BOARD OF STUDIES

NEW SOUTH WALES

2003

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find the coordinates of the point P that divides the interval joining $(-3, 4)$ and $(5, 6)$ internally in the ratio $1 : 3$. **2**

(b) Solve $\dfrac{3}{x-2} \leq 1$. **3**

(c) Evaluate $\lim\limits_{x \to 0} \dfrac{3x}{\sin 2x}$. **2**

(d) A curve has parametric equations $x = \dfrac{t}{2}$, $y = 3t^2$. Find the Cartesian equation for this curve. **2**

(e) Use the substitution $u = x^2 + 1$ to evaluate **3**

$$\int_0^2 \frac{x}{\left(x^2+1\right)^3}\, dx\,.$$

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Sketch the graph of $y = 3\cos^{-1} 2x$. Your graph must clearly indicate the domain and the range. **2**

(b) Find $\dfrac{d}{dx}\left(x\tan^{-1} x\right)$. **2**

(c) Evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{2-x^2}}\,dx$. **2**

(d) Find the coefficient of x^4 in the expansion of $\left(2+x^2\right)^5$. **2**

(e) (i) Express $\cos x - \sin x$ in the form $R\cos(x+\alpha)$, where α is in radians. **2**

(ii) Hence, or otherwise, sketch the graph of $y = \cos x - \sin x$ for $0 \le x \le 2\pi$. **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) How many nine-letter arrangements can be made using the letters of the word ISOSCELES? **2**

(b) A particle moves in a straight line and its position at time t is given by

$$x = 4\sin\left(2t + \frac{\pi}{3}\right).$$

(i) Show that the particle is undergoing simple harmonic motion. **2**

(ii) Find the amplitude of the motion. **1**

(iii) When does the particle first reach maximum speed after time $t = 0$? **1**

(c) (i) Explain why the probability of getting a sum of 5 when one pair of fair dice is tossed is $\frac{1}{9}$. **1**

(ii) Find the probability of getting a sum of 5 at least twice when a pair of dice is tossed 7 times. **2**

(d) Use mathematical induction to prove that **3**

$$\frac{1}{1\times 3} + \frac{1}{3\times 5} + \frac{1}{5\times 7} + \cdots + \frac{1}{(2n-1)(2n+1)} = \frac{n}{2n+1}$$

for all positive integers n.

Marks

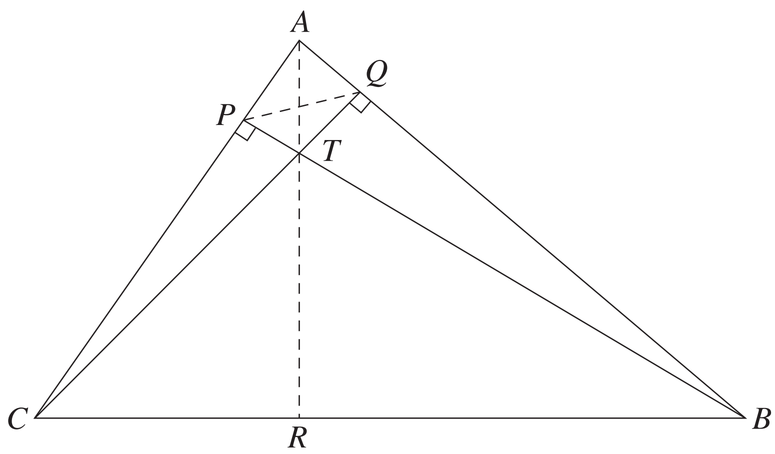
Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) A committee of 6 is to be chosen from 14 candidates. In how many different ways can this be done? **1**

(b) The function $f(x) = \sin x - \dfrac{2x}{3}$ has a zero near $x = 1.5$. Taking $x = 1.5$ as a first approximation, use one application of Newton's method to find a second approximation to the zero. Give your answer correct to three decimal places. **3**

(c) It is known that two of the roots of the equation $2x^3 + x^2 - kx + 6 = 0$ are reciprocals of each other. Find the value of k. **2**

(d)



In the diagram, CQ and BP are altitudes of the triangle ABC. The lines CQ and BP intersect at T, and AT is produced to meet CB at R.

Copy or trace the diagram into your writing booklet.

(i) Explain why $CPQB$ is a cyclic quadrilateral. **1**

(ii) Explain why $PAQT$ is a cyclic quadrilateral. **1**

(iii) Prove that $\angle TAQ = \angle QCB$. **2**

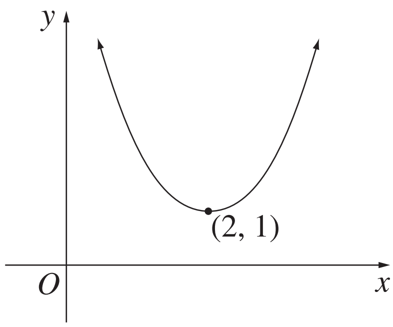
(iv) Prove that $AR \perp CB$. **2**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int \cos^2 3x \, dx$. **2**

(b) The graph of $f(x) = x^2 - 4x + 5$ is shown in the diagram.



(i) Explain why $f(x)$ does not have an inverse function. **1**

(ii) Sketch the graph of the inverse function, $g^{-1}(x)$, of $g(x)$, where $g(x) = x^2 - 4x + 5, \ x \le 2$. **1**

(iii) State the domain of $g^{-1}(x)$. **1**

(iv) Find an expression for $y = g^{-1}(x)$ in terms of x. **2**

(c) Dr Kool wishes to find the temperature of a very hot substance using his thermometer, which only measures up to 100°C. Dr Kool takes a sample of the substance and places it in a room with a surrounding air temperature of 20°C, and allows it to cool.

After 6 minutes the temperature of the substance is 80°C, and after a further 2 minutes it is 50°C. If $T(t)$ is the temperature of the substance after t minutes, then Newton's law of cooling states that T satisfies the equation

$$\frac{dT}{dt} = k(T - A),$$

where k is a constant and A is the surrounding air temperature.

(i) Verify that $T = A + Be^{kt}$ satisfies the above equation. **1**

(ii) Show that $k = -\frac{\log_e 2}{2}$, and find the value of B. **3**

(iii) Hence find the initial temperature of the substance. **1**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) The acceleration of a particle P is given by the equation

$$\frac{d^2x}{dt^2} = 8x\left(x^2+4\right),$$

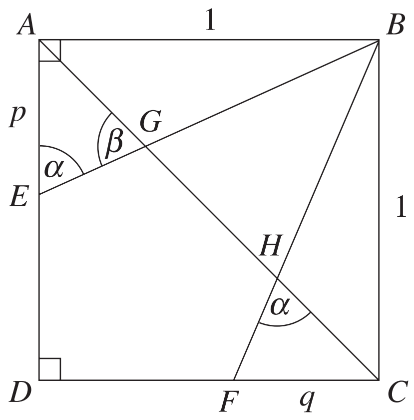
where x metres is the displacement of P from a fixed point O after t seconds.

Initially the particle is at O and has velocity $8\ \mathrm{m\,s^{-1}}$ in the positive direction.

(i) Show that the speed at any position x is given by $2(x^2+4)\ \mathrm{m\,s^{-1}}$. **3**

(ii) Hence find the time taken for the particle to travel 2 metres from O. **2**

(b)



In the diagram, $ABCD$ is a unit square. Points E and F are chosen on AD and DC respectively, such that $\angle AEG = \angle FHC$, where G and H are the points at which BE and BF respectively cut the diagonal AC.

Let $AE = p$, $FC = q$, $\angle AEG = \alpha$ and $\angle AGE = \beta$.

(i) Express α in terms of p, and β in terms of q. **2**

(ii) Prove that $p + q = 1 - pq$. **2**

(iii) Show that the area of the quadrilateral $EBFD$ is given by **1**

$$1 - \frac{p}{2} + \frac{p-1}{2(1+p)}\ .$$

(iv) What is the maximum value of the area of $EBFD$? **2**

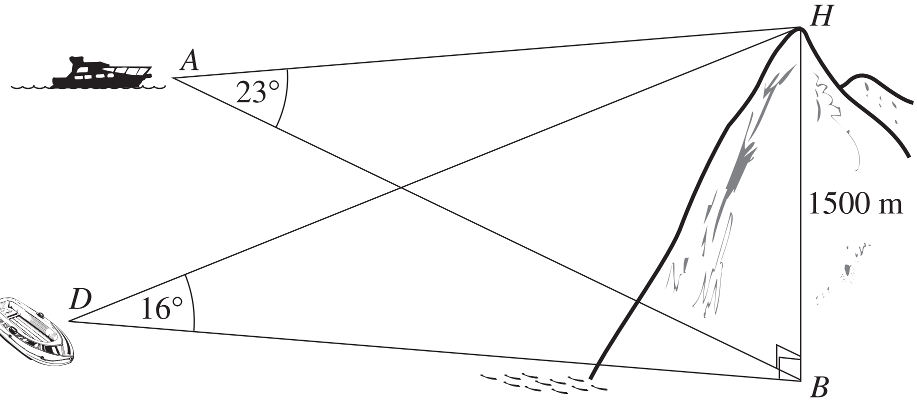
Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) David is in a life raft and Anna is in a cabin cruiser searching for him. They are in contact by mobile telephone. David tells Anna that he can see Mt Hope. From David's position the mountain has a bearing of 109°, and the angle of elevation to the top of the mountain is 16°. **4**

Anna can also see Mt Hope. From her position it has a bearing of 139°, and the top of the mountain has an angle of elevation of 23°.

The top of Mt Hope is 1500 m above sea level.



Find the distance and bearing of the life raft from Anna's position.

Question 7 continues on page 9

Marks

Question 7 (continued)

(b) A particle is projected from the origin with velocity $v\ \mathrm{m\,s^{-1}}$ at an angle α to the horizontal. The position of the particle at time t seconds is given by the parametric equations

$$x = vt\cos\alpha$$
$$y = vt\sin\alpha - \frac{1}{2}gt^2 ,$$

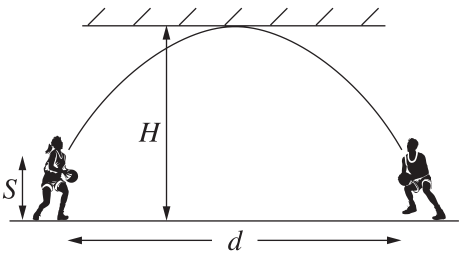
where $g\ \mathrm{m\,s^{-2}}$ is the acceleration due to gravity. (You are NOT required to derive these.)

(i) Show that the maximum height reached, h metres, is given by **2**

$$h = \frac{v^2\sin^2\alpha}{2g}.$$

(ii) Show that it returns to the initial height at $x = \frac{v^2}{g}\sin 2\alpha$. **2**

(iii) Chris and Sandy are tossing a ball to each other in a long hallway. The ceiling height is H metres and the ball is thrown and caught at shoulder height, which is S metres for both Chris and Sandy. **4**



The ball is thrown with a velocity $v\ \mathrm{m\,s^{-1}}$. Show that the maximum separation, d metres, that Chris and Sandy can have and still catch the ball is given by

$$d = 4 \times \sqrt{(H-S)\left(\frac{v^2}{2g}\right) - (H-S)^2}, \quad \text{if } v^2 \ge 4g(H-S), \quad \text{and}$$

$$d = \frac{v^2}{g}, \qquad \text{if } v^2 \le 4g(H-S).$$

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$