2015 Notes from the Marking Centre – Mathematics Extension 2

Introduction

This document has been produced for the teachers and candidates of the Stage 6 Mathematics Extension 2 course. It contains comments on candidate responses to the 2015 Higher School Certificate examination, highlighting their strengths in particular parts of the examination and indicating where candidates need to improve.

This document should be read along with:

- the *Mathematics Extension 2 Stage 6 Syllabus*
- the 2015 Higher School Certificate Mathematics Extension 2 examination
- the marking guidelines
- *Advice for HSC students about examinations*
- other support documents developed by the Board of Studies, Teaching and Educational Standards NSW to assist in the teaching and learning of Mathematics in Stage 6.

Question 11

(b)(ii) A common problem was:

- not taking into account that z was in the second quadrant and finding $arg(z) = -\pi/6$ instead of $arg(z) = 5\pi/6$

(b)(iii) Almost all candidates realised that $arg(z/w) = arg(z) - arg(w)$.

A common problem was:

- finding z/w in mod– arg form and not explicitly giving the value of $arg(z/w)$.

(d) A common problem was:

- not labelling the x- and y-intercepts.

(e) A common problem was:

- making errors when making dy/dx the subject or in the numerical evaluation.

(f) The simplest method was to use the 't' formulae. Another common method was to show that $\cot\theta + \operatorname{cosec}\theta = (\cos\theta + 1)/\sin\theta$ and then to use the fact that $\cos\theta = 2\cos^2(\theta/2) - 1$ and that $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$.

A common problem was:

- not stating that $t = \tan(\theta/2)$ when using the 't' formulae.

(f)(ii) This part was found to be more challenging than part (i)

Most of the candidates who did not successfully complete part (i) still attempted this part.

Common problems were:

- not realising that $a = 2$ when giving an answer in the form $a \ln|\sin(\theta/2)| + c$
- omitting the absolute value signs.

Question 12

(a) Common problems were:

- drawing very small diagrams, showing no scale or markings on the axes
- drawing separate diagrams (the instructions were to draw on the same Argand diagram), making the common attributes of the three complex numbers difficult to see.

(i)
In better responses, candidates showed the argument of $\frac{\pi}{4}$ and the modulus of 2 clearly marked on the vector. Some plotted *z* at the point $(\sqrt{2}, \sqrt{2}i)$, or stated the Cartesian form $\sqrt{2} + i\sqrt{2}$ or Mod-arg form: 2 $cis\,\frac{\pi}{4}$ or clearly showed the point on the circle radius 2 units from the origin.

A common problem was:

- showing only a point in the first quadrant of the plane with no markings on the point or the axes or the location of $\frac{\pi}{4}$, so its position could not be confirmed

(ii) The better responses had a scale on the imaginary (*y*) axis, with u ($= z^2$) clearly marked at $4i$ or (0, 4).

(iii) This was the most challenging of the three Argand diagrams.

The geometrical link between parts (i) and (ii) was usually evident in responses which used a single diagram. Responses in which geometry was used, showing all three points with required conjugates and negatives, were generally successful.

In responses using separate graphs, calculation errors were often made in finding the correct co-ordinates of *v*, $-\sqrt{2} + (4 + \sqrt{2})i$.

(b)(i) The better responses showed that the roots occur in conjugate pairs $(a \pm ib)$ and $(a \pm 2ib)$ and in adding the four roots the value of $a = 1$ was found. This then led to the values of *b*, and then to the four final roots.

Common problems were:

- simply writing a = 1 with no explanation of the method(s) used
- attempting to find the product of the roots first, creating some very difficult algebra that was often incomplete
- not answering the question, but stating the factors of the polynomial instead.

(ii) This question was answered well by candidates who had stated the four roots in part (i)

Most found one factor as required, but then, unnecessarily continued to find the second factor. Many minor errors occurred in the algebraic process.

Some candidates who were not successful in finding the roots demonstrated that, by using $a - ib$ and $a + ib$ as roots and multiplying the relevant linear factors, leads to the expression $x^2 - 2ax + a^2 + b^2$ which is a quadratic polynomial with real co-efficients.

(c)(i) Most candidates found that the most direct method was to perform long division to find the oblique asymptote. Some candidates successfully used partial fractions (a more difficult approach) or a rearrangement of the numerator.

A common problem was:

- having completed the division correctly, not recognising that the quotient x – 6 was the part required to form the equation of the asymptote y = x – 6.

(ii) This part was attempted well by the candidates who had successfully found the equation of the oblique asymptote in part (i) and included the vertical asymptote at $x = 1$.

Common problems were:

- for candidates who could not identify the correct oblique asymptote, difficulty with the limits as their graph approached incorrect asymptotes
- not providing the required information on diagrams, as required by the instruction ‘clearly indicating all intercepts and asymptotes’.

(d) Well answered by most candidates, who could state the radius of the typical shell as $r = 3 - x$, then use area and then the volume of the shell to obtain the correct integral.

With the actual integration, the better responses immediately used a substitution of either $u = x + 1$ or $u^2 = x + 1$.

Common problems were:

- difficulty with a negative answer when x – 3 was used as the radius of the cylindrical shell
- omitting (or losing in the process) the 2π multiple
- using other substitutions or a number of other methods which involved much more complex algebra often led to errors in working.

Question 13

(a)(i) This question was attempted well. Most responses included the substitution of the co-ordinates of Q into the equation of H_2. -1 .
Another successful technique was to substitute $x = a\tan\theta$ into the equation of H_2 to confirm $y = b\sec\theta$.

(ii) In the better responses, candidates calculated the gradient and simplified it to $-\frac{b}{a}$ before using the ‘point-gradient’ equation or the ‘gradient-intercept’ equation of a straight line.

A common problem was:

- difficulty reaching the target equation (due to algebraic errors) when $m_{PQ} = \frac{b\tan\theta - b\sec\theta}{a\sec\theta - a\tan\theta}$ was not simplified before attempting to use it.

(iii) There were a great variety of techniques presented for this question.

Using the given information, many responses included calculations of the length of the interval PQ and the perpendicular distance from O to PQ. These provided the length of the base and the perpendicular height of triangle OPQ.

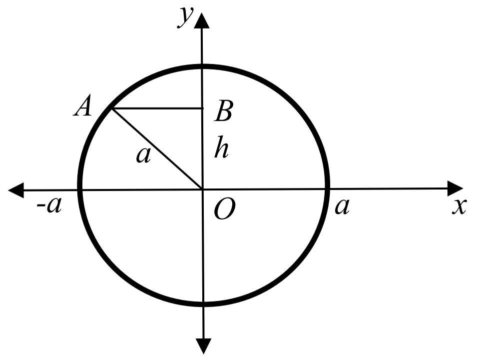
Candidates who showed recognition that $\sqrt{(a\sec\theta - a\tan\theta)^2 + (b\tan\theta - b\sec\theta)^2} = \sqrt{(a^2+b^2)(\sec\theta - \tan\theta)^2}$ were usually successful in achieving the desired result.

Another successful method included finding the difference between areas of ΔOQX and ΔOPX (where X is the x-intercept of the line PQ). Other similar methods were also used quite successfully.

Common problems were:

- not having a diagram to support a solution or method
- in the simplification of the expression for the length of the interval PQ
- difficulty in the algebra when attempting to evaluate OP, OQ and $\angle QOP$
- making the incorrect assumption that ΔOPQ was isosceles and incorrectly used the distance from O to the midpoint of PQ as the perpendicular height.

(b)(i) Responses that recognised using part or all of the circle $x^2 + y^2 = a^2$ were often successful in showing, using either algebra or a diagram, the desired result.



(ii) Recognising that AB^2 was required, and integrating with respect to h (from $h = 0$ to $h = a$) led to the simple integral $\int_0^a \left(a^2 - h^2\right) dh$ to evaluate.

(c)(i) The chain rule $\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$, where $S = 4\pi r^2$ and $\frac{dS}{dt} = \left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$, led to many successful responses.

Implicit differentiation was also often successfully used.

(ii) The chain rule $\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$, where $V = \frac{4}{3}\pi r^3$ and using the expression for $\frac{dr}{dt}$ from part (i) was used by most candidates to show $\frac{dV}{dt} = \frac{r}{2}\left(\frac{4}{3}\pi\right)^{1/3}$.

In the successful responses, candidates expressed this in the form $\frac{dV}{dt} = \frac{1}{2}\left(\frac{4}{3}\pi r^3\right)^{1/3}$ by showing an understanding that $r = \left(r^3\right)^{1/3}$.

Implicit differentiation was sometimes successfully used.

(iii) In responses where candidates rearranged $\frac{dV}{dt} = \frac{1}{2}V^{1/3}$ and then integrated $\frac{dt}{dV} = 2V^{-\frac{1}{3}}$ were generally successful in progressing towards the correct solution.

Question 14

(a)(i) Most candidates differentiated the expression successfully.

Common problems were:

- errors when using the chain rule, for example:
  $u = \sin^{n-1}\theta \Rightarrow \frac{du}{d\theta} = (n-1)\sin^{n-2}\theta$ or $\frac{du}{d\theta} = (n-1)\cos^{n-2}\theta$
- incorrect signs.

(ii) Majority of the candidates who used the correct expression from part (i) completed the solution successfully.

Other methods to achieve the required result included using a rearrangement and integration by parts.

In a significant number of responses, candidates split the integral into two separate integrals before applying integration by parts to one. For example:

$\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta = \int_0^{\frac{\pi}{2}} (\sin^{n-2}\theta - \cos^2\theta \sin^{n-2}\theta) d\theta$ or $\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta = \int_0^{\frac{\pi}{2}} \sin\theta \times \sin^{n-1}\theta d\theta$.

Common problems were:

- not showing the limits of the definite integral of the derivative ie
  $\left[\sin^{n-1}\theta\cos\theta\right]_0^{\frac{\pi}{2}} = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta d\theta - n\int_0^{\frac{\pi}{2}} \sin^n\theta d\theta$
- making an assumption that the definite integral was zero
- not selecting u and dv correctly or giving incorrect integration (primitives)
- using mathematical induction.

(iii) Many candidates used the recurrence formula to complete this part without having successfully completing parts (i) and (ii).

(b)(i) This part was attempted very well by the majority of the candidates.

(ii) This part was attempted successfully by many of the candidates.

The more successful method involved substituting the roots into the original polynomial equation, then adding the three equations to achieve the result ie $q = 3$ in minimal steps.

Candidates who chose to expand $(\alpha + \beta + \gamma)^3$ met with varying degrees of success.

(iii) Some candidates adopted a similar method used in part (ii) by substitution and/or addition and they were very successful.

For example $\alpha^3 = p\alpha - q$ becomes $\alpha^4 = p\alpha^2 - q\alpha$ etc

Many candidates chose to expand $\left(\alpha^2 + \beta^2 + \gamma^2\right)^2$ in order to achieve $\alpha^4 + \beta^4 + \gamma^4$ and were not as successful, particularly those who showed poor algebraic skills.

(c)(i) Candidates who resolved forces horizontally and vertically usually were successful in achieving the required result.

Division of the two equations achieved the desired result with least errors.

A common problem was:

- errors made with negative signs when using other methods of combining the two equations

(ii) Many candidates substituted and achieved an equation for μ in terms of $\tan\theta$.

Justifying why $\mu < 1$ appeared to be challenging. Some candidates assumed $\frac{\tan\theta + \mu}{1 - \mu\tan\theta} < 1$ and concluded that $\mu < 1$.

Question 15

(a)(i) The majority of candidates used the differential equation $\ddot{x} = -k\dot{x}^2$ to derive the required equation.

In the less successful responses, candidates experienced difficulties with the appropriate sign in the differential equation.

Common problems were:

- not being able to find $\int \frac{dv}{v^2}$
- not initially applying the information that particle A had unit mass to the differential equation
- difficulties going from $t = \frac{1}{kv} - \frac{1}{ku}$ to $\frac{1}{v} = kt + \frac{1}{u}$
- using other incorrect differential equations, including $\ddot{x} = \pm kv$ or $\ddot{x} = \pm g \pm kv^2$ or $\ddot{x} = \pm v \pm kv^2$ or $\ddot{x} = \pm u \pm kv^2$ as well as $\ddot{x} = F - kv^2$ for an unspecified constant F.

(ii) The vast majority of candidates used the differential equation $\frac{dw}{dt} = -k\,w^2 - g$ to achieve the required result.

Many candidates showed a correct integration step down to $\int \frac{dx}{a^2+x^2} = \frac{1}{a}tan^{-1}\left(\frac{x}{a}\right)$, however, other candidates showed less familiar rearrangements of constants.

Common problems were:

- simply writing down the correct primitive implied by the answer given in this question
- similar errors with signs and confusion with the use of $v\frac{dv}{dx}$ rather than $\frac{dv}{dt}$ for acceleration, as was the case in (a)(i)
- using incorrect differential equations such as $\ddot{x} = g - kv^2$ as well as the use of $v\frac{dv}{dx}$ integrating successfully, however, then copying down the result given in the question.

(iii) Almost all candidates successfully achieved the required result.

Common problems were:

- not being clear about B being at rest so $w = 0$

- not showing the substitution, but rather quoting the required result.

(iv) The majority of candidates were again successful in obtaining the required result.

A considerable number stated the final result as $V \rightarrow \frac{2}{\pi}\sqrt{\frac{k}{g}}$ rather than as $V \rightarrow \frac{2}{\pi}\sqrt{\frac{g}{k}}$.

A common problem was:

- writing $\frac{1}{\infty}$ rather than $\lim_{u \to \infty} \frac{1}{u} = 0$, or assuming that nothing needs to be written when the result is zero. Attention to detail is required in answering the question.

(b)(i) There were many approaches used, such as:

- stating that $x \geq 0 \Rightarrow 1 + x \geq 0$, and then multiplying throughout the inequality by $1 + x$ and showing that the resulting inequality was true since $x \geq 0$
  showing that $1 - x - \frac{1}{1+x} \leq 0$ and that $\frac{1}{1+x} - 1 \leq 0$

Common problems were:

- incorrectly believing that $1 - x \leq 1$ and $\frac{1}{1+x} \leq 1$ is sufficient to prove that the given inequality is true
- showing that $\frac{1}{1+x} \leq 1$, but then not being able to show that $1 - x \leq \frac{1}{1+x}$
- multiplying throughout the inequality by $(1 + x)^2$ and then expanding. As a result, often there were algebraic errors or the expressions were insufficiently simplified to complete a proof
- in attempting differential calculus for each inequality, many errors were made when differentiating or by failing to consider the position at $x = 0$

in giving a graphical interpretation of the inequality, there was insufficient evidence provided to support the graphs.

(ii) Candidates who stated that $\int_0^{1/n}(1 - x)dx \leq \int_0^{1/n}\frac{dx}{1+x} \leq \int_0^{1/n} 1\, dx$ were able to achieve the required result.

Common problems were:

- not recognising that the required inequality was the result of a definite rather than an indefinite integral
- using the indefinite integral but not considering the constant of integration
- approaching this part by finding
$$\int_{n-1}^{n}(1 - x)\, dx \leq \int_{n-1}^{n}\frac{dx}{1 + x} \leq \int_{n-1}^{n} 1\, dx$$
- simply substituting $\frac{1}{2n}$ for x in the inequality $1 - x \leq \frac{1}{1+x} \leq 1$
- attempting to use $y = \frac{1}{1+x}$ or $y = \frac{1}{x}$ with inscribed and circumscribed rectangles. Such attempts were usually not successful as the LHS inequality was not as sharp as the desired inequality in the question.

(iii) In the better responses, candidates showed that as $\rightarrow \infty$, $1 - \frac{1}{2n} \rightarrow 1$ so that:

$$1 \leq \lim_{n \to \infty} n \ln\left(1 + \frac{1}{n}\right) \leq 1$$

and hence that

$$\lim_{n \to \infty} \ln\left[\left(1+\frac{1}{n}\right)^n\right] = 1 \Rightarrow \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = e$$

Common problems were:

- simply stating that as $n \to \infty$, $\frac{1}{2n} \to 0$ and as a result

$$\lim_{n \to \infty} \ln\left[\left(1+\frac{1}{n}\right)^n\right] = 1$$

This approach failed to show that $\lim_{n \to \infty} n \ln\left(1+\frac{1}{n}\right)$ was squeezed/pinched/sandwiched between 1 and 1, and as a result $\lim_{n \to \infty} \ln\left[\left(1+\frac{1}{n}\right)^n\right] = 1 \Rightarrow \lim_{n \to \infty}\left(1+\frac{1}{n}\right)^n = e$.

- incorrect statements indicating a lack of understanding of limits in general, such as $1+\frac{1}{n} \to e^{\frac{1}{n}}$.

(c)(i) The most successful approach was to start with $(x-y)^2 \geq 0$.
Other successful approaches included:

- replacing x by x^2 and y by y^2 leading to $\sqrt{x^2 y^2} \leq \frac{x^2+y^2}{2} \Rightarrow \sqrt{xy} \leq \sqrt{\frac{x^2+y^2}{2}}$
- starting with $\sqrt{xy} \leq \frac{x+y}{2}$
- showing that $\sqrt{\frac{x^2+y^2}{2}} \geq \frac{x+y}{2}$ and hence from the given inequality that $\sqrt{\frac{x^2+y^2}{2}} \geq \sqrt{xy}$.

A common problem was:

- incorrect algebraic manipulation

(ii) Successful candidates used various approaches to prove the required result starting with one of the following

$$\sqrt{abcd} = \sqrt{ab} \times \sqrt{cd} \leq \sqrt{\left(\frac{a^2+b^2}{2}\right)\left(\frac{c^2+d^2}{2}\right)}$$

or $$\sqrt{abcd} = \sqrt{ab} \times \sqrt{cd} \leq \frac{a+b}{2} \times \frac{c+d}{2}$$

or $$\sqrt{\sqrt{abcd}} \leq \sqrt{\frac{ab+cd}{2}}$$

Candidates are reminded to show all the necessary steps required in the proof.

For example, it is not sufficient to go from $\sqrt{abcd} \leq \sqrt{\frac{ab+cd}{2}}$ to

$\sqrt{abcd} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$

without explanation.

The response needs to at least include

$$a^2+b^2 \geq 2ab,\ c^2+d^2 \geq 2cd.$$

Common problems were:

- combining the results for $\sqrt{ab}$ and $\sqrt{cd}$

- simply adding the results $\sqrt{\mathrm{ab}} \leq \sqrt{\frac{\mathrm{a}^2+\mathrm{b}^2}{2}}$ and $\sqrt{\mathrm{cd}} \leq \sqrt{\frac{\mathrm{c}^2+\mathrm{d}^2}{2}}$ or adding the results$\sqrt{\mathrm{ab}} \leq \frac{\mathrm{a+b}}{2}$ and $\sqrt{\mathrm{cd}} \leq \frac{\mathrm{c+d}}{2}$
- poor algebraic skills and/or misquoting the result in (c)(i)
- a number of candidates attempted the following proof which was incorrect

$$\sqrt{abcd} \leq \sqrt{\left(\frac{a^2+b^2}{2}\right)\left(\frac{c^2+d^2}{2}\right)}$$

$$\sqrt{abcd} \leq \sqrt{\left(\frac{a^2c^2+a^2d^2+b^2c^2+b^2d^2}{4}\right)}$$

Now let $x = ac, y = ad, z = bc, w = bd \therefore xyzw = a^2b^2c^2d^2$ and

$$\sqrt{xyzw} = abcd \Longrightarrow \sqrt[4]{xyzw} = \sqrt{abcd}$$

$$\sqrt[4]{xyzw} = \sqrt{abcd} \leq \sqrt{\left(\frac{a^2c^2+a^2d^2+b^2c^2+b^2d^2}{4}\right)}$$

$$= \sqrt{\left(\frac{x^2+y^2+z^2+w^2}{4}\right)}$$

Replace x, y, z, w by a, b, c, d

$$\sqrt[4]{abcd} \leq \sqrt{\left(\frac{a^2+b^2+c+d^2}{4}\right)}$$

This proof was incorrect as x, y, z, w are not independent. Indeed $\frac{x}{y} = \frac{c}{d} = \frac{z}{w}$.

Question 16

(a)(i) In the majority of successful responses, candidates used a combinatorial or counting argument by determining the number of ways of placing exactly one black counter in each column divided by the number of ways of placing the black counters amongst all the cells without restriction, that is $\dfrac{3^5}{\binom{15}{5}}$.

Some successful responses used a 'successive outcome' approach by multiplying probabilities and obtaining $\frac{15}{15} \times \frac{12}{14} \times \frac{9}{13} \times \frac{6}{12} \times \frac{3}{11}$ or similar.

(ii) Common problems were:

- not providing clear arguments in deriving the formula for P_n
- giving a generic explanation such as 'number of favourable outcomes divided by the number of possible outcomes'
- showing that the formula holds for the special case in part (a)(i) and concluding that the formula therefore holds in general.

(iii) This part was challenging.

In many responses, candidates were able to write P_n as a quotient involving factorials but then gave spurious arguments for the limit by claiming 'dominance' of either the numerator

or denominator.
Responses including 0, ∞ or 1 were thus common.
In the successful responses, candidates first obtained an expression for P_n with q terms in the denominator before progressing to the correct solution.

(b)(i) This part was generally attempted well. The majority of responses included an explicit application of De Moivre's theorem and the real part of $\cos(2n\alpha) + i\sin(2n\alpha)$ was equated with the real part in the binomial expansion of $(\cos\alpha + i\sin\alpha)^{2n}$.
In some responses, however, the proof was not convincing as it did not include enough terms in the binomial expansion.

(ii) The majority of candidates recognised the connection between $T_{2n}(x)$ and $\cos(2n\alpha)$ by either replacing α with $\cos^{-1} x$ or replacing $\cos\alpha$ with x in part (i).

A common problem was:

- including the writing of $\sin(\cos^{-1} x) = 1 - x^2$ or $\sin^4(\cos^{-1} x) = 1 - \cos^4(\cos^{-1} x)$.

(iii) The majority of candidates who attempted this part considered the roots of $\cos(2n\cos^{-1}(x)) = 0$. However, a significant number of candidates were not able to solve this trigonometric equation correctly.
Of the candidates who recognised that $\cos\left(\frac{\pi}{4n}\right) \times \cos\left(\frac{3\pi}{4n}\right) \times \cdots \times \cos\left(\frac{(4n-1)\pi}{4n}\right)$ is the product of roots of $T_{2n}(x)$, many then did not progress to the correct solution, giving wrong answers such as:

$(-1)^n$ or $(-1)^n(1-x^2)^n$ or $\dfrac{(-1)^n}{\binom{2n}{0} - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + \ldots + (-1)^n\binom{2n}{2n}}$

(iv) The better responses involved substituting $x = \frac{1}{\sqrt{2}}$ into part (ii) or substituting $\alpha = \frac{\pi}{4}$ into part (i).
Some successful responses did not use previous parts of the question. One such response, for example, involved comparing the real part of $(1+i)^{2n}$ via De Moivre's theorem and its binomial expansion.

Common problems were:

- substituting $x = i$ or $x = 0$ or $x = 1$ in part (ii).
- attempting a proof of the result by mathematical induction.