BOARD OF STUDIES
NEW SOUTH WALES

2003
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^1 \frac{e^x}{\left(1+e^x\right)^2}\,dx$. **2**

(b) Use integration by parts to find **3**

$$\int x^3 \log_e x \, dx .$$

(c) By completing the square and using the table of standard integrals, find **2**

$$\int \frac{dx}{\sqrt{x^2-2x+5}} .$$

(d) (i) Find the real numbers a and b such that **2**

$$\frac{5x^2-3x+13}{(x-1)\left(x^2+4\right)} \equiv \frac{a}{x-1} + \frac{bx-1}{x^2+4}.$$

(ii) Find $\int \frac{5x^2-3x+13}{(x-1)\left(x^2+4\right)}\,dx$. **2**

(e) Use the substitution $x = 3\sin\theta$ to evaluate **4**

$$\int_0^{\frac{3}{\sqrt{2}}} \frac{dx}{\left(9-x^2\right)^{\frac{3}{2}}} .$$

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 2 + i$ and $w = 1 - i$.

Find, in the form $x + iy$,

(i) $z\overline{w}$ **1**

(ii) $\dfrac{4}{z}$. **1**

(b) Let $\alpha = -1 + i$.

(i) Express α in modulus-argument form. **2**

(ii) Show that α is a root of the equation $z^4 + 4 = 0$. **1**

(iii) Hence, or otherwise, find a real quadratic factor of the polynomial $z^4 + 4$. **2**

(c) Sketch the region in the complex plane where the inequalities **3**

$$|z - 1 - i| < 2 \quad \text{and} \quad 0 < \arg(z - 1 - i) < \frac{\pi}{4}$$

hold simultaneously.

(d) By applying de Moivre's theorem and by also expanding $(\cos\theta + i\sin\theta)^5$, express $\cos 5\theta$ as a polynomial in $\cos\theta$. **3**

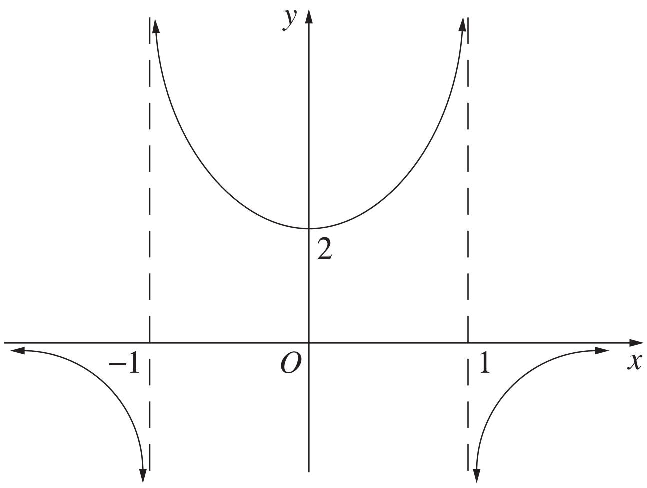
(e) Suppose that the complex number z lies on the unit circle, and $0 \le \arg(z) \le \dfrac{\pi}{2}$. **2**

Prove that $2\arg(z + 1) = \arg(z)$.

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = f(x) + |f(x)|$ **2**

(iii) $y = (f(x))^2$ **1**

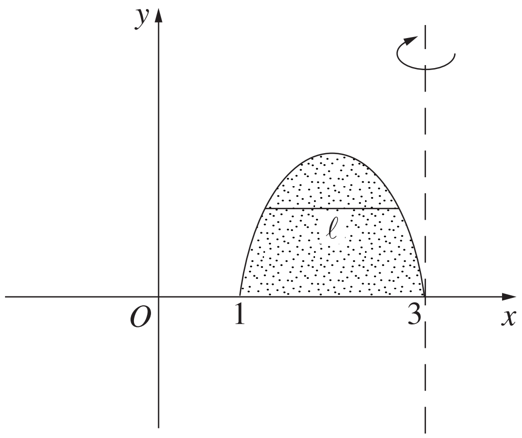
(iv) $y = e^{f(x)}$. **2**

(b) Find the eccentricity, foci and the equations of the directrices of the ellipse **3**

$$\frac{x^2}{9} + \frac{y^2}{4} = 1 .$$

Question 3 continues on page 5

(c) The region bounded by the curve $y = (x-1)(3-x)$ and the x-axis is rotated about the line $x = 3$ to form a solid. When the region is rotated, the horizontal line segment ℓ at height y sweeps out an annulus.



(i) Show that the area of the annulus at height y is given by $4\pi\sqrt{1-y}$. **3**

(ii) Find the volume of the solid. **2**

End of Question 3

Please turn over

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) A particle P of mass m moves with constant angular velocity ω on a circle of radius r. Its position at time t is given by:

$$x = r\cos\theta$$
$$y = r\sin\theta,$$

where $\theta = \omega t$.

(i) Show that there is an inward radial force of magnitude $mr\omega^2$ acting on P. **3**

(ii) A telecommunications satellite, of mass m, orbits Earth with constant angular velocity ω at a distance r from the centre of Earth. The gravitational force exerted by Earth on the satellite is $\dfrac{Am}{r^2}$, where A is a constant. By considering all other forces on the satellite to be negligible, show that **1**

$$r = \sqrt[3]{\frac{A}{\omega^2}}.$$

(b) (i) Derive the equation of the tangent to the hyperbola **2**

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point $P\,(a\sec\theta,\, b\tan\theta)$.

(ii) Show that the tangent intersects the asymptotes of the hyperbola at the points **2**

$$A\left(\frac{a\cos\theta}{1-\sin\theta}, \frac{b\cos\theta}{1-\sin\theta}\right) \quad \text{and} \quad B\left(\frac{a\cos\theta}{1+\sin\theta}, \frac{-b\cos\theta}{1+\sin\theta}\right).$$

(iii) Prove that the area of the triangle OAB is ab. **4**

Question 4 continues on page 7

Marks

Question 4 (continued)

(c) A hall has n doors. Suppose that n people each choose any door at random to enter the hall.

(i) In how many ways can this be done? **1**

(ii) What is the probability that at least one door will not be chosen by any of the people? **2**

End of Question 4

Please turn over

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) Let α, β and γ be the three roots of $x^3 + px + q = 0$, and define s_n by

$$s_n = \alpha^n + \beta^n + \gamma^n \quad \text{for} \quad n = 1, 2, 3, \ldots$$

(i) Explain why $s_1 = 0$, and show that $s_2 = -2p$ and $s_3 = -3q$. **3**

(ii) Prove that for $n > 3$ **2**

$$s_n = -ps_{n-2} - qs_{n-3}\,.$$

(iii) Deduce that **2**

$$\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{2}\right)\left(\frac{\alpha^3 + \beta^3 + \gamma^3}{3}\right).$$

Question 5 continues on page 9

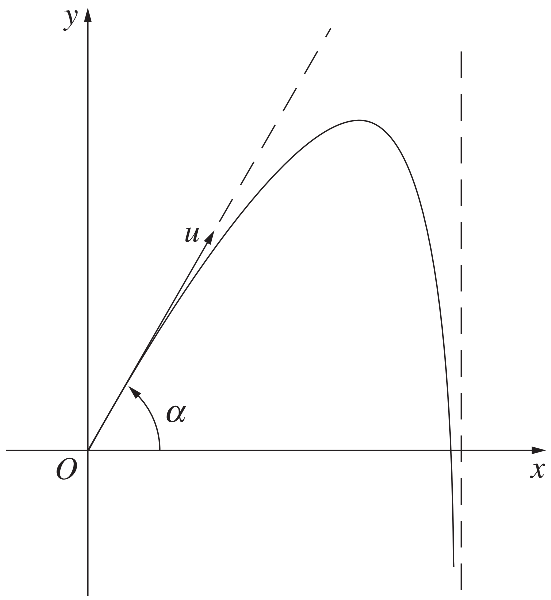
Marks

Question 5 (continued)

(b) A particle of mass m is thrown from the top, O, of a very tall building with an initial velocity u at an angle α to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -k\dot{x} \quad \text{and} \quad \ddot{y} = -k\dot{y} - g,$$

where k is a constant and the acceleration due to gravity is g. (You are NOT required to show these.)



(i) Derive the result $\dot{x} = ue^{-kt}\cos\alpha$ from the relevant equation of motion. **2**

(ii) Verify that $\dot{y} = \frac{1}{k}\left(\left(ku\sin\alpha + g\right)e^{-kt} - g\right)$ satisfies the appropriate equation of motion and initial condition. **2**

(iii) Find the value of t when the particle reaches its maximum height. **2**

(iv) What is the limiting value of the horizontal displacement of the particle? **2**

End of Question 5

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Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove the identity $\cos(a+b)x + \cos(a-b)x = 2\cos ax \cos bx$. **1**

(ii) Hence find $\int \cos 3x \cos 2x \, dx$. **2**

(b) A sequence s_n is defined by $s_1 = 1$, $s_2 = 2$ and, for $n > 2$,

$$s_n = s_{n-1} + (n-1)s_{n-2} .$$

(i) Find s_3 and s_4. **1**

(ii) Prove that $\sqrt{x} + x \geq \sqrt{x(x+1)}$ for all real numbers $x \geq 0$. **2**

(iii) Prove by induction that $s_n \geq \sqrt{n!}$ for all integers $n \geq 1$. **3**

(c) (i) Let x and y be real numbers such that $x \geq 0$ and $y \geq 0$. **1**

Prove that $\dfrac{x+y}{2} \geq \sqrt{xy}$.

(ii) Suppose that a, b, c are real numbers. **2**

Prove that $a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2$.

(iii) Show that $a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + b^2ac + c^2ab$. **2**

(iv) Deduce that if $a + b + c = d$, then $a^4 + b^4 + c^4 \geq abcd$. **1**

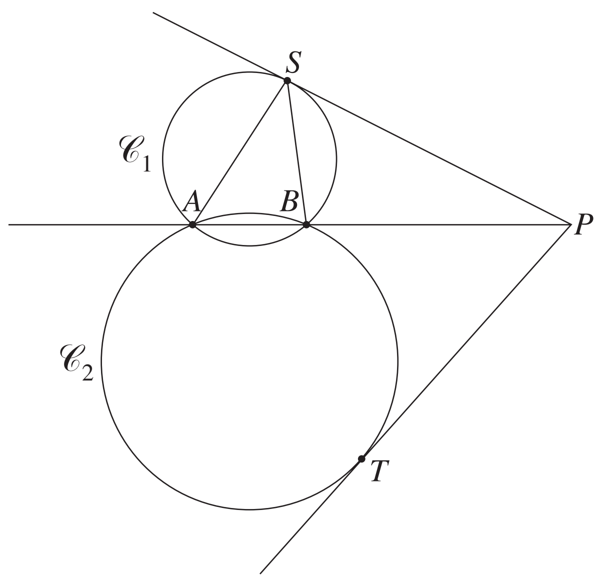
Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) The region bounded by $0 \le x \le \sqrt{3}$, $0 \le y \le x\left(3 - x^2\right)$ is rotated about the y-axis to form a solid. **3**

Use the method of cylindrical shells to find the volume of the solid.

(b)



Two circles $\mathscr{C}_1$ and $\mathscr{C}_2$ intersect at the points A and B. Let P be a point on AB produced and let PS and PT be tangents to $\mathscr{C}_1$ and $\mathscr{C}_2$ respectively, as shown in the diagram.

Copy or trace the diagram into your writing booklet.

(i) Prove that $\Delta ASP \parallel\!| \Delta SBP$. **2**

(ii) Hence, prove that $SP^2 = AP \times BP$ and deduce that $PT = PS$. **2**

(iii) The perpendicular to SP drawn from S meets the bisector of $\angle SPT$ at D. Prove that DT passes through the centre of $\mathscr{C}_2$. **3**

Question 7 continues on page 13

Marks

Question 7 (continued)

(c) Suppose that α is a real number with $0 < \alpha < \pi$.

Let $P_n = \cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{8}\right)\ldots\cos\left(\frac{\alpha}{2^n}\right)$.

(i) Show that $P_n \sin\left(\frac{\alpha}{2^n}\right) = \frac{1}{2} P_{n-1} \sin\left(\frac{\alpha}{2^{n-1}}\right)$. **2**

(ii) Deduce that $P_n = \dfrac{\sin\alpha}{2^n \sin\left(\frac{\alpha}{2^n}\right)}$. **1**

(iii) Given that $\sin x < x$ for $x > 0$, show that **2**

$$\frac{\sin\alpha}{\cos\left(\frac{\alpha}{2}\right)\cos\left(\frac{\alpha}{4}\right)\cos\left(\frac{\alpha}{8}\right)\ldots\cos\left(\frac{\alpha}{2^n}\right)} < \alpha .$$

End of Question 7

Please turn over

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose that $\omega^3 = 1, \omega \neq 1$, and k is a positive integer.

(i) Find the two possible values of $1 + \omega^k + \omega^{2k}$. **2**

(ii) Use the binomial theorem to expand $(1 + \omega)^n$ and $(1 + \omega^2)^n$, where n is a positive integer. **1**

(iii) Let ℓ be the largest integer such that $3\ell \leq n$. **2**

Deduce that

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots + \binom{n}{3\ell} = \frac{1}{3}\left(2^n + (1+\omega)^n + \left(1+\omega^2\right)^n\right).$$

(iv) If n is a multiple of 6, prove that **2**

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots + \binom{n}{n} = \frac{1}{3}\left(2^n + 2\right).$$

Question 8 continues on page 15

Marks

Question 8 (continued)

(b) Suppose that π could be written in the form $\frac{p}{q}$, where p and q are positive integers.

Define the family of integrals I_n for $n = 0, 1, 2, \ldots$ by

$$I_n = \frac{q^{2n}}{n!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^n \cos x\, dx.$$

You are given that $I_0 = 2$ and $I_1 = 4q^2$. (Do NOT prove this.)

(i) Use integration by parts twice to show that, for $n \geq 2$, **3**

$$I_n = \frac{2q^{2n}}{(n-1)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\frac{\pi^2}{4} - x^2\right)^{n-1} \cos x\, dx - \frac{4q^{2n}}{(n-2)!}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2\left(\frac{\pi^2}{4} - x^2\right)^{n-2} \cos x\, dx.$$

(ii) By writing x^2 as $\frac{\pi^2}{4} - \left(\frac{\pi^2}{4} - x^2\right)$ where appropriate, deduce that **1**

$$I_n = (4n-2)q^2 I_{n-1} - p^2q^2 I_{n-2}, \text{ for } n \geq 2.$$

(iii) Explain briefly why I_n is an integer for $n = 0, 1, 2, \ldots$ **1**

(iv) Prove that **2**

$$0 < I_n < \frac{p}{q}\left(\frac{p}{2}\right)^{2n}\frac{1}{n!} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

(v) Given that $\frac{p}{q}\left(\frac{p}{2}\right)^{2n}\frac{1}{n!} < 1$, if n is sufficiently large, deduce that π is irrational. **1**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$