BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

2000
MATHEMATICS
3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time allowed—Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1 Use a SEPARATE Writing Booklet. **Marks**

(a) Differentiate $x\sin^{-1}x$. **2**

(b) Find the acute angle between the lines $y = 2x - 1$ and $y = \frac{1}{3}x + 1$. **2**

(c) Find the value of k if $x - 3$ is a factor of $P(x) = x^3 - 3kx + 6$. **2**

(d) Evaluate $\displaystyle\int_0^{\sqrt{3}} \frac{4}{x^2 + 9}\,dx$. **3**

(e) Solve $\dfrac{5}{x+2} \leq 1$. **3**

QUESTION 2 Use a SEPARATE Writing Booklet. **Marks**

(a) How many arrangements of the letters of the word HOCKEYROO are possible? **2**

(b) Find the coefficient of x^6 in the expansion of $(5 + 2x^2)^7$. **3**

(c) Solve the equation $\cos 2\theta = \sin \theta, \quad 0 \le \theta \le 2\pi$. **4**

(d) Use the substitution $u = 2 + x$ to find $\displaystyle\int \frac{x}{\sqrt{2 + x}}\,dx$. **3**

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QUESTION 3 Use a SEPARATE Writing Booklet. **Marks**

(a) Use the definition $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ **2**

to find the derivative of x^3 where $x = a$.

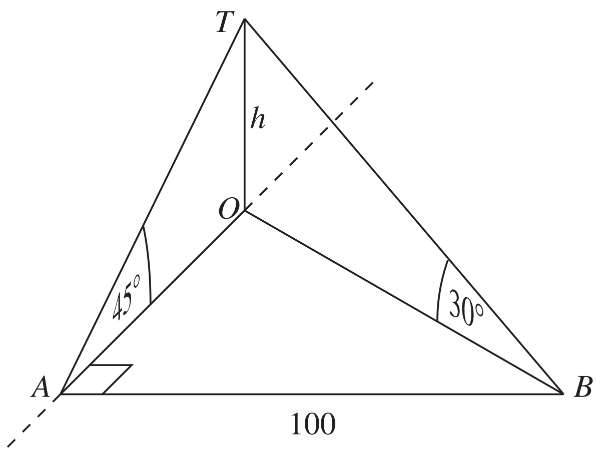
(b) Consider the function $f(x) = 3\tan^{-1} x$. **5**

(i) State the range of the function $y = f(x)$.

(ii) Sketch the graph of $y = f(x)$.

(iii) Find the gradient of the tangent to the curve $y = f(x)$ at $x = \dfrac{1}{\sqrt{3}}$.

(c) **5**



A surveyor stands at a point A, which is due south of a tower OT of height h m. The angle of elevation of the top of the tower from A is 45°. The surveyor then walks 100 m due east to point B, from where she measures the angle of elevation of the top of the tower to be 30°.

(i) Express the length of OB in terms of h.

(ii) Show that $h = 50\sqrt{2}$.

(iii) Calculate the bearing of B from the base of the tower.

QUESTION 4 Use a SEPARATE Writing Booklet. **Marks**

(a) Use mathematical induction to prove that **3**

$$1+3+6+\ldots+\frac{1}{2}n(n+1)=\frac{1}{6}n(n+1)(n+2)$$

for all integers $n = 1, 2, 3, \ldots$.

(b) We wish to find the interest rate r such that **3**

$$(1+r)\left\{(1+r)^{24}-1\right\}-50r=0\,.$$

Use one step of Newton's method to estimate r. Take $r_1 = 0{\cdot}06$ as the first approximation.

(c) The polynomial $P(x) = x^3 + px^2 + qx + r$ has real roots $\sqrt{k}, -\sqrt{k}$ and α. **4**

(i) Explain why $\alpha + p = 0$.

(ii) Show that $k\alpha = r$.

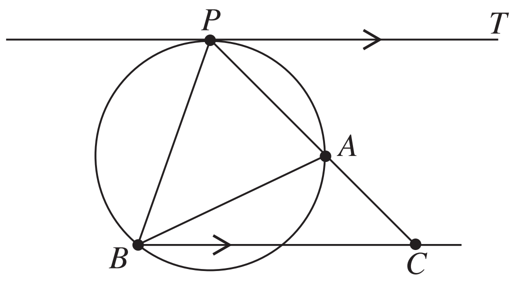
(iii) Show that $pq = r$.

(d) A particle is moving in simple harmonic motion about a fixed point O. Its amplitude is 3 cm and its period is 4π seconds. **2**

Find its speed at the point O.

QUESTION 5 Use a SEPARATE Writing Booklet. **Marks**

(a) **4**



In the diagram, A, P and B are points on the circle. The line PT is tangent to the circle at P, and PA is produced to C so that BC is parallel to PT.

Copy the diagram into your Writing Booklet.

(i) Show that $\angle PBA = \angle PCB$.

(ii) Deduce that $PB^2 = PA \times PC$.

(b) Consider the function $f(x) = \dfrac{x}{x+2}$. **8**

(i) Show that $f'(x) > 0$ for all x in the domain.

(ii) State the equation of the horizontal asymptote of $y = f(x)$.

(iii) Without using any further calculus, sketch the graph of $y = f(x)$.

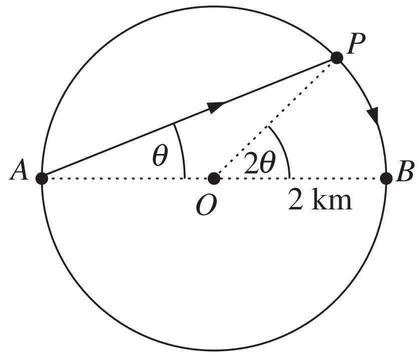
(iv) Explain why $f(x)$ has an inverse function $f^{-1}(x)$.

(v) Find an expression for $f^{-1}(x)$.

(vi) Write down the domain of $f^{-1}(x)$.

QUESTION 6 Use a SEPARATE Writing Booklet. **Marks**

(a) **8**



The diagram shows a circular lake, centre O, of radius 2 km with diameter AB. Pat can row at 3 km/h and can walk at 4 km/h and wishes to travel from A to B as quickly as possible. Pat considers the strategy of rowing direct from A to a point P and then walking around the edge of the lake to B. Let $\angle PAB = \theta$ radians, and let the time taken for Pat to travel from A to B by this route be T hours.

(i) Show that

$$T = \frac{1}{3}(4\cos\theta + 3\theta) \ .$$

(ii) Find the value of θ for which $\dfrac{dT}{d\theta} = 0$.

(iii) To what point P, if any, should Pat row to minimise T? Give reasons for your answer.

(b) A standard pack of 52 cards consists of 13 cards of each of the four suits: spades, hearts, clubs and diamonds. **4**

(i) In how many ways can six cards be selected without replacement so that exactly two are spades and four are clubs? (Assume that the order of selection of the six cards is not important.)

(ii) In how many ways can six cards be selected without replacement if at least five must be of the same suit? (Assume that the order of selection of the six cards is not important.)

QUESTION 7 Use a SEPARATE Writing Booklet. **Marks**

(a) The amount of fuel F in litres required *per hour* to propel a plane in level flight at constant speed u km/h is given by **4**

$$F = Au^3 + \frac{B}{u},$$

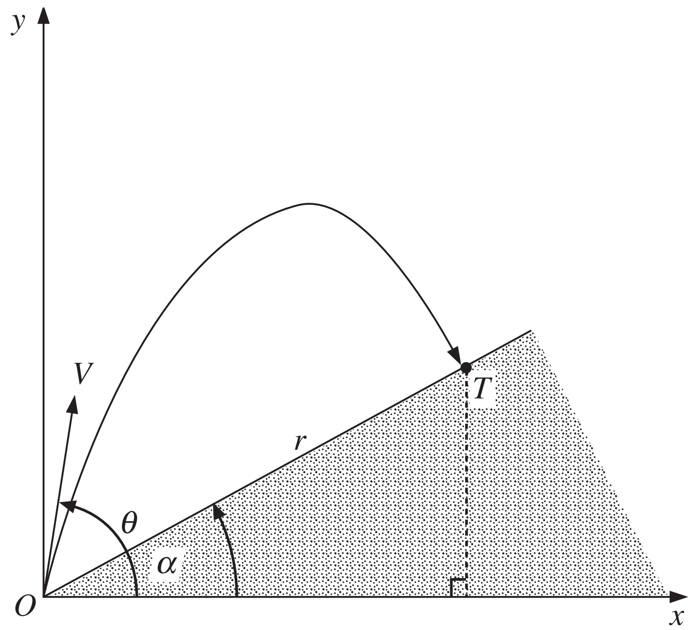
where A and B are positive constants.

(i) Show that a pilot wishing to remain in level flight for as long a period as possible should fly at

$$\left(\frac{B}{3A}\right)^{\frac{1}{4}} \text{ km/h}.$$

(ii) Show that a pilot wishing to fly as far as possible in level flight should fly approximately 32% faster than the speed given in part (i).

(b) **8**



The diagram shows an inclined plane that makes an angle of α radians with the horizontal. A projectile is fired from O, at the bottom of the incline, with a speed of V m s^{-1} at an angle of elevation θ to the horizontal, as shown.

With the above axes, you may assume that the position of the projectile is given by

$$x = Vt\cos\theta$$
$$y = Vt\sin\theta - \tfrac{1}{2}gt^2,$$

where t is the time, in seconds, after firing, and g is the acceleration due to gravity.

QUESTION 7 (Continued) **Marks**

For simplicity we assume that the unit of length has been chosen so that

$$\frac{2V^2}{g} = 1 \ .$$

(i) Show that the path of the trajectory of the projectile is

$$y = x\tan\theta - x^2\sec^2\theta \ .$$

(ii) Show that the range of the projectile, $r = OT$ metres, up the inclined plane is given by

$$r = \frac{\sin(\theta - \alpha)\cos\theta}{\cos^2\alpha} \ .$$

(iii) Hence, or otherwise, deduce that the maximum range, R metres, up the incline is

$$R = \frac{1}{2(1 + \sin\alpha)} \ .$$

(You may assume that $2\sin A\cos B = \sin(A + B) + \sin(A - B)$.)

(iv) Consider the trajectory of the projectile for which the maximum range R is achieved.

Show that, for this trajectory, the initial direction is perpendicular to the direction at which the projectile hits the inclined plane.

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$