 5$ and $\frac{dy}{dx} = \frac{-y}{x}$ for $xy = 6$.

A number of students solved the equations simultaneously, getting co-ordinates of $(3, 2)$ and $(-3, -2)$ and proceeded to use these to correctly show the tangents were perpendicular.

A common problem was:

- writing $\frac{x}{y} \times \frac{-y}{x} = -1$ but not making any reference to the co-ordinates of the point P$(x_0, y_0)$ as specified in the question.

(d)(i) Most candidates showed the correct working.

A common problem was:

- writing $\int_0^1 \frac{1}{x^2+1}\, dx = [\tan x]_0^1$ instead of $[\tan^{-1} x]_0^1$.

(d)(ii) The most common approach was writing $I_n + I_{n-1} = \int_0^1 \frac{x^{2n}}{x^2+1} + \frac{x^{2n-2}}{x^2+1}\, dx$, which after factorisation and cancellation reduced to $\int_0^1 x^{2n-2}\, dx = \frac{1}{2n-1}$.

Another method which was usually done well was $\int_0^1 \frac{x^{2n}}{x^2+1}\, dx = \int_0^1 \frac{x^{2n-2}(x^2+1)}{x^2+1}\, dx - \int_0^1 \frac{x^{2n-2}}{x^2+1}\, dx$. This resulted in the statement $I_n = \int_0^1 x^{2n-2}\, dx - I_{n-1}$ which eventually yields the correct answer of $I_n + I_{n-1} = \frac{1}{2n-1}$.

A common problem was:

- using the method of integration by parts, which led to a very convoluted proof (done poorly by all but a very few students).

(d)(iii) This part of the question was done successfully using part (ii), commencing with $I_1 + I_0 = \frac{1}{2\times1-1}$, hence $I_1 = 1 - \frac{\pi}{4}$ and then using $I_2 + I_1 = \frac{1}{2\times2-1}$ which eventually leads to the correct answer.

Some candidates used the method of long division where $\frac{x^4}{x^2+1} = x^2 - 1 + \frac{1}{x^2+1}$ which leads to a correct answer of $\frac{\pi}{4} - \frac{2}{3}$.

A common problem was:

- attempting to evaluate $\int_0^1 \frac{x^4}{x^2+1} dx$ as $I_4$, not $I_2$ as required.

## Question 13

(a) Candidates displayed good knowledge of the identities and were able to show that $dx = \frac{2dt}{1+t^2}$. Most candidates were able to change the limits successfully. Some candidates used their algebraic skills to simplify the integral to obtain $\int_{\frac{1}{\sqrt{3}}}^{1} \frac{2dt}{9t^2+6t+1} = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2dt}{(3t+1)^2}$ and successfully evaluated the integral.

Common problems were:

- not reducing the denominator to $9t^2 + 6t + 1$

- not recognising that the denominator was a perfect square
- not correctly finding the primitive of $(3t + 1)^{-2}$

(b) Common problems were:

- not being able to find the height of the trapezium (common incorrect heights found (using Pythagoras Theorem) included: $\frac{\sqrt{5}y}{2}$, $\frac{\sqrt{3}y}{4}$, $\frac{y}{2}$, $\frac{y}{2\sqrt{2}}$, $\frac{\sqrt{5}x}{2}$, $\frac{\sqrt{3}x}{4}$ and $\frac{x}{2\sqrt{2}}$.)
- not knowing the formula for area of a trapezium (a technique used, often unsuccessfully, for finding the area of the trapezium included dividing the area into a combination of a square and triangles or into three equilateral triangles
- confusing the $x$ and $y$ values
- introducing their own pronumeral for the length of the base.

(c)(i) Almost all candidates gained the mark allocated for this part of the question. The substitution of the coordinates of $P$ into the equation of the hyperbola was the approach adopted by most candidates. A small number of candidates substituted for either $x$ or $y$ and then proceeded to obtain the appropriate expression for the other coordinate or abscissa.

(c)(ii) The majority of candidates performed very well in this section of the question. There were many approaches such as:

- finding the gradient of $PQ$ and the gradient of the tangent at $M$ and stating that they were equal and that $P, M$ and $Q$ were collinear
- finding the equation of the tangent at $M$ and showing that the coordinates of $P$ satisfied the equation and stating that $P, M$ and $Q$ were collinear
- showing that the equations of $PQ$ and the tangent at $M$ were the same
- showing that the point of intersection of the hyperbola and $y = \frac{b}{a}x$ was $P\ (at, bt)$
- showing that the point of intersection of the hyperbola and $y = -\frac{b}{a}x$ was $Q\ \left(\frac{a}{t}, -\frac{b}{t}\right)$.

Common problems were:

- finding the equation for $PQ$ and then substituting into the hyperbola, attempting to show that the resulting quadratic had equal roots (such candidates had little success in showing that $\Delta = 0$)
- finding the equation of $PQ$ and then substituting the coordinates of $M$ into $PQ$ but failing to see that even though $M$ satisfied $PQ$, this was not sufficient to show that $PQ$ was the tangent at $M$.

(c)(iii) Almost all students were able to gain at least one mark in this part.
A common problem was:

- quoting the relationship $b^2 = a^2(1 - e^2)$ rather than $b^2 = a^2(e^2 - 1)$.

One of the most efficient approach approaches was:

$$OP \times OQ = \sqrt{a^2t^2 + b^2t^2} \times \sqrt{\frac{a^2}{t^2} + \frac{b^2}{t^2}}$$
$$= t \times \sqrt{a^2 + b^2} \times \frac{1}{t} \times \sqrt{a^2 + b^2}$$

$$= t \times \sqrt{a^2e^2} \times \frac{1}{t} \times \sqrt{a^2e^2}\ , \qquad \text{since } b^2 = a^2(e^2 - 1)$$
$$= aet \times \frac{ae}{t}$$
$$= a^2e^2$$
$$= OS^2$$

(c)(iv) Most candidates achieved full marks on this part. Candidates who used $\frac{0 - \frac{b[e^2-1]}{2e}}{ae - \frac{a[e^2+1]}{2e}}$ for the slope of $MS$ obtained $-\frac{b[e^2-1]}{a(e^2-1)}$ and avoided considering the expression $\frac{b[e^2-1]}{a(1-e^2)}$, where some failed to see that it simplified to $-\frac{b}{a}$.

Common problems were:

- being unable to derive the expression for the slope of $MS$
- not recognising that that $t = e$.

## Question 14

(a)(i) Most candidates used the multiple root theorem, but an alternative method used was to use long division to prove a multiplicity of 3.

A common problem was:

- not showing that $P(1) = 0$ and $P''(1) = 0$ for a root of multiplicity of 3.

(a)(ii) Most candidates were successful in finding the complex roots. Methods used by candidates included the use of the sum and product of roots, long division or by inspection to find the quadratic $x^2 + 3x + 6 = 0$.

A common problem was:

- including incorrect negative signs in the quadratic equation.

(b)(i) Many candidates used the formula for the angle between two lines, substituted correctly, used algebraic techniques and trigonometric facts to achieve the required result.

Common problems were:

- using the gradient of the tangent instead of the gradient of the normal
- using Cartesian form instead of parametric form
- losing negatives when substituting
- using an incorrect formula.

(b)(ii) In successful responses, candidates recognised that $\sin\theta\cos\theta$ could be replaced by $\frac{1}{2}\sin 2\theta$ and then used the fact that the maximum occurs when $\sin\theta = 1$.

A common problem was:

- using differentiation but not showing that the result was a maximum at $\theta = \frac{\pi}{4}$.

(c)(i) In most responses, candidates quoted the relevant expression for the resolution of forces and substituted $k = \frac{F}{300^2}$ to show that the equation of motion was as stated.

The use of a force diagram would have assisted the candidate in achieving the desired result.

A common problem was:

- making errors in deriving the expression for the resultant force.

(c)(ii) Candidates who successfully completed this part replaced $\ddot{x}$ by $\frac{dv}{dt}$, rearranged the expression, completed the partial fractions, integrated correctly to achieve log expressions and then substituted to achieve the solution.

Common problems were:

- making errors when integrating to achieve logs
- incorrect rearrangements
- incorrect substitution
- poor algebraic manipulation skills.

## Question 15

(a) Most candidates successfully considered the expansion of $(a+b+c)^2$. Although some candidates then moved on and used the condition $a \le b \le c$ to prove the result, many were unable to link these ideas.

Common problems were:

- confusion with inequalities signs
- stating 'replace' or 'substitute' $a$ for $b$ in $2ab$ rather than 'as $a \le b$ then $2a^2 \le 2ab$'
- using incorrect approaches such as considering AM or GM type inequalities eg $a^2 + b^2 \ge 2ab$ leading to $3a^2 + 3b^2 + 3c^2 \ge 1$

(b)(i) Candidates mostly used de Moivre's theorem to show this result although a number unsuccessfully attempted mathematical induction.

Common problems were:

- careless algebraic errors due to poor notation, lack of brackets and incorrect translation of cis notation to $\cos\theta + i\sin\theta$.
- leaving $(1-i)^n$ as $\left(\sqrt{2}\right)^n\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$

(b)(ii) Candidates struggled with this part. Many who successfully used de Moivre's theorem in (b)(i) unsuccessfully attempted mathematical induction in this part. Candidates rarely linked this part to part (b)(i) by considering the real parts only in the expansion of $(1+i)^n + (1-i)^n$.

Common problems were:

- poor use of notation
- not showing enough terms in their expansions of $(1+i)^n$ and $(1-i)^n$ to establish the pattern of terms with odd and even powers of $i$
- not explaining carefully when considering the real parts of the expansion and the imaginary parts to be discarded

- not dealing with $i^n$ carefully to find the last term of $\binom{n}{n}$ in each expansion.

(c)(i) While many candidates were very efficient in their algebraic reasoning, first writing $T\cos^2\phi$ as $\frac{mv^2}{l}$ before eliminating $T$ from their equations, others used convoluted algebraic processes to successfully show the result.

Common problems were:

- using incorrect signs on the forces, for example $T\sin\phi = mg - kv^2$
- commencing with $T\cos\phi = mr\omega^2$ which although correct, complicated the reasoning
- squaring $T\cos\phi = \frac{mv^2}{r}$, which made eliminating $T$ from the equations more difficult
- using incorrect equations transposing $\sin\phi$ and $\cos\phi$, or stating $r = l\sin\phi$ or $r = l$ or $T = l$.

(c)(ii) Many candidates tried to link this part to the previous part, substituting $\frac{lk}{m} - \frac{lg}{v^2}$ for $\frac{\sin\phi}{\cos^2\phi}$ rather than using the implication as given in the question to derive the quadratic inequality.

Common problems were:

- using incorrect signs, including incorrect inequality signs
- not giving a convincing argument as to why $\sin\phi$ is between the roots, rather stating the required inequality as $\sin\phi > 0$

(c)(iii) Most candidates used a calculus approach and although many were successful, a significant number were unable to find the correct derivative.

Common problems were:

- using a variety of equivalent expressions such as $\sin\phi\sec^2\phi$ for $\frac{\sin\phi}{\cos^2\phi}$ and subsequently finding the incorrect derivative or not being able to show that the derivative is positive in an equivalent form
- simply stating that the derivative is positive without justification
- showing that stationary points exist at $\phi = \pm\frac{\pi}{2}$ (often not realising the function and derivative are not defined for these values), sometimes with a correct justification as to why the function is increasing within the domain
- attempting to prove the result graphically

(c)(iv) This part was done poorly with candidates unable to link the previous parts of the question to form a valid mathematical argument.

Common problems were:

- using a physical description of velocity being the upward lifting force, ignoring the downward component (of $T$) also involving velocity, so that if $v$ increases the plane moves up and $\phi$ increases.
- confusing increasing/decreasing with limits.

## Question 16

(a)(i) Most candidates applied the theorem: the angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.

(a)(ii)The majority of candidates struggled with this part.

Common problems were:

- incorrectly naming angles
- using fallacious reasoning by implicitly assuming *APC* or *DPB* to be on a straight line to show that points *A*, *P* and *C* are collinear – for example, some candidates claimed angles *RPC* and *APX* to be vertically opposite equal angles before 'showing' *A*, *P* and C are collinear.

(a)(iii) Most candidates correctly identified (with reasoning) equal angles in the same segment to show that *ABCD* is a cyclic quadrilateral.
A common problem was:

- incorrect or no geometric reasoning.

(b)(i) Many candidates did not attempt this part.

Common problems were:

- not correctly simplifying the middle term
- making errors in calculating the geometric sum.

(b)(ii) In better responses, candidates integrated throughout the inequality in part (b)(i) using the correct limits of 0 and 1. Less successful responses found primitives in the inequality followed by substituting $x$ with 1.

(b)(iii) In successful responses, candidates noted that $\frac{\pm 1}{2n+1} \to 0$ as $n \to \infty$.

(c) Many candidates struggled with this part. The majority of candidates attempted the question by making a substitution without reaching a final answer. There were a variety of successful responses which typically proceeded from a substitution to an eventual integration by parts.