BOARD OF STUDIES

NEW SOUTH WALES

2002

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\lim_{x \to 0} \frac{\sin 3x}{x}$. **1**

(b) Find $\frac{d}{dx}\left(3x^2 \ln x\right)$ for $x > 0$. **2**

(c) Use the table of standard integrals to evaluate $\int_0^{\frac{\pi}{6}} \sec 2x \tan 2x \, dx$. **2**

(d) State the domain and range of the function $f(x) = 3\sin^{-1}\left(\frac{x}{2}\right)$. **2**

(e) The variable point $\left(3t, 2t^2\right)$ lies on a parabola. Find the Cartesian equation for this parabola. **2**

(f) Use the substitution $u = 1 - x^2$ to evaluate $\int_2^3 \frac{2x}{\left(1 - x^2\right)^2} \, dx$. **3**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Solve $2^x = 3$. **2**

Express your answer correct to two decimal places.

(b) Find the general solution to $2\cos x = \sqrt{3}$. **2**

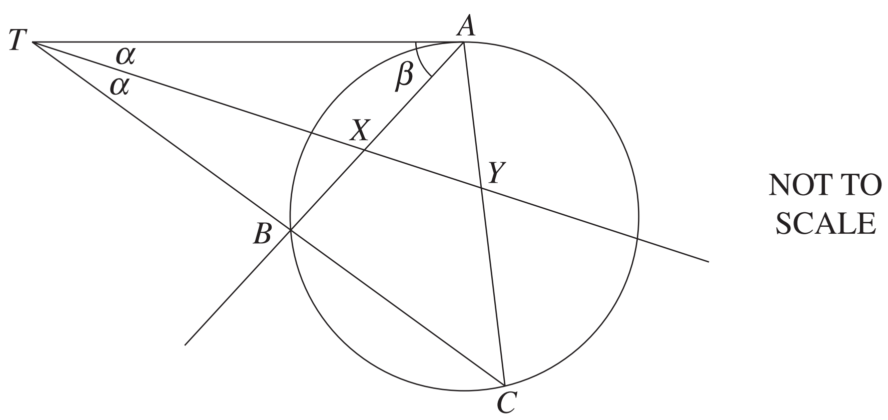
Express your answer in terms of π.

(c) Suppose $x^3 - 2x^2 + a \equiv (x+2)Q(x) + 3$ where $Q(x)$ is a polynomial. **2**

Find the value of a.

(d) Evaluate $2\int_0^{\frac{\pi}{4}} \sin^2 4x \, dx$. **3**

(e)



In the diagram the points A, B and C lie on the circle and CB produced meets the tangent from A at the point T. The bisector of the angle ATC intersects AB and AC at X and Y respectively. Let $\angle TAB = \beta$.

Copy or trace the diagram into your writing booklet.

(i) Explain why $\angle ACB = \beta$. **1**

(ii) Hence prove that triangle AXY is isosceles. **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Seven people are to be seated at a round table.

(i) How many seating arrangements are possible? **1**

(ii) Two people, Kevin and Jill, refuse to sit next to each other. How many seating arrangements are then possible? **2**

(b) (i) Show that $f(x) = e^x - 3x^2$ has a root between $x = 3.7$ and $x = 3.8$. **1**

(ii) Starting with $x = 3.8$, use one application of Newton's method to find a better approximation for this root. Write your answer correct to three significant figures. **3**

(c) A household iron is cooling in a room of constant temperature 22°C. At time t minutes its temperature T decreases according to the equation

$$\frac{dT}{dt} = -k(T - 22) \text{ where } k \text{ is a positive constant.}$$

The initial temperature of the iron is 80°C and it cools to 60°C after 10 minutes.

(i) Verify that $T = 22 + Ae^{-kt}$ is a solution of this equation, where A is a constant. **1**

(ii) Find the values of A and k. **2**

(iii) How long will it take for the temperature of the iron to cool to 30°C? Give your answer to the nearest minute. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Lyndal hits the target on average 2 out of every 3 shots in archery competitions. During a competition she has 10 shots at the target.

(i) What is the probability that Lyndal hits the target exactly 9 times? Leave your answer in unsimplified form. **1**

(ii) What is the probability that Lyndal hits the target fewer than 9 times? Leave your answer in unsimplified form. **2**

(b) The polynomial $P(x) = x^3 - 2x^2 + kx + 24$ has roots α, β, γ.

(i) Find the value of $\alpha + \beta + \gamma$. **1**

(ii) Find the value of $\alpha\beta\gamma$. **1**

(iii) It is known that two of the roots are equal in magnitude but opposite in sign. **2**

Find the third root and hence find the value of k.

(c) A particle, whose displacement is x, moves in simple harmonic motion such that $\ddot{x} = -16x$. At time $t = 0$, $x = 1$ and $\dot{x} = 4$.

(i) Show that, for all positions of the particle, **2**

$$|\dot{x}| = 4\sqrt{2 - x^2}\,.$$

(ii) What is the particle's greatest displacement? **1**

(iii) Find x as a function of t. You may assume the general form for x. **2**

Marks

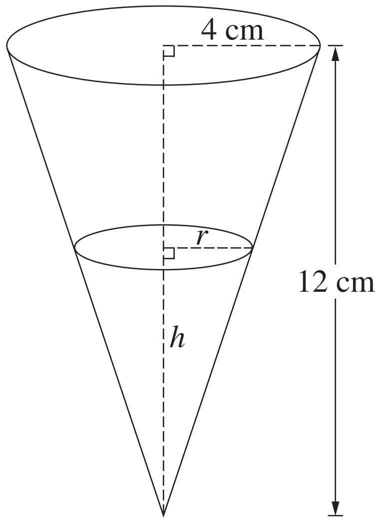
Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Use the principle of mathematical induction to show that **3**

$$2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (n^2 + 1)n! = n(n+1)!$$

for all positive integers n.

(b)



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The diagram shows a conical drinking cup of height 12 cm and radius 4 cm. The cup is being filled with water at the rate of 3 cm^3 per second. The height of water at time t seconds is h cm and the radius of the water's surface is r cm.

(i) Show that $r = \frac{1}{3}h$. **1**

(ii) Find the rate at which the height is increasing when the height of water is 9 cm. (Volume of cone = $\frac{1}{3}\pi r^2 h$.) **3**

(c) Consider the function

$$f(x) = 2\sin^{-1}\sqrt{x} - \sin^{-1}(2x-1) \text{ for } 0 \le x \le 1 .$$

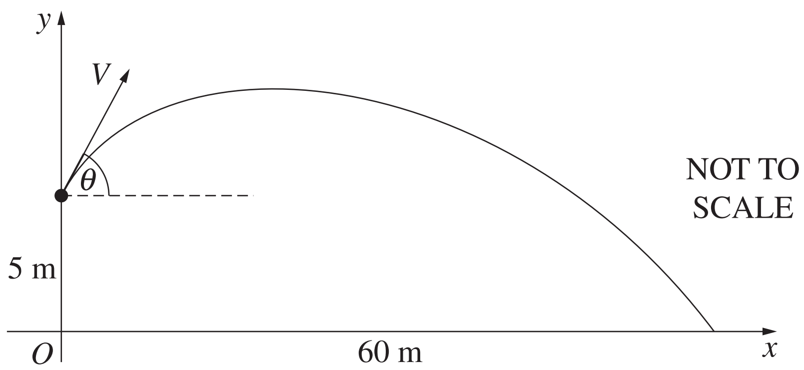
(i) Show that $f'(x) = 0$ for $0 < x < 1$. **3**

(ii) Sketch the graph of $y = f(x)$. **2**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a)



An angler casts a fishing line so that the sinker is projected with a speed V m s^{-1} from a point 5 metres above a flat sea. The angle of projection to the horizontal is θ, as shown.

Assume that the equations of motion of the sinker are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -10,$$

referred to the coordinate axes shown.

(i) Let (x, y) be the position of the sinker at time t seconds after the cast, and before the sinker hits the water. **2**

It is known that $x = Vt\cos\theta$.

Show that $y = Vt\sin\theta - 5t^2 + 5$.

(ii) Suppose the sinker hits the sea 60 metres away as shown in the diagram. **3**

Find the value of V if $\theta = \tan^{-1}\dfrac{3}{4}$.

(iii) For the cast described in part (ii), find the maximum height above sea level that the sinker achieved. **2**

Question 6 continues on page 9

Marks

Question 6 (continued)

(b) Let n be a positive integer.

(i) By considering the graph of $y = \frac{1}{x}$ show that **2**

$$\frac{1}{n+1} < \int_{n}^{n+1} \frac{dx}{x} < \frac{1}{n}.$$

(ii) Hence deduce that **3**

$$\left(1+\frac{1}{n}\right)^{n} < e < \left(1+\frac{1}{n}\right)^{n+1}.$$

End of Question 6

Please turn over

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Let $g(x) = e^x + \frac{1}{e^x}$ for all real values of x and let $f(x) = e^x + \frac{1}{e^x}$ for $x \le 0$.

(i) Sketch the graph $y = g(x)$ and explain why $g(x)$ does not have an inverse function. **2**

(ii) On a separate diagram, sketch the graph of the inverse function $y = f^{-1}(x)$. **1**

(iii) Find an expression for $y = f^{-1}(x)$ in terms of x. **3**

(b) The coefficient of x^k in $(1 + x)^n$, where n is a positive integer, is denoted by c_k (so $c_k = {}^nC_k$).

(i) Show that **3**

$$c_0 + 2c_1 + 3c_2 + \ldots + (n+1)c_n = (n+2)2^{n-1}.$$

(ii) Find the sum **3**

$$\frac{c_0}{1.2} - \frac{c_1}{2.3} + \frac{c_2}{3.4} - \ldots + (-1)^n \frac{c_n}{(n+1)(n+2)} .$$

Write your answer as a simple expression in terms of n.

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$