BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2014 HSC Mathematics Extension 1
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	D
2	A
3	C
4	D
5	B
6	B
7	A
8	D
9	C
10	C

Section II

Question 11 (a)

Criteria	Marks
• Provides a correct solution	3
• Obtains a quadratic in x, or equivalent merit	2
• Attempts to solve quadratic in $x + \frac{2}{x}$, or equivalent merit	1

Sample answer:

$$\text{Let } n = x + \frac{2}{x}$$

$$n^2 - 6n + 9 = 0$$

$$(n - 3)^2 = 0 \qquad \therefore n = 3$$

$$\text{Now, } x + \frac{2}{x} = 3$$

$$x^2 - 3x + 2 = 0$$

$$(x - 2)(x - 1) = 0$$

$$x = 2 \text{ or } x = 1$$

Question 11 (b)

Criteria	Marks
• Provides a correct solution	2
• Finds a correct expression for the probability of rain on 1 or 2 days, or equivalent merit	1

Sample answer:

$P(\text{rain}) = 0.1 \qquad P\left(\widetilde{\text{rain}}\right) = 1 - 0.1 = 0.9$

Probability that it rains on fewer than 3 days is

$$\binom{30}{0}(0.9)^{30} + \binom{30}{1} \times 0.9^{29} \times 0.1 + \binom{30}{2} 0.9^{28} (0.1)^2$$

Question 11 (c)

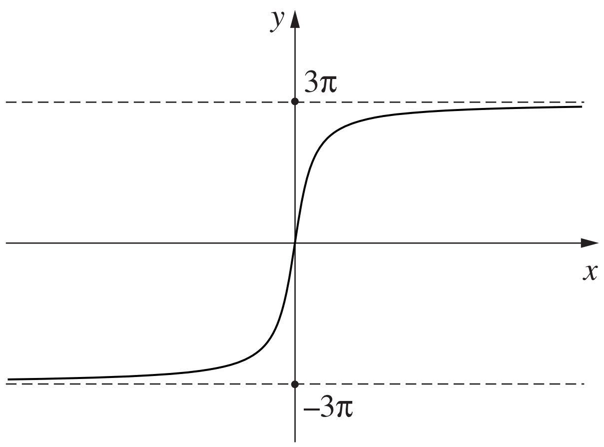
Criteria	Marks
• Draws a correct sketch	2
• Draws a graph showing the correct range or shape, or equivalent merit	1

Sample answer:

$y = 6\tan^{-1} x$

$$\frac{-\pi}{2} < \frac{y}{6} < \frac{\pi}{2}$$

$$-3\pi < y < 3\pi$$



Question 11 (d)

Criteria	Marks
• Provides a correct solution	3
• Obtains a correct primitive, or equivalent merit	2
• Attempts the given substitution, or equivalent merit	1

Sample answer:

$$\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$$

$x = u^2 + 1 \quad | \quad x = 2 \quad u = 1$

$dx = 2u\,du \quad | \quad x = 5 \quad u = 2$

$$\int_1^2 \frac{u^2+1}{\cancel{u}}\,2\cancel{u}\,du$$

$$= 2\left[\frac{u^3}{3} + u\right]_1^2$$

$$= 2\left[\frac{8}{3} + 2 - \frac{1}{3} - 1\right]$$

$$= 2\left[\frac{10}{3}\right]$$

$$= \frac{20}{3}$$

Question 11 (e)

Criteria	Marks
• Provides a correct solution	3
• Obtains one correct interval, or equivalent merit	2
• Obtains a quadratic inequality, or equivalent merit	1

Sample answer:

Multiply by x^2:

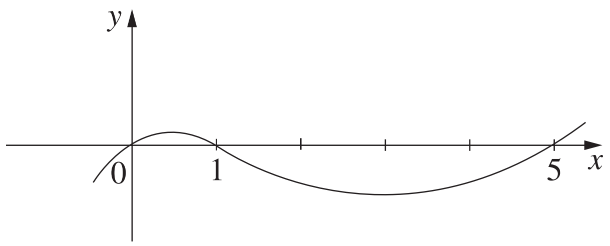
$$\frac{x^2\left(x^2+5\right)}{x} > 6 \times x^2$$

$$\text{ie } x\left(x^2+5\right) > 6x^2$$

$$x\left(x^2+5\right) - 6x^2 > 0$$

$$x\left(x^2-6x+5\right) > 0$$

$$x(x-5)(x-1) > 0$$



$\therefore 0 < x < 1$ or $x > 5$

Question 11 (f)

Criteria	Marks
• Finds the correct derivative	2
• Uses an appropriate differentiation rule, or equivalent merit	1

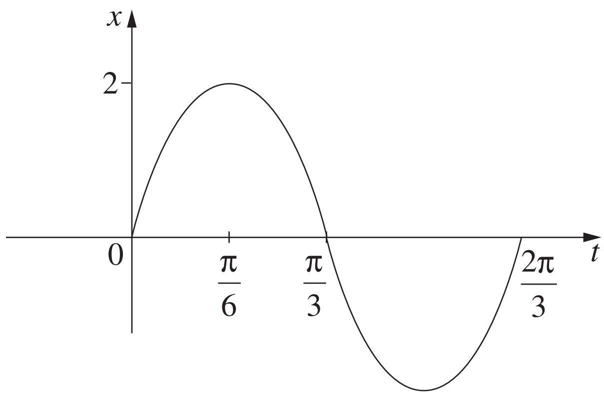
Sample answer:

$$y = \frac{e^x \ln x}{x}$$

$$\frac{dy}{dx} = \frac{x\left(e^x.\frac{1}{x} + e^x \ln x\right) - e^x \ln x.1}{x^2}$$

$$= \frac{x\left(\frac{e^x}{x} + e^x \ln x\right) - e^x \ln x}{x^2}$$

Question 12 (a)



Question 12 (a) (i)

Criteria	Marks
• Provides a correct answer	1

Sample answer:

Distance travelled is (2 + 2) m, ie 4 m.

Question 12 (a) (ii)

Criteria	Marks
• Provides a correct solution	2
• Identifies when the particle is first at rest, or equivalent merit	1

Sample answer:

When at rest,

$\dot{x} = 0$ ie $6\cos 3t = 0$

$\therefore \cos 3t = 0$

$3t = \dfrac{\pi}{2}$

$t = \dfrac{\pi}{6}$

$\therefore$ First at rest when $t = \dfrac{\pi}{6}$

$$\begin{aligned} \text{Now, } x &= 2\sin 3t \\ \dot{x} &= 6\cos 3t \\ \ddot{x} &= -18\sin 3t \end{aligned}$$

when $t = \dfrac{\pi}{6}$

$$\begin{aligned} \ddot{x} &= -18\sin\frac{\pi}{2} \\ &= -18 \times 1 \end{aligned}$$

acceleration $= -18\ \text{m/s}^2$

Question 12 (b)

Criteria	Marks
• Provides a correct solution	3
• Obtains correct integral expression for volume in terms of $\cos 8x$, or equivalent merit	2
• Obtains integral expression for volume of the form $k\int_0^{\frac{\pi}{8}} \cos^2 4x \;\, dx$, or equivalent merit	1

Sample answer:

$$y = \cos 4x$$

$$V = \int_0^{\frac{\pi}{8}} \pi y^2 \, dx$$

$$= \int_0^{\frac{\pi}{8}} \pi \cos^2 4x \, dx$$

$$= \pi \int_0^{\frac{\pi}{8}} \frac{1 + \cos 8x}{2} \, dx$$

$$= \frac{\pi}{2}\left[x + \frac{\sin 8x}{8}\right]_0^{\frac{\pi}{8}}$$

$$= \frac{\pi}{2}\left[\frac{\pi}{8} + 0 - 0 - 0\right]$$

$$\text{volume} = \frac{\pi^2}{16} \text{ unit}^3$$

Question 12 (c)

Criteria	Marks
• Provides a correct solution	3
• Obtains expression for v^2 possibly involving an undetermined constant	2
• Attempts to use $a = \frac{1}{2}\frac{d\left(v^2\right)}{dx}$ or equivalent merit	1

Sample answer:

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = \ddot{x} = 2 - e^{\frac{-x}{2}}$$

$$\frac{1}{2}v^2 = \int 2 - e^{-\frac{x}{2}}\,dx$$

$$= 2x - \frac{e^{\frac{-x}{2}}}{-\frac{1}{2}} + c$$

$$= 2x + 2e^{\frac{-x}{2}} + c$$

When $x = 0, v = 4$

$$\frac{1}{2}(16) = 0 + 2 + c$$

$$c = 6$$

$$\therefore \frac{1}{2}v^2 = 2x + 2e^{-\frac{x}{2}} + 6$$

$$v^2 = 4x + 4e^{\frac{-x}{2}} + 12$$

Question 12 (d)

Criteria	Marks
• Provides a correct solution	2
• Attempts to use the binomial theorem.	1

Sample answer:

Consider $(1+x)^n$. From the binomial theorem:

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \ldots + \binom{n}{n}x^n$$

Let $x = -1$

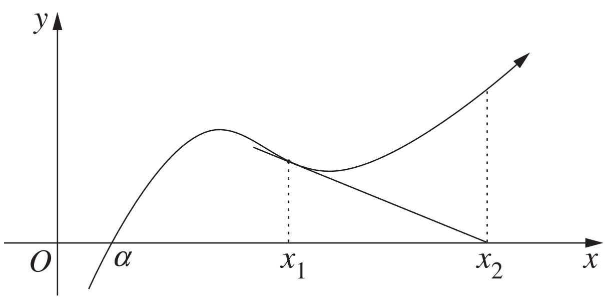
$$(1+-1)^n = 0 = \binom{n}{0} + \binom{n}{1}(-1) + \binom{n}{2}(-1)^2 + \ldots + \binom{n}{n}(-1)^n$$

$$0 = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \ldots + (-1)^n\binom{n}{n}$$

Question 12 (e)

Criteria	Marks
• Provides a valid explanation	1

Sample answer:



The tangent at $x = x_1$ is drawn, and it intersects the x axis at x_2.

As shown in the diagram, x_2 is further away from α than x_1.

Question 12 (f)

Criteria	Marks
• Provides a correct solution	3
• Obtains A and B, or equivalent merit	2
• Shows $A - B = 2$, or equivalent merit	1

Sample answer:

$$T = A - Be^{-0.03t}$$

As $t \to \infty$, $\quad T \to 23$

$\therefore A = 23$

$$\therefore T = 23 - Be^{-0.03t}$$

$t = 0 \qquad T = 2$

$$2 = 23 - B$$

$$B = 21$$

$\therefore T = 23 - 21e^{-0.03t}$

Now, when $T = 0$

$$10 = 23 - 21e^{-0.03t}$$

$21e^{-0.03t} = 13$

$$e^{-0.03t} = \frac{13}{21}$$

$$-0.03t = \ln\left(\frac{13}{21}\right)$$

$$t = \frac{\ln\left(\frac{13}{21}\right)}{-0.03}$$

$$= 15.985\ldots !$$

$$t \approx 16 \text{ minutes}$$

Question 13 (a)

Criteria	Marks
• Provides a correct proof	3
• Uses inductive assumption, or equivalent merit	2
• Establishes case for $n = 1$, or equivalent merit	1

Sample answer:

$2^n + (-1)^{n+1}$

$n = 1$, $2^1 + (-1)^2 = 2 + 1 = 3$ which is divisible by 3

Assume true for $n = k$

ie Assume $2^k + (-1)^{k+1}$ is divisible by 3

ie $2^k + (-1)^{k+1} = 3M$ where M is an integer.

We need to prove true for $n = k + 1$

ie That $2^{k+1} + (-1)^{k+2}$ is divisible by 3

$$\begin{aligned} 2^{k+1} + (-1)^{k+2} &= 2\left(2^k\right) + (-1)^{k+2} \\ &= 2\left[3M + (-1)^{k+2}\right] + (-1)^{k+2} \\ &= 3\left[2M + (-1)^{k+2}\right] \end{aligned}$$

$$\begin{aligned} 2^k &= 3M - (-1)^{k+1} \\ &= 3M + (-1)^{k+2} \end{aligned}$$

Which is a multiple of 3 since M and k are integers.

ie if $2^k + (-1)^{k+1}$ is divisible by 3, then

$2^{k+1} + (-1)^{k+1}$ is divisible by 3

$\therefore$ By the principle of mathematical induction

$2^n + (-1)^{n+1}$ is divisible by 3 for all $n \geq 1$.

Question 13 (b) (i)

Criteria	Marks
• Provides a correct solution	2
• Uses Pythagoras' Theorem and attempts to differentiate, or equivalent merit	1

Sample answer:

$$
\begin{aligned}
L &= \sqrt{40^2 + x^2} \\
\frac{dL}{dx} &= \frac{1}{2}.\frac{1}{\sqrt{40^2 + x^2}}.2x \\
&= \frac{x}{\sqrt{40^2 + x^2}} \\
&= \frac{x}{L} \\
&= \cos\theta
\end{aligned}
$$

Question 13 (b) (ii)

Criteria	Marks
• Provides a correct solution	1

Sample answer:

$$
\begin{aligned}
\frac{dL}{dt} &= \frac{dL}{dx}\cdot\frac{dx}{dt} \\
&= \frac{dL}{dx}\cdot 3 \\
&= 3\cos\theta \text{ (from part (i)).}
\end{aligned}
$$

Question 13 (c) (i)

Criteria	Marks
• Provides a correct solution	2
• Establishes one coordinate, or equivalent merit	1

Sample answer:

$P\left(2at, at^2\right) \quad S(0, a)$

$t^2 : 1$

$$Q = \left[\frac{0 \times t^2 + 2at}{t^2 + 1}, \frac{a \times t^2 + 1 \times at^2}{t^2 + 1}\right]$$

$$Q = \left[\frac{2at}{t^2 + 1}, \frac{2at^2}{t^2 + 1}\right]$$

Question 13 (c) (ii)

Criteria	Marks
• Provides a correct solution	1

Sample answer:

$$m_{OQ} = \frac{\dfrac{2at^2}{t^2+1} - 0}{\dfrac{2at}{t^2+1} - 0}$$

$$= \frac{2at^2}{t^2+1} \times \frac{t^2+1}{2at}$$

$$= t$$

Question 13 (c) (iii)

Criteria	Marks
• Provides a correct solution	3
• Obtains an equation in x, y and a	2
• Attempts to use part (ii), or equivalent merit	1

Sample answer:

$$x = \frac{2at}{t^2+1} \qquad y = \frac{2at^2}{t^2+1}$$

$m_{OQ} = t = \dfrac{y}{x}$ (from part (ii))

$$\therefore x = \frac{2a\dfrac{y}{x}}{\dfrac{y^2}{x^2}+1}$$

$$x = \frac{2axy}{y^2+x^2}$$

$$\cancel{x}\left(x^2+y^2\right) = 2a\cancel{x}y, \text{ for } x \neq 0$$

$$x^2+y^2 = 2ay$$

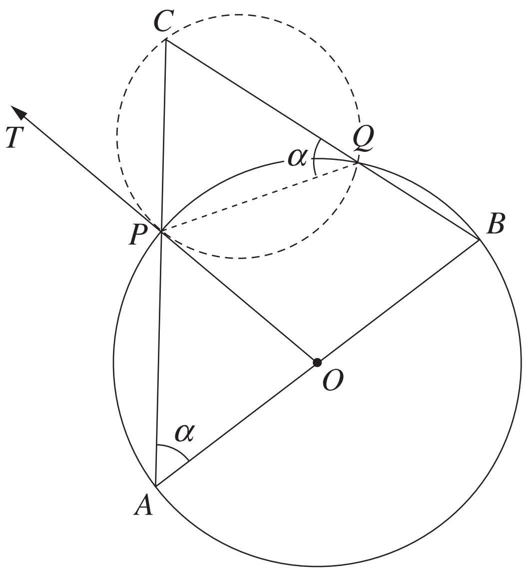
$$x^2+y^2-2ay = 0$$

$$x^2+y^2-2ay+a^2 = a^2$$

$$x^2+(y-a)^2 = a^2$$

which is a circle of centre $(0,a)$ and radius a units and point Q lies on this circle.

Question 13 (d)



Question 13 (d) (i)

Criteria	Marks
• Provides a correct explanation	1

Sample answer:

$\angle BAC = \angle CQP$

($ABQP$ is a cyclic quadrilateral; exterior angle equals interior opposite angle)

Question 13 (d) (ii)

Criteria	Marks
• Provides a correct proof	2
• Observes that $\triangle OPA$ is isosceles, or equivalent merit	1

Sample answer:

From (i) $\angle OAP = \angle PQC = \alpha$

$\angle OAP = \angle OPA = \alpha$ ($OA = OP$ radii; angles opposite equal sides of $\triangle OAP$ are equal)

$\angle OPA = \angle TPC = \alpha$ (vertically opposite angles equal)

$\therefore \angle TPC = \angle PQC = \alpha$

$\therefore$ OP is a tangent to circle through P, Q, C, since the angle between a tangent (OP) and a chord (CP) is equal to the angle in the alternate segment.

Question 14 (a) (i)

Criteria	Marks
• Provides a correct solution	2
• Attempts to eliminate t, or equivalent merit	1

Sample answer:

$$x = Vt\cos\theta, y = -\frac{1}{2}gt^2 + Vt\sin\theta$$

$$t = \frac{x}{V\cos\theta}$$

$$y = -\frac{1}{2}g \cdot \frac{x^2}{V^2\cos^2\theta} + V.\frac{x}{V\cos\theta}.\sin\theta$$

$$= \frac{-gx^2}{2V^2}\sec^2\theta + x\tan\theta$$

Question 14 (a) (ii)

Criteria	Marks
• Provides a correct solution	3
• Obtains the x coordinate of P, or equivalent merit	2
• Obtains an equation in x, or equivalent merit	1

Sample answer:

P is the point of intersection of

$$y = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta \quad \text{and } y = -x$$

$$\therefore -x = x\tan\theta - \frac{gx^2}{2V^2}\sec^2\theta$$

$$x = \frac{gx^2}{2V^2}\sec^2\theta - x\tan\theta$$

$$x(1+\tan\theta) = \frac{gx^2}{2V^2}\sec^2\theta$$

$$x\left[\frac{gx\sec^2\theta}{2V^2} - (1+\tan\theta)\right] = 0$$

$$\therefore x = 0 \text{ or } \quad \frac{gx\sec^2\theta}{2V^2} = 1+\tan\theta$$

$$\frac{gx}{2V^2\cos^2\theta} = 1 + \frac{\sin\theta}{\cos\theta}$$

$$x = \frac{2V^2}{g}\cos^2\theta + \frac{2V^2}{g}\frac{\sin\theta.\cos^2\theta}{\cos\theta} \qquad (\cos\theta \neq 0)$$

$$= \frac{2V^2}{g}\cos\theta(\cos\theta + \sin\theta)$$

Since $D^2 = 2x^2$, $x = \dfrac{D}{\sqrt{2}}$

$$\therefore \frac{D}{\sqrt{2}} = \frac{2V^2}{g}\cos\theta(\sin\theta + \cos\theta)$$

$$D = 2\sqrt{2}\frac{V^2}{g}\cos\theta(\sin\theta + \cos\theta)$$

Question 14 (a) (iii)

Criteria	Marks
• Provides a correct solution	2
• Uses double-angle formulae, or equivalent merit	1

Sample answer:

$$D = 2\sqrt{2}\frac{V^2}{g}\left[\cos^2\theta + \sin\theta\cos\theta\right]$$

$$= 2\sqrt{2}\frac{V^2}{g}\left[\frac{1+\cos 2\theta}{2} + \frac{1}{2}\sin 2\theta\right]$$

$$= \frac{2\sqrt{2}V^2}{2g}\left[1 + \cos 2\theta + \sin 2\theta\right]$$

$$\frac{dD}{d\theta} = \sqrt{2}\frac{V^2}{g}\left[-2\sin 2\theta + 2\cos 2\theta\right]$$

$$= 2\sqrt{2}\frac{V^2}{g}\left[\cos 2\theta - \sin 2\theta\right]$$

Question 14 (a) (iv)

Criteria	Marks
• Provides a correct solution	3
• Obtains $\theta = \frac{\pi}{8}$, or equivalent merit	2
• Attempts to solve $\frac{dD}{d\theta} = 0$, or equivalent merit	1

Sample answer:

D has a maximum value when $\frac{dD}{d\theta} = 0$

ie $\sin 2\theta = \cos 2\theta$

$$\tan 2\theta = 1$$

$$2\theta = \frac{\pi}{4}$$

$$\theta = \frac{\pi}{8}$$

$$\frac{d^2D}{d\theta^2} = 2\sqrt{2}\frac{V^2}{g}(-2\sin 2\theta - 2\cos 2\theta)$$

$$= -4\sqrt{2}\frac{V^2}{g}(\sin 2\theta + \cos 2\theta)$$

for $\theta = \frac{\pi}{8}$, $\sin 2\theta + \cos 2\theta > 0$

$$\therefore \frac{d^2D}{d\theta^2} < 0$$

$\therefore D$ is maximum when $\theta = \frac{\pi}{8}$

Question 14 (b) (i)

Criteria	Marks
• Provides a correct solution	2
• Shows probability that A wins on second turn is rq, or equivalent merit	1

Sample answer:

$$\begin{aligned}
P(A \text{ wins on 1st or 2nd}) &= p + rq \\
&= p + r\big(1 - (p + r)\big) \\
&= p + r - pr - r^2 \\
&= p(1 - r) + r(1 - r) \\
&= (1 - r)(p + r)
\end{aligned}$$

Question 14 (b) (ii)

Criteria	Marks
• Provides a correct solution	3
• Observes limiting probability is a geometric series, or equivalent merit	2
• Finds probability that A wins on third turn, or equivalent merit	1

Sample answer:

$$\begin{aligned}
P(A \text{ wins eventually}) &= p + rq + r^2 p + r^3 q + \ldots \\
&= p\left(1 + r^2 + r^4 + \ldots\right) + rq\left(1 + r^2 + r^4 + \ldots\right) \\
&= (p + rq)\left(1 + r^2 + r^4 + \ldots\right) \\
&= (p + rq) \times \frac{1}{1 - r^2} \\
&= \cancel{(1 - r)}\,(p + r) \times \frac{1}{(1 + r)\,\cancel{(1 - r)}} \qquad \text{(using part(i))} \\
&= \frac{p + r}{1 + r}
\end{aligned}$$

Mathematics Extension 1

2014 HSC Examination Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	2.8	PE3
2	1	5.7	PE2
3	1	17.3	HE3
4	1	6.6	PE6, PE2
5	1	16.3	PE3
6	1	15.5	HE4
7	1	14.4	HE3
8	1	18.1	PE3
9	1	16.2	PE3
10	1	4.3	PE2, P4

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	3	9.4	PE3, P4
11 (b)	2	18.2	HE3
11 (c)	2	15.3	HE4
11 (d)	3	11.5	HE6
11 (e)	3	1.4E	PE3
11 (f)	2	8.8, 12.5	P7, H3, PE5
12 (a) (i)	1	14.4	HE3
12 (a) (ii)	2	14.4	HE3
12 (b)	3	11.4, 13.6	H8
12 (c)	3	14.3E	HE5
12 (d)	2	17.3E	HE3, HE7
12 (e)	1	16.4	HE7, PE6
12 (f)	3	14.2E	HE3
13 (a)	3	7.4	HE2, PE6
13 (b) (i)	2	5.1, 8.8	P4, P7, PE6
13 (b) (ii)	1	14.1E	HE5, PE6
13 (c) (i)	2	6.7E, 9.6	PE3
13 (c) (ii)	1	9.6	PE3
13 (c) (iii)	3	9.6	PE3
13 (d) (i)	1	2.10	PE3, PE6
13 (d) (ii)	2	2.9, 2.10	PE3, PE6
14 (a) (i)	2	14.3E	HE3

Question	Marks	Content	Syllabus outcomes
14 (a) (ii)	3	14.3E	HE3
14 (a) (iii)	2	14.3, 5.7, 13.7	HE3, H5
14 (a) (iv)	3	5.2, 10.6	HE3, H5
14 (b) (i)	2	3.3	H5, HE7, PE6
14 (b) (ii)	3	3.3	H5, HE7, PE6