BOARD OF STUDIES

NEW SOUTH WALES

2001

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int_0^{\frac{\pi}{4}} \tan^3 x \sec^2 x \, dx$. **2**

(b) By completing the square, find $\int \frac{dx}{\sqrt{x^2 - 4x + 1}}$. **2**

(c) Use integration by parts to evaluate **3**

$$\int_e^4 \frac{\ln x}{x^2} \, dx \, .$$

(d) Use the substitution $u = \sqrt{x-1}$ to evaluate **4**

$$\int_2^3 \frac{1+x}{\sqrt{x-1}} \, dx \, .$$

(e) (i) Find real numbers a and b such that **2**

$$\frac{5x^2 - 3x + 1}{(x^2+1)(x-2)} \equiv \frac{ax+1}{x^2+1} + \frac{b}{x-2} \, .$$

(ii) Find $\int \frac{5x^2 - 3x + 1}{(x^2+1)(x-2)} \, dx$. **2**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 2 + 3i$ and $w = 1 + i$. **2**

Find zw and $\frac{1}{w}$ in the form $x + iy$.

(b) (i) Express $1 + \sqrt{3}i$ in modulus-argument form. **2**

(ii) Hence evaluate $\left(1 + \sqrt{3}i\right)^{10}$ in the form $x + iy$. **2**

(c) Sketch the region in the complex plane where the inequalities **3**

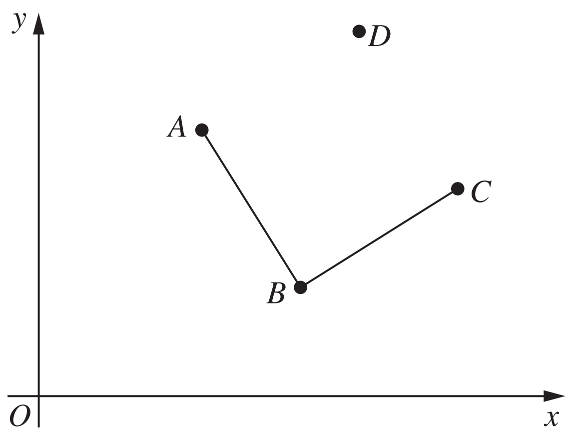
$$|z + 1 - 2i| \le 3 \text{ and } -\frac{\pi}{3} \le \arg z \le \frac{\pi}{4}$$

both hold.

(d) Find all solutions of the equation $z^4 = -1$. **3**

Give your answers in modulus-argument form.

(e)



In the diagram the vertices of a triangle ABC are represented by the complex numbers z_1, z_2 and z_3, respectively. The triangle is isosceles and right-angled at B.

(i) Explain why $(z_1 - z_2)^2 = -(z_3 - z_2)^2$. **2**

(ii) Suppose D is the point such that $ABCD$ is a square. Find the complex number, expressed in terms of z_1, z_2 and z_3, that represents D. **1**

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the hyperbola $\mathcal{H}$ with equation $\dfrac{x^2}{9} - \dfrac{y^2}{16} = 1$.

(i) Find the points of intersection of $\mathcal{H}$ with the x axis, and the eccentricity and the foci of $\mathcal{H}$. **3**

(ii) Write down the equations of the directrices and the asymptotes of $\mathcal{H}$. **2**

(iii) Sketch $\mathcal{H}$. **1**

(b) The numbers α, β and γ satisfy the equations

$$\alpha + \beta + \gamma = 3$$

$$\alpha^2 + \beta^2 + \gamma^2 = 1$$

$$\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = 2 \quad .$$

(i) Find the values of $\alpha\beta + \beta\gamma + \gamma\alpha$ and $\alpha\beta\gamma$. **3**

Explain why α, β and γ are the roots of the cubic equation

$$x^3 - 3x^2 + 4x - 2 = 0.$$

(ii) Find the values of α, β and γ. **2**

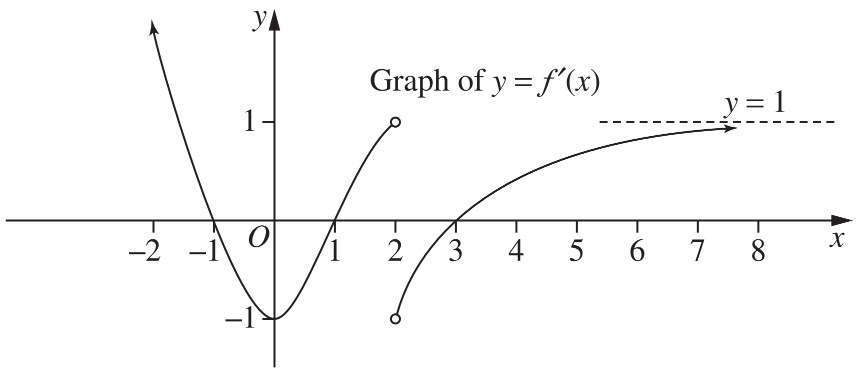
(c) The area under the curve $y = \sin x$ between $x = 0$ and $x = \pi$ is rotated about the y axis. **4**

Use the method of cylindrical shells to find the volume of the resulting solid of revolution.

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a)

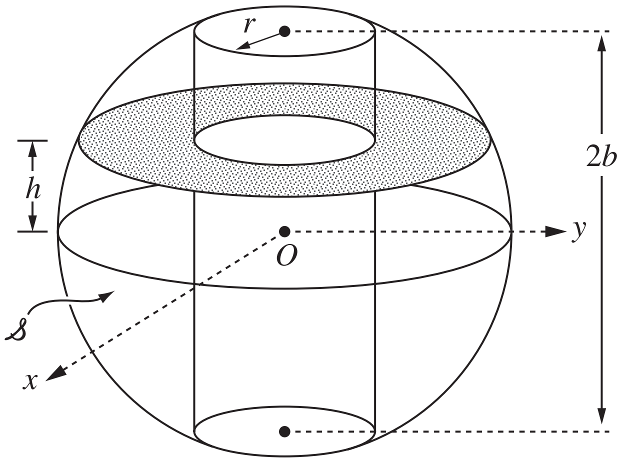


The diagram shows a sketch of $y = f'(x)$, the derivative function of $y = f(x)$. The curve $y = f'(x)$ has a horizontal asymptote $y = 1$.

(i) Identify and classify the turning points of the curve $y = f(x)$. **3**

(ii) Sketch the curve $y = f(x)$ given that $f(0) = 0 = f(2)$ and $y = f(x)$ is continuous. On your diagram, clearly identify and label any important features. **4**

(b)



A cylindrical hole of radius r is bored through a sphere of radius R. The hole is perpendicular to the xy plane and its axis passes through the origin O, which is the centre of the sphere. The resulting solid is denoted by $\mathcal{S}$. The cross-section of $\mathcal{S}$ shown in the diagram is distance h from the xy plane.

(i) Show that the area of the cross-section shown above is $\pi(R^2 - h^2 - r^2)$. **2**

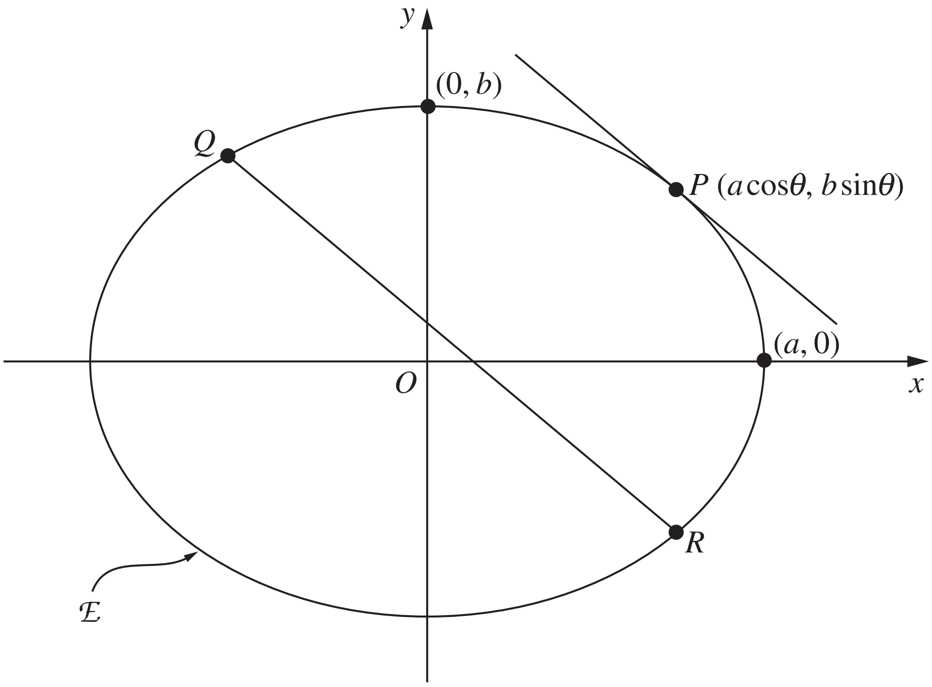
(ii) Find the volume of $\mathcal{S}$, and express your answer in terms of b alone, where $2b$ is the length of the hole. **3**

(c) Use differentiation to show that $\tan^{-1}\dfrac{x}{x+1} + \tan^{-1}\dfrac{1}{2x+1}$ is constant for $2x + 1 > 0$. What is the exact value of the constant? **3**

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a)



Consider the ellipse $\mathcal{E}$, with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, and the points $P\left(a\cos\theta, b\sin\theta\right)$, $Q\left(a\cos(\theta+\varphi), b\sin(\theta+\varphi)\right)$ and $R\left(a\cos(\theta-\varphi), b\sin(\theta-\varphi)\right)$ on $\mathcal{E}$.

(i) Show that the equation of the tangent to $\mathcal{E}$ at the point P is **2**

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 .$$

(ii) Show that the chord QR is parallel to the tangent at P. **2**

(iii) Deduce that OP bisects the chord QR. **3**

Question 5 continues on page 7

Marks

Question 5 (continued)

(b) A submarine of mass m is travelling underwater at maximum power. At maximum power, its engines deliver a force F on the submarine. The water exerts a resistive force proportional to the square of the submarine's speed v.

(i) Explain why **1**

$$\frac{dv}{dt} = \frac{1}{m}\left(F - kv^2\right)$$

where k is a positive constant.

(ii) The submarine increases its speed from v_1 to v_2. Show that the distance travelled during this period is **3**

$$\frac{m}{2k}\log_e\left(\frac{F - kv_1^{\,2}}{F - kv_2^{\,2}}\right).$$

(c) A class of 22 students is to be divided into four groups consisting of 4, 5, 6 and 7 students.

(i) In how many ways can this be done? Leave your answer in unsimplified form. **2**

(ii) Suppose that the four groups have been chosen. **2**

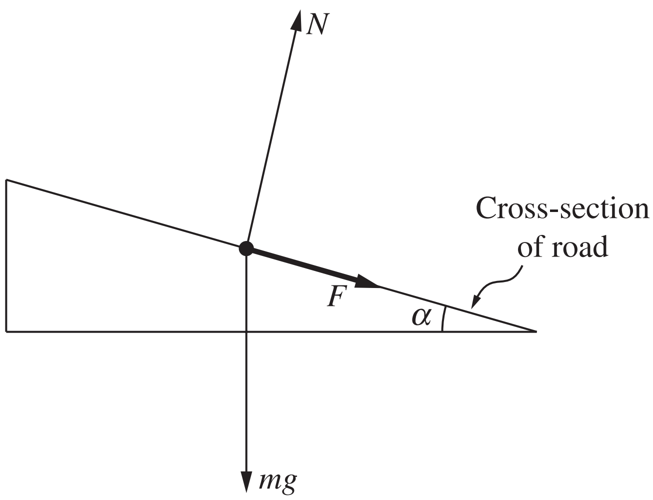
In how many ways can the 22 students be arranged around a circular table if the students in each group are to be seated together? Leave your answer in unsimplified form.

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a)



A road contains a bend that is part of a circle of radius r. At the bend, the road is banked at an angle α to the horizontal. A car travels around the bend at constant speed v. Assume that the car is represented by a point of mass m, and that the forces acting on the car are the gravitational force mg, a sideways friction force F (acting down the road as drawn) and a normal reaction N to the road.

(i) By resolving the horizontal and vertical components of force, find expressions for $F\cos\alpha$ and $F\sin\alpha$. **3**

(ii) Show that $F = \dfrac{m\left(v^2 - gr\tan\alpha\right)}{r}\cos\alpha$. **2**

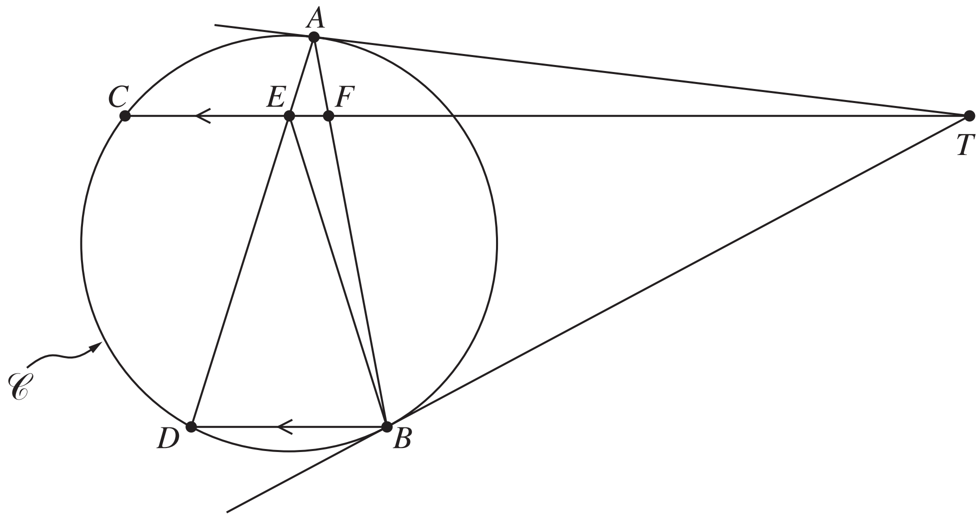
(iii) Suppose that the radius of the bend is 200 m and that the road is banked to allow cars to travel at 100 kilometres per hour with no sideways friction force. Assume that the value of g is $9.8\ \text{m s}^{-2}$. **2**

Find the value of angle α, giving full reasons for your answer.

Question 6 continues on page 9

Question 6 (continued)

(b)



In the diagram, $\mathscr{C}$ is a circle with exterior point T. From T, tangents are drawn to the points A and B on $\mathscr{C}$ and a line TC is drawn, meeting the circle at C. The point D is the point on $\mathscr{C}$ such that BD is parallel to TC. The line TC cuts the line AB at F and the line AD at E.

Copy or trace the diagram into your writing booklet.

(i) Prove that ΔTFA is similar to ΔTAE. **3**

(ii) Deduce that $TE.TF = TB^2$. **2**

(iii) Show that ΔEBT is similar to ΔBFT. **2**

(iv) Prove that ΔDEB is isosceles. **1**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose that $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ where θ is real.

(i) Find $|z|$. **1**

(ii) Show that the imaginary part of the geometric series **3**

$$1 + z + z^2 + z^3 + \ldots = \frac{1}{1-z}$$

is $\dfrac{2\sin\theta}{5 - 4\cos\theta}$.

(iii) Find an expression for **2**

$$1 + \frac{1}{2}\cos\theta + \frac{1}{2^2}\cos 2\theta + \frac{1}{2^3}\cos 3\theta + \ldots$$

in terms of $\cos\theta$.

(b) Consider the equation $x^3 - 3x - 1 = 0$, which we denote by (*).

(i) Let $x = \frac{p}{q}$ where p and q are integers having no common divisors other than +1 and –1. Suppose that x is a root of the equation $ax^3 - 3x + b = 0$, where a and b are integers. **4**

Explain why p divides b and why q divides a. Deduce that (*) does not have a rational root.

(ii) Suppose that r, s and d are rational numbers and that $\sqrt{d}$ is irrational. Assume that $r + s\sqrt{d}$ is a root of (*). **4**

Show that $3r^2s + s^3d - 3s = 0$ and show that $r - s\sqrt{d}$ must also be a root of (*).

Deduce from this result and part (i), that no root of (*) can be expressed in the form $r + s\sqrt{d}$ with r, s and d rational.

(iii) Show that one root of (*) is $2\cos\frac{\pi}{9}$. **1**

(You may assume the identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.)

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $2ab \le a^2 + b^2$ for all real numbers a and b. **3**

Hence deduce that $3(ab + bc + ca) \le (a + b + c)^2$ for all real numbers a, b and c.

(ii) Suppose a, b and c are the sides of a triangle. Explain why $(b - c)^2 \le a^2$. **4**

Deduce that $(a + b + c)^2 \le 4(ab + bc + ca)$.

(b) (i) Explain why, for $\alpha > 0$, **2**

$$\int_0^1 x^{\alpha} e^x \, dx < \frac{3}{\alpha + 1} .$$

(You may assume $e < 3$.)

(ii) Show, by induction, that for $n = 0, 1, 2, \ldots$ there exist integers a_n and b_n such that **2**

$$\int_0^1 x^n e^x \, dx = a_n + b_n e \ .$$

(iii) Suppose that r is a positive rational, so that $r = \dfrac{p}{q}$ where p and q are positive integers. Show that, for all integers a and b, either **2**

$$|a + br| = 0 \quad \text{or} \quad |a + br| \ge \frac{1}{q} \ .$$

(iv) Prove that e is irrational. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$