BOARD OF STUDIES
NEW SOUTH WALES

2004

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) The radius of Mars is approximately 3 397 000 m. Write this number in scientific notation, correct to two significant figures. **2**

(b) Differentiate $x^4 + 5x^{-1}$ with respect to x. **2**

(c) Solve $\dfrac{x-5}{3} - \dfrac{x+1}{4} = 5$. **2**

(d) Find integers a and b such that $\left(3-\sqrt{2}\right)^2 = a - b\sqrt{2}$. **2**

(e) A packet contains 12 red, 8 green, 7 yellow and 3 black jellybeans. One jellybean is selected from the packet at random. **2**

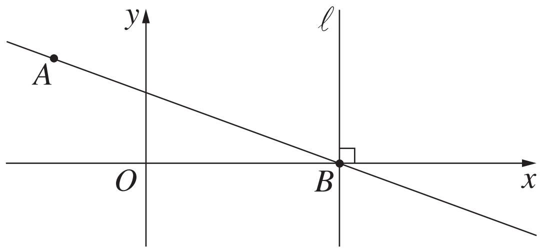
What is the probability that the selected jellybean is red or yellow?

(f) Find the values of x for which $|x+1| \le 5$. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the points $A(-1, 3)$ and $B(2, 0)$.
The line ℓ is drawn perpendicular to the x-axis through the point B.



NOT TO SCALE

(i) Calculate the length of the interval AB. **1**

(ii) Find the gradient of the line AB. **1**

(iii) What is the size of the acute angle between the line AB and the line ℓ? **1**

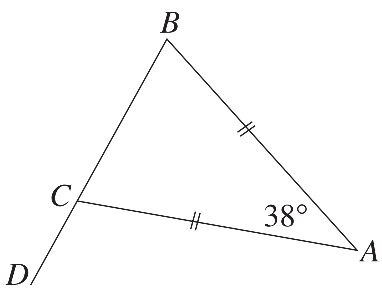
(iv) Show that the equation of the line AB is $x + y - 2 = 0$. **1**

(v) Copy the diagram into your writing booklet and shade the region defined by $x + y - 2 \leq 0$. **1**

(vi) Write down the equation of the line ℓ. **1**

(vii) The point C is on the line ℓ such that AC is perpendicular to AB. Find the coordinates of C. **2**

(b) **2**



NOT TO SCALE

In the diagram, ABC is an isosceles triangle with $AB = AC$ and $\angle BAC = 38°$. The line BC is produced to D.

Copy or trace the diagram into your writing booklet.

Find the size of $\angle ACD$. Give reasons for your answer.

(c) For what values of k does $x^2 - kx + 4 = 0$ have no real roots? **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Differentiate with respect to x:

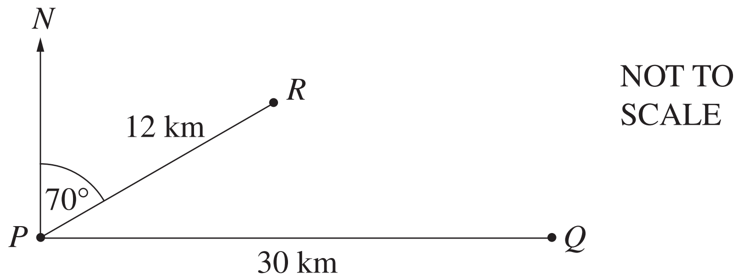
(i) $x^2 \log_e x$ **2**

(ii) $(1+\sin x)^5$. **2**

(b) (i) Evaluate $\int_1^2 e^{3x}\,dx$. **2**

(ii) Find $\int \frac{x}{x^2-3}\,dx$. **2**

(c)



The diagram shows a point P which is 30 km due west of the point Q.
The point R is 12 km from P and has a bearing from P of 070°.

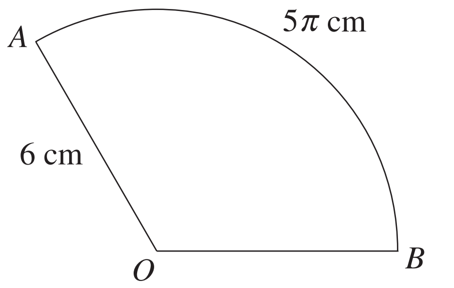
(i) Find the distance of R from Q. **2**

(ii) Find the bearing of R from Q. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) **2**



NOT TO SCALE

AOB is a sector of a circle, centre O and radius 6 cm.
The length of the arc AB is 5π cm.

Calculate the exact area of the sector AOB.

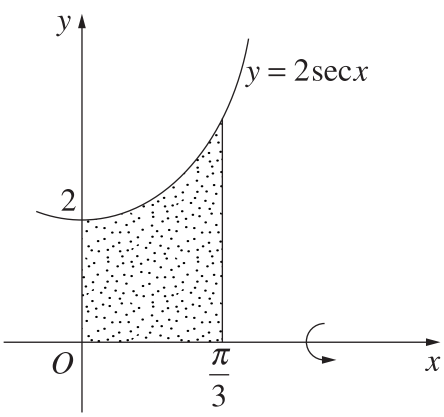
(b) Consider the function $f(x) = x^3 - 3x^2$.

(i) Find the coordinates of the stationary points of the curve $y = f(x)$ and determine their nature. **3**

(ii) Sketch the curve showing where it meets the axes. **2**

(iii) Find the values of x for which the curve $y = f(x)$ is concave up. **2**

(c) **3**



In the diagram, the shaded region is bounded by the curve $y = 2\sec x$, the coordinate axes and the line $x = \frac{\pi}{3}$. The shaded region is rotated about the x-axis.

Calculate the exact volume of the solid of revolution formed.

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Clare is learning to drive. Her first lesson is 30 minutes long. Her second lesson is 35 minutes long. Each subsequent lesson is 5 minutes longer than the lesson before.

(i) How long will Clare's twenty-first lesson be? **1**

(ii) How many hours of lessons will Clare have completed after her twenty-first lesson? **2**

(iii) During which lesson will Clare have completed a total of 50 hours of driving lessons? **2**

(b) A particle moves along a straight line so that its displacement, x metres, from a fixed point O is given by $x = 1 + 3\cos 2t$, where t is measured in seconds.

(i) What is the initial displacement of the particle? **1**

(ii) Sketch the graph of x as a function of t for $0 \le t \le \pi$. **2**

(iii) Hence, or otherwise, find when AND where the particle first comes to rest after $t = 0$. **2**

(iv) Find a time when the particle reaches its maximum speed. What is this speed? **2**

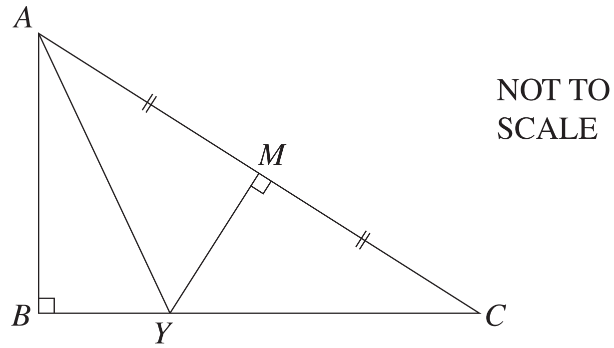
Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Solve the following equation for x: **2**

$$e^{2x} + 3e^{x} - 10 = 0.$$

(b)



The diagram shows a right-angled triangle ABC with $\angle ABC = 90°$. The point M is the midpoint of AC, and Y is the point where the perpendicular to AC at M meets BC.

(i) Show that $\Delta AYM \equiv \Delta CYM$. **2**

(ii) Suppose that it is also given that AY bisects $\angle BAC$. Find the size of $\angle YCM$ and hence find the exact ratio $MY : AC$. **3**

(c) In a game, a turn involves rolling two dice, each with faces marked 0, 1, 2, 3, 4 and 5. The score for each turn is calculated by multiplying the two numbers uppermost on the dice.

(i) What is the probability of scoring zero on the first turn? **2**

(ii) What is the probability of scoring 16 or more on the first turn? **1**

(iii) What is the probability that the sum of the scores in the first two turns is less than 45? **2**

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\sum_{n=2}^{4} n^2$. **1**

(b) At the beginning of 1991 Australia's population was 17 million. At the beginning of 2004 the population was 20 million.

Assume that the population P is increasing exponentially and satisfies an equation of the form $P = Ae^{kt}$, where A and k are constants, and t is measured in years from the beginning of 1991.

(i) Show that $P = Ae^{kt}$ satisfies $\frac{dP}{dt} = kP$. **1**

(ii) What is the value of A? **1**

(iii) Find the value of k. **2**

(iv) Predict the year during which Australia's population will reach 30 million. **2**

(c) Betty decides to set up a trust fund for her grandson, Luis. She invests \$80 at the beginning of each month. The money is invested at 6% per annum, compounded monthly.

The trust fund matures at the end of the month of her final investment, 25 years after her first investment. This means that Betty makes 300 monthly investments.

(i) After 25 years, what will be the value of the first \$80 invested? **2**

(ii) By writing a geometric series for the value of all Betty's investments, calculate the final value of Luis' trust fund. **3**

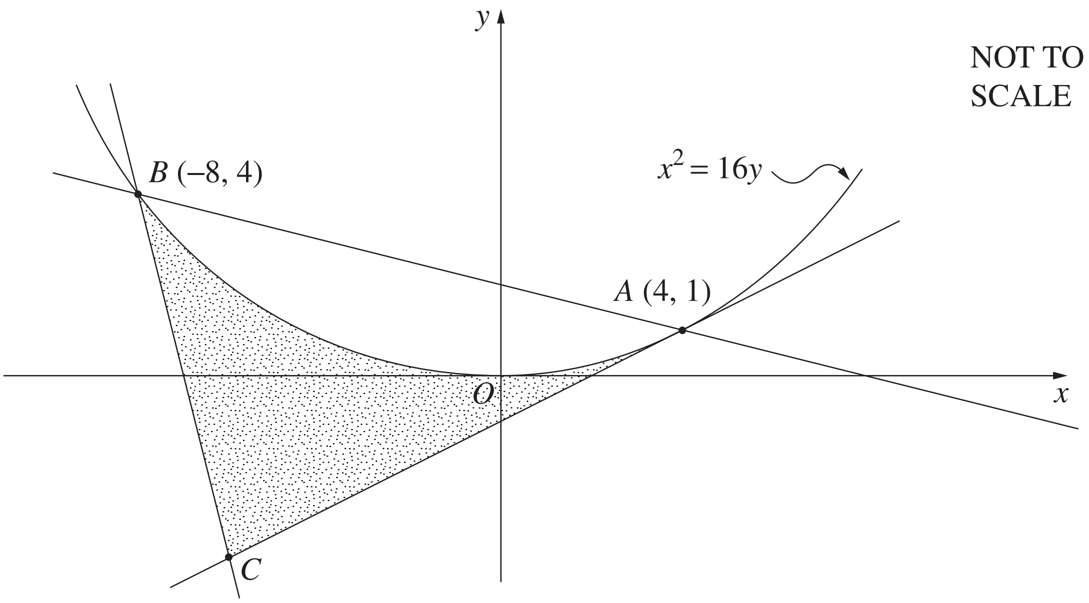
Marks

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $\cos\theta\tan\theta = \sin\theta$. **1**

(ii) Hence solve $8\sin\theta\cos\theta\tan\theta = \operatorname{cosec}\theta$ for $0 \le \theta \le 2\pi$. **2**

(b) The diagram shows the graph of the parabola $x^2 = 16y$. The points $A\,(4, 1)$ and $B\,(-8, 4)$ are on the parabola, and C is the point where the tangent to the parabola at A intersects the directrix.



(i) Write down the equation of the directrix of the parabola $x^2 = 16y$. **1**

(ii) Find the equation of the tangent to the parabola at the point A. **2**

(iii) Show that C is the point $(-6, -4)$. **1**

(iv) Given that the equation of the line AB is $y = 2 - \dfrac{x}{4}$, find the area bounded by the line AB and the parabola. **2**

(v) Hence, or otherwise, find the shaded area bounded by the parabola, the tangent at A and the line BC. **3**

Marks

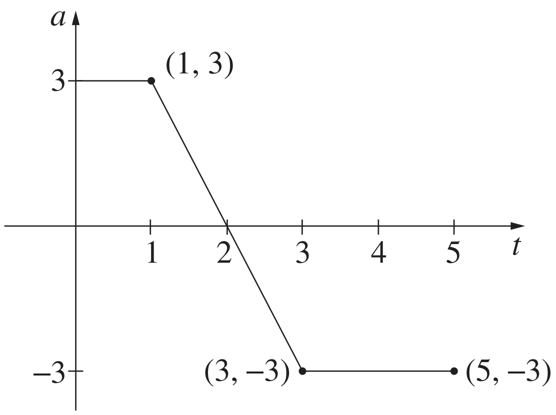
Question 9 (12 marks) Use a SEPARATE writing booklet.

(a) Consider the geometric series $1 - \tan^2\theta + \tan^4\theta - \ldots$

(i) When the limiting sum exists, find its value in simplest form. **2**

(ii) For what values of θ in the interval $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$ does the limiting sum of the series exist? **2**

(b) A particle moves along the x-axis. Initially it is at rest at the origin. The graph shows the acceleration, a, of the particle as a function of time t for $0 \le t \le 5$.



(i) Write down the time at which the velocity of the particle is a maximum. **1**

(ii) At what time during the interval $0 \le t \le 5$ is the particle furthest from the origin? Give brief reasons for your answer. **2**

(c) Consider the function $f(x) = \dfrac{\log_e x}{x}$, for $x > 0$.

(i) Show that the graph of $y = f(x)$ has a stationary point at $x = e$. **2**

(ii) By considering the gradient on either side of $x = e$, or otherwise, show that the stationary point is a maximum. **1**

(iii) Use the fact that the maximum value of $f(x)$ occurs at $x = e$ to deduce that $e^x \ge x^e$ for all $x > 0$. **2**

Marks

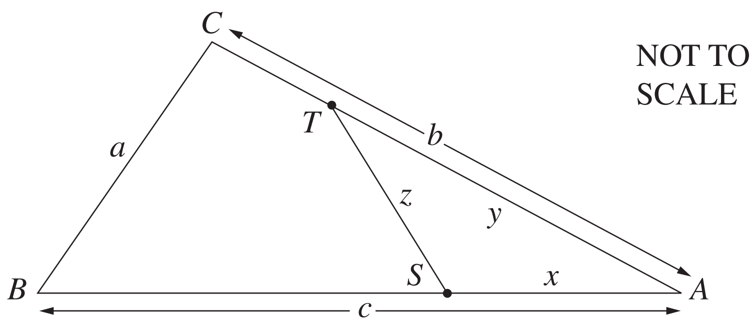
Question 10 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Use Simpson's rule with 3 function values to find an approximation to the area under the curve $y = \frac{1}{x}$ between $x = a$ and $x = 3a$, where a is positive. **2**

(ii) Using the result in part (i), show that **1**

$$\ln 3 \doteqdot \frac{10}{9}.$$

(b)



The diagram shows a triangular piece of land ABC with dimensions $AB = c$ metres, $AC = b$ metres and $BC = a$ metres, where $a \le b \le c$.

The owner of the land wants to build a straight fence to divide the land into two pieces of equal area. Let S and T be points on AB and AC respectively so that ST divides the land into two pieces of equal area.

Let $AS = x$ metres, $AT = y$ metres and $ST = z$ metres.

(i) Show that $xy = \frac{1}{2}bc$. **1**

(ii) Use the cosine rule in triangle AST to show that **2**

$$z^2 = x^2 + \frac{b^2c^2}{4x^2} - bc\cos A.$$

(iii) Show that the value of z^2 in the equation in part (ii) is a minimum when $x = \sqrt{\frac{bc}{2}}$. **4**

(iv) Show that the minimum length of the fence is $\sqrt{\frac{(P-2b)(P-2c)}{2}}$ metres, where $P = a + b + c$. **2**

(You may assume that the value of x given in part (iii) is feasible.)

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$