BOSTES
BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2016 HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–18

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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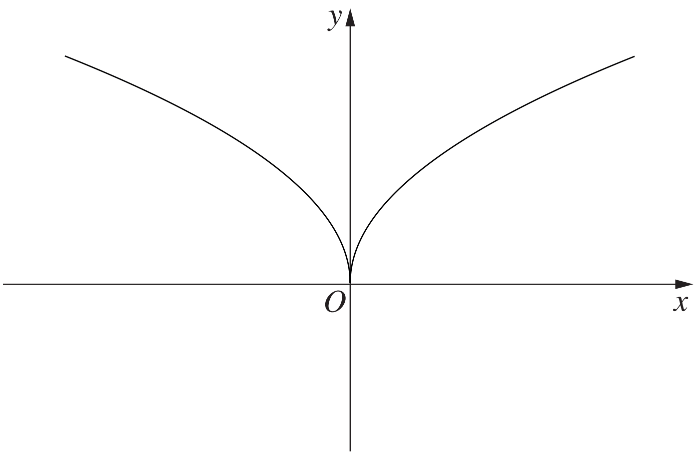
1 Which equation best represents the following graph?



(A) $y = \sqrt{x}$

(B) $|y| = \sqrt{x}$

(C) $y = \sqrt{|x|}$

(D) $|y| = \sqrt{|x|}$

2 Which polynomial has a multiple root at $x = 1$?

(A) $x^5 - x^4 - x^2 + 1$

(B) $x^5 - x^4 - x - 1$

(C) $x^5 - x^3 - x^2 + 1$

(D) $x^5 - x^3 - x + 1$

3 The sum of the eccentricities of two different conics is $\frac{3}{4}$.

Which pair of conics could this be?

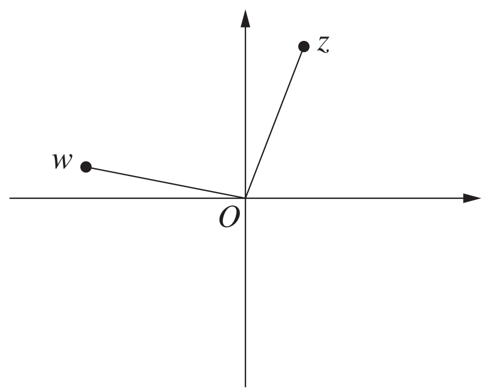
(A) Circle and ellipse

(B) Ellipse and parabola

(C) Parabola and hyperbola

(D) Hyperbola and circle

4 The Argand diagram shows the complex numbers z and w, where z lies in the first quadrant and w lies in the second quadrant.



Which complex number could lie in the 3rd quadrant?

(A) $-w$

(B) $2iz$

(C) $\overline{z}$

(D) $w - z$

5 Multiplying a non-zero complex number by $\dfrac{1-i}{1+i}$ results in a rotation about the origin on an Argand diagram.

What is the rotation?

(A) Clockwise by $\dfrac{\pi}{4}$

(B) Clockwise by $\dfrac{\pi}{2}$

(C) Anticlockwise by $\dfrac{\pi}{4}$

(D) Anticlockwise by $\dfrac{\pi}{2}$

6 Let $p(x) = 1 + x + x^2 + x^3 + \cdots + x^{12}$.

What is the coefficient of x^8 in the expansion of $p(x + 1)$?

(A) 1

(B) 495

(C) 715

(D) 1287

7 The hyperbola with equation $xy = 8$ is the hyperbola $x^2 - y^2 = k$ referred to different axes.

What is the value of k?

(A) 2

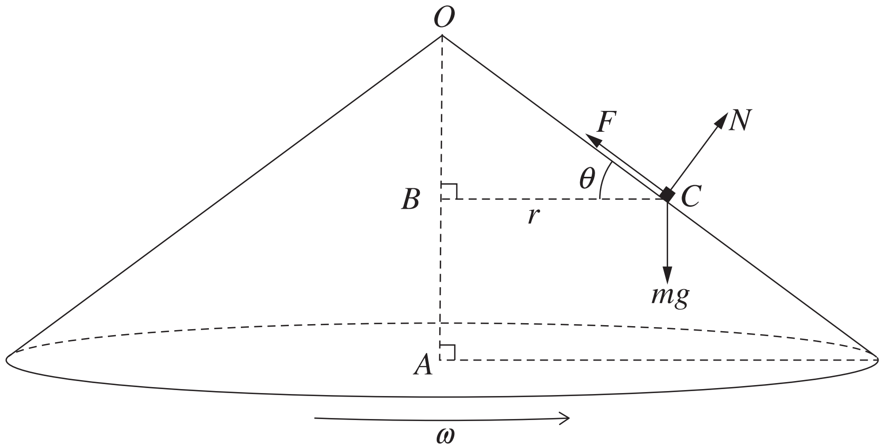
(B) 4

(C) 8

(D) 16

8 A small object of mass m kg sits on a rotating conical surface at C, r metres from the axis OA and with $\angle OCB = \theta$, as shown in the diagram.

The surface is rotating about its axis with angular velocity ω. The forces acting on the object are gravity, a normal reaction force N and a frictional force F, which prevents the object from sliding down the surface.



Which pair of statements is correct?

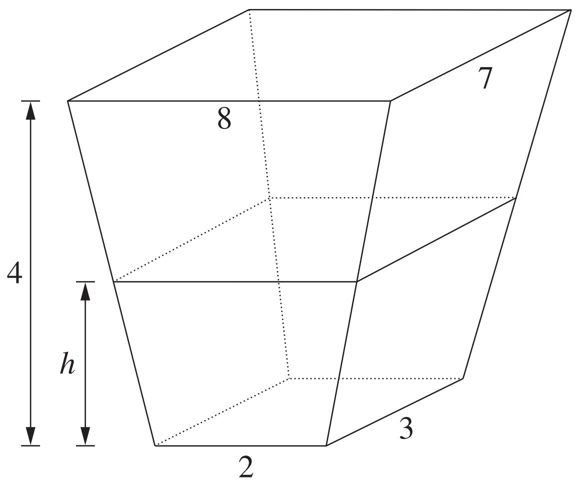
(A) $F\cos\theta + N\sin\theta = mr\omega^2$
$F\sin\theta + N\cos\theta = mg$

(B) $F\cos\theta + N\sin\theta = mr\omega^2$
$F\sin\theta - N\cos\theta = mg$

(C) $F\cos\theta - N\sin\theta = mr\omega^2$
$F\sin\theta + N\cos\theta = mg$

(D) $F\cos\theta - N\sin\theta = mr\omega^2$
$F\sin\theta - N\cos\theta = mg$

9 The diagram shows the dimensions of a polyhedron with parallel base and top. A slice taken at height h parallel to the base is a rectangle.



NOT TO SCALE

What is a correct expression for the volume of the polyhedron?

(A) $\displaystyle\int_0^4 (h+3)\left(\frac{3h}{2}+2\right) dh$

(B) $\displaystyle\int_0^4 \left(\frac{5h}{4}+3\right)\left(\frac{3h}{2}+2\right) dh$

(C) $\displaystyle\int_0^4 (h+3)\left(\frac{5h}{4}+2\right) dh$

(D) $\displaystyle\int_0^4 \left(\frac{5h}{4}+3\right)\left(\frac{5h}{4}+2\right) dh$

10 Suppose that $x+\dfrac{1}{x}=-1$.

What is the value of $x^{2016}+\dfrac{1}{x^{2016}}$?

(A) 1

(B) 2

(C) $\dfrac{2\pi}{3}$

(D) $\dfrac{4\pi}{3}$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = \sqrt{3} - i$.

(i) Express z in modulus–argument form. **2**

(ii) Show that z^6 is real. **1**

(iii) Find a positive integer n such that z^n is purely imaginary. **1**

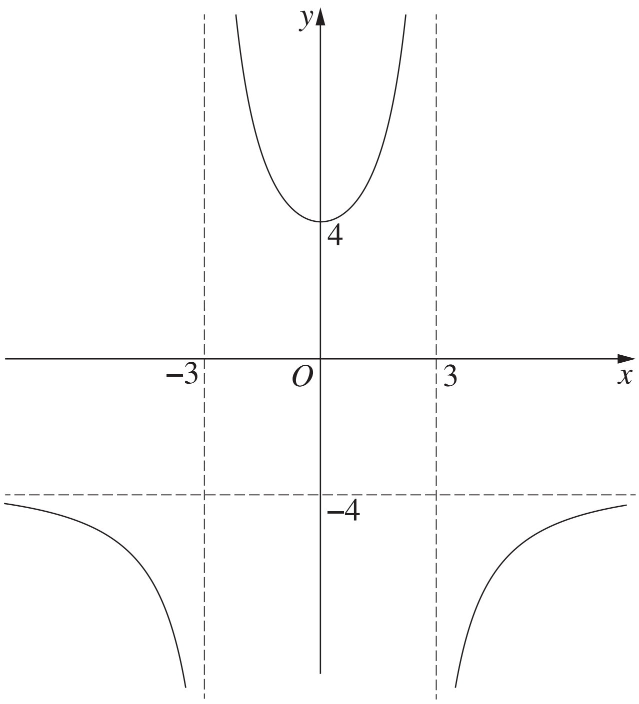
(b) Find $\int x\,e^{-2x}\,dx$. **3**

(c) Find $\dfrac{dy}{dx}$ for the curve given by $x^3 + y^3 = 2xy$, leaving your answer in terms of x and y. **2**

Question 11 continues on page 8

Question 11 (continued)

(d) The diagram shows the graph of $y = f(x)$.



Draw a separate half-page diagram for each of the following functions, showing all asymptotes and intercepts.

(i) $y = \sqrt{f(x)}$ **2**

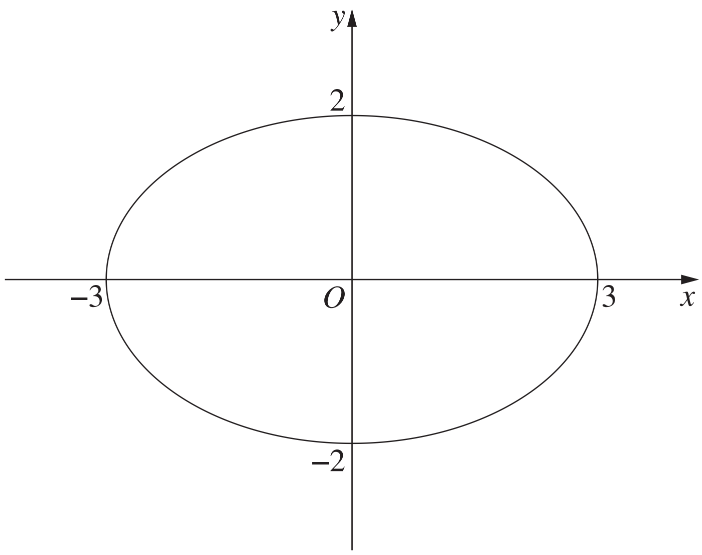
(ii) $y = \dfrac{1}{f(x)}$ **2**

(e) State the domain and range of the function $f(x) = x\sin^{-1}\left(\dfrac{x}{2}\right)$. **2**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows an ellipse.



(i) Write an equation for the ellipse. **1**

(ii) Find the eccentricity of the ellipse. **1**

(iii) Write the coordinates of the foci of the ellipse. **1**

(iv) Write the equations of the directrices of the ellipse. **1**

(b) (i) Differentiate $x f(x) - \int x f'(x)\,dx$. **1**

(ii) Hence, or otherwise, find $\int \tan^{-1} x\,dx$. **2**

Question 12 continues on page 10

Question 12 (continued)

(c) Let $z = \cos\theta + i\sin\theta$.

(i) By considering the real part of z^4, show that $\cos 4\theta$ is **2**

$$\cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta.$$

(ii) Hence, or otherwise, find an expression for $\cos 4\theta$ involving only powers of $\cos\theta$. **1**

(d) (i) Show that the equation of the normal to the hyperbola $xy = c^2$, $c \neq 0$, at $P\left(cp, \frac{c}{p}\right)$ is given by $px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right)$. **2**

(ii) The normal at P meets the hyperbola again at $Q\left(cq, \frac{c}{q}\right)$. **3**

Show that $q = -\frac{1}{p^3}$.

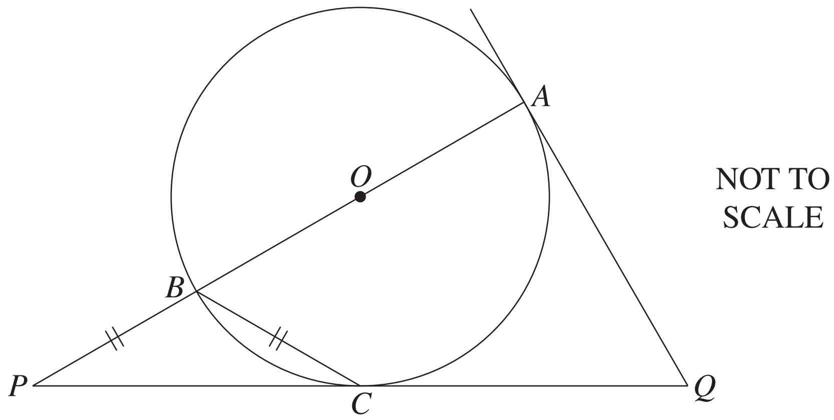
End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The function $f(x) = x^x$ is defined and positive for all $x > 0$. **3**

By differentiating $\ln(f(x))$, find the value of x at which $f(x)$ has a minimum.

(b) The circle centred at O has diameter AB. A point P on AB produced is chosen so that PC is a tangent to the circle at C and $BP = BC$. The tangents to the circle at A and C meet at Q. **4**



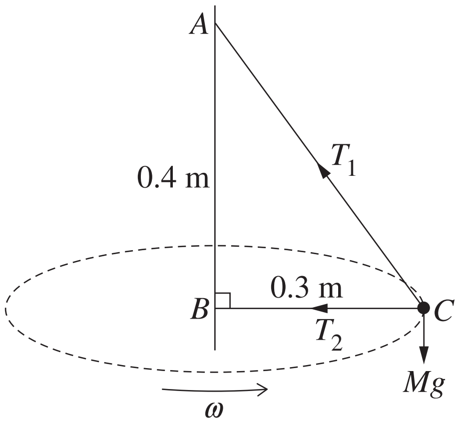
Copy or trace the diagram into your writing booklet.

Prove that $OP = OQ$.

Question 13 continues on page 12

(c) The ends of a string are attached to points A and B, with A directly above B. The points A and B are 0.4 m apart.

An object of mass M kg is fixed to the string at C. The object moves in a horizontal circle with centre B and radius 0.3 m, as shown in the diagram.



The tensions in the string from the object to points A and B are T_1 and T_2 respectively. The object rotates with constant angular velocity ω. You may assume that the acceleration due to gravity is $g = 10 \text{ m s}^{-2}$.

(i) Show that $T_2 = 0.3M\left(\omega^2 - 25\right)$. **3**

(ii) For what range of values of ω is $T_2 > T_1$? **1**

(d) Suppose $p(x) = ax^3 + bx^2 + cx + d$ with a, b, c and d real, $a \neq 0$.

(i) Deduce that if $b^2 - 3ac < 0$ then $p(x)$ cuts the x-axis only once. **2**

(ii) If $b^2 - 3ac = 0$ and $p\left(-\dfrac{b}{3a}\right) = 0$, what is the multiplicity of the root $x = -\dfrac{b}{3a}$? **2**

End of Question 13

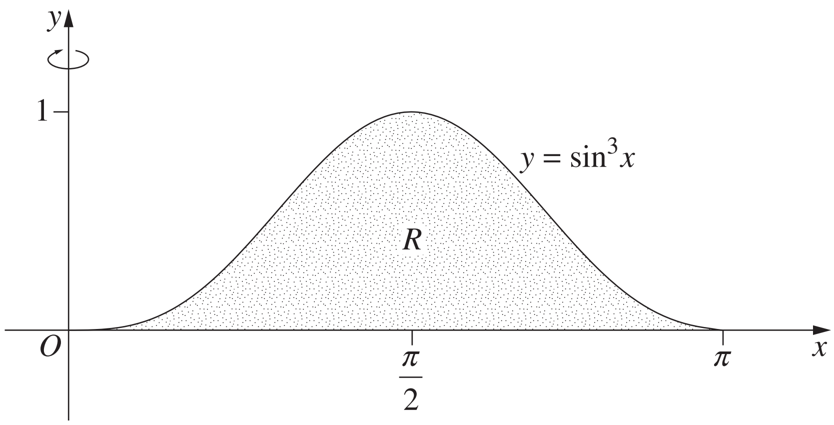
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\displaystyle\int \sin^3 x \, dx = \frac{1}{3}\cos^3 x - \cos x + C$. **1**

(ii) Using a graphical approach, or otherwise, explain why **1**

$\displaystyle\int_0^{\pi} \cos^{2n-1} x \, dx = 0$, for all positive integers n.

(iii) The diagram shows the region R enclosed by $y = \sin^3 x$ and the x-axis for $0 \le x \le \pi$. **3**



Using the method of cylindrical shells and the results in parts (i) and (ii), find the exact volume of the solid formed when R is rotated about the y-axis.

Question 14 continues on page 14

(b) Let $I_n = \int_0^1 \frac{x^n}{\left(x^2+1\right)^2}\,dx$, for $n = 0, 1, 2, \ldots$.

(i) Using a suitable substitution, show that $I_0 = \frac{\pi}{8} + \frac{1}{4}$. **3**

(ii) Show that $I_0 + I_2 = \frac{\pi}{4}$. **1**

(iii) Find I_4. **3**

(c) Show that $x\sqrt{x} + 1 \ge x + \sqrt{x}$, for $x \ge 0$. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^3 - 3x + 1 = 0$ has roots α, β and γ. **2**

Find a cubic equation with integer coefficients that has roots α^2, β^2 and γ^2.

(b) A particle is initially at rest at the point B which is b metres to the right of O. The particle then moves in a straight line towards O.

For $x \neq 0$, the acceleration of the particle is given by $\dfrac{-\mu^2}{x^2}$, where x is the distance from O and μ is a positive constant.

(i) Prove that $\dfrac{dx}{dt} = -\mu\sqrt{2}\sqrt{\dfrac{b-x}{bx}}$. **2**

(ii) Using the substitution $x = b\cos^2\theta$, show that the time taken to reach a distance d metres to the right of O is given by **3**

$$t = \frac{b\sqrt{2b}}{\mu}\int_0^{\cos^{-1}\sqrt{\frac{d}{b}}} \cos^2\theta \, d\theta.$$

It can be shown that $t = \dfrac{1}{\mu}\sqrt{\dfrac{b}{2}}\left(\sqrt{bd - d^2} + b\cos^{-1}\sqrt{\dfrac{d}{b}}\right)$. (Do NOT prove this.)

(iii) What is the limiting time taken for the particle to reach O? **1**

Question 15 continues on page 16

(c) (i) Use partial fractions to show that **2**

$$\frac{3!}{x(x+1)(x+2)(x+3)} = \frac{1}{x} - \frac{3}{x+1} + \frac{3}{x+2} - \frac{1}{x+3}.$$

(ii) Suppose that for n a positive integer **3**

$$\frac{n!}{x(x+1)\ldots(x+n)} = \frac{a_0}{x} + \frac{a_1}{x+1} + \cdots + \frac{a_k}{x+k} + \cdots + \frac{a_n}{x+n}.$$

Show that $a_k = (-1)^k \binom{n}{k}$.

(iii) Hence, or otherwise, find the limiting sum of **2**

$$1 - \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} - \frac{1}{4}\binom{n}{3} + \cdots + \frac{(-1)^n}{n+1}.$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) The complex numbers $z = \cos\theta + i\sin\theta$ and $w = \cos\alpha + i\sin\alpha$, where $-\pi < \theta \leq \pi$ and $-\pi < \alpha \leq \pi$, satisfy **3**

$$1 + z + w = 0.$$

By considering the real and imaginary parts of $1 + z + w$, or otherwise, show that 1, z and w form the vertices of an equilateral triangle in the Argand diagram.

(ii) Hence, or otherwise, show that if the three non-zero complex numbers $2i$, z_1 and z_2 satisfy **2**

$$|2i| = |z_1| = |z_2| \quad \text{AND} \quad 2i + z_1 + z_2 = 0$$

then they form the vertices of an equilateral triangle in the Argand diagram.

(b) (i) The complex numbers 0, u and v form the vertices of an equilateral triangle in the Argand diagram. **2**

Show that $u^2 + v^2 = uv$.

(ii) Give an example of non-zero complex numbers u and v, so that 0, u and v form the vertices of an equilateral triangle in the Argand diagram. **1**

Question 16 continues on page 18

(c) In a group of n people, each has one hat, giving a total of n different hats. They place their hats on a table. Later, each person picks up a hat, not necessarily their own.

A situation in which none of the n people picks up their own hat is called a derangement.

Let $D(n)$ be the number of possible derangements.

(i) Tom is one of the n people. In some derangements Tom finds that he and one other person have each other's hat. **1**

Show that, for $n > 2$, the number of such derangements is $(n-1)\,D(n-2)$.

(ii) By also considering the remaining possible derangements, show that, for $n > 2$, **2**

$$D(n) = (n-1)\left[D(n-1) + D(n-2)\right].$$

(iii) Hence, show that $D(n) - nD(n-1) = -\left[D(n-1) - (n-1)D(n-2)\right]$, for $n > 2$. **1**

(iv) Given $D(1) = 0$ and $D(2) = 1$, deduce that $D(n) - nD(n-1) = (-1)^n$, for $n > 1$. **1**

(v) Prove by mathematical induction, or otherwise, that for all integers $n \geq 1$, $D(n) = n!\sum_{r=0}^{n}\frac{(-1)^r}{r!}$. **2**

End of paper

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REFERENCE SHEET

– Mathematics –

– Mathematics Extension 1 –

– Mathematics Extension 2 –

Mathematics

Factorisation

$$a^2 - b^2 = (a+b)(a-b)$$

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2)$$

$$a^3 - b^3 = (a-b)(a^2 + ab + b^2)$$

Angle sum of a polygon

$$S = (n-2) \times 180^\circ$$

Equation of a circle

$$(x-h)^2 + (y-k)^2 = r^2$$

Trigonometric ratios and identities

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta}$$

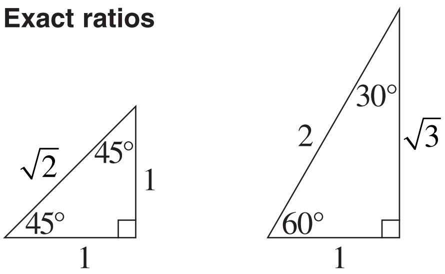
$$\sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sin^2\theta + \cos^2\theta = 1$$

Exact ratios



Sine rule

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Cosine rule

$$c^2 = a^2 + b^2 - 2ab\cos C$$

Area of a triangle

$$\text{Area} = \frac{1}{2}ab\sin C$$

Distance between two points

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Perpendicular distance of a point from a line

$$d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2 + b^2}}$$

Slope (gradient) of a line

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

Point–gradient form of the equation of a line

$$y - y_1 = m(x - x_1)$$

***n*th term of an arithmetic series**

$$T_n = a + (n-1)d$$

Sum to *n* terms of an arithmetic series

$$S_n = \frac{n}{2}\left[2a + (n-1)d\right] \quad \text{or} \quad S_n = \frac{n}{2}(a + l)$$

***n*th term of a geometric series**

$$T_n = ar^{n-1}$$

Sum to *n* terms of a geometric series

$$S_n = \frac{a(r^n - 1)}{r - 1} \quad \text{or} \quad S_n = \frac{a(1 - r^n)}{1 - r}$$

Limiting sum of a geometric series

$$S = \frac{a}{1-r}$$

Compound interest

$$A_n = P\left(1 + \frac{r}{100}\right)^n$$

Differentiation from first principles

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

Derivatives

If $y = x^n$, then $\frac{dy}{dx} = nx^{n-1}$

If $y = uv$, then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

If $y = \frac{u}{v}$, then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

If $y = F(u)$, then $\frac{dy}{dx} = F'(u)\frac{du}{dx}$

If $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$

If $y = \log_e f(x) = \ln f(x)$, then $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

If $y = \sin f(x)$, then $\frac{dy}{dx} = f'(x)\cos f(x)$

If $y = \cos f(x)$, then $\frac{dy}{dx} = -f'(x)\sin f(x)$

If $y = \tan f(x)$, then $\frac{dy}{dx} = f'(x)\sec^2 f(x)$

Solution of a quadratic equation

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Sum and product of roots of a quadratic equation

$$\alpha + \beta = -\frac{b}{a} \qquad \alpha\beta = \frac{c}{a}$$

Equation of a parabola

$$(x-h)^2 = \pm 4a(y-k)$$

Integrals

$$\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$$\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + C$$

$$\int \frac{f'(x)}{f(x)} \, dx = \ln|f(x)| + C$$

$$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C$$

$$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

$$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

Trapezoidal rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a) + f(b)\right]$$

Simpson's rule (one application)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Logarithms – change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Angle measure

$180° = \pi$ radians

Length of an arc

$l = r\theta$

Area of a sector

$\text{Area} = \frac{1}{2}r^2\theta$

Angle sum identities

$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$

$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

***t* formulae**

If $t = \tan\frac{\theta}{2}$, then

$$\sin\theta = \frac{2t}{1+t^2}$$

$$\cos\theta = \frac{1-t^2}{1+t^2}$$

$$\tan\theta = \frac{2t}{1-t^2}$$

General solution of trigonometric equations

$\sin\theta = a, \qquad \theta = n\pi + (-1)^n \sin^{-1} a$

$\cos\theta = a, \qquad \theta = 2n\pi \pm \cos^{-1} a$

$\tan\theta = a, \qquad \theta = n\pi + \tan^{-1} a$

Division of an interval in a given ratio

$$\left(\frac{mx_2 + nx_1}{m+n},\ \frac{my_2 + ny_1}{m+n}\right)$$

Parametric representation of a parabola

For $x^2 = 4ay$,

$$x = 2at, \quad y = at^2$$

At $\left(2at,\ at^2\right)$,

tangent: $y = tx - at^2$

normal: $x + ty = at^3 + 2at$

At $\left(x_1,\ y_1\right)$,

tangent: $xx_1 = 2a\left(y + y_1\right)$

normal: $y - y_1 = -\frac{2a}{x_1}\left(x - x_1\right)$

Chord of contact from $\left(x_0,\ y_0\right)$: $xx_0 = 2a\left(y + y_0\right)$

Acceleration

$$\frac{d^2x}{dt^2} = \frac{dv}{dt} = v\frac{dv}{dx} = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

Simple harmonic motion

$x = b + a\cos(nt + \alpha)$

$\ddot{x} = -n^2(x - b)$

Further integrals

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

Sum and product of roots of a cubic equation

$$\alpha + \beta + \gamma = -\frac{b}{a}$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = \frac{c}{a}$$

$$\alpha\beta\gamma = -\frac{d}{a}$$

Estimation of roots of a polynomial equation

Newton's method

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

Binomial theorem

$$(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k} = \sum_{k=0}^{n}\binom{n}{k}a^{n-k}b^k$$