BOARD OF STUDIES
NEW SOUTH WALES

2004

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Indicate the region on the number plane satisfied by $y \geq |x+1|$. **2**

(b) Solve $\dfrac{4}{x+1} < 3$. **3**

(c) Let A be the point $(3, -1)$ and B be the point $(9, 2)$. **2**

Find the coordinates of the point P which divides the interval AB externally in the ratio $5:2$.

(d) Find $\displaystyle\int_0^1 \frac{dx}{\sqrt{4-x^2}}$. **2**

(e) Use the substitution $u = x - 3$ to evaluate **3**

$$\int_3^4 x\sqrt{x-3}\, dx.$$

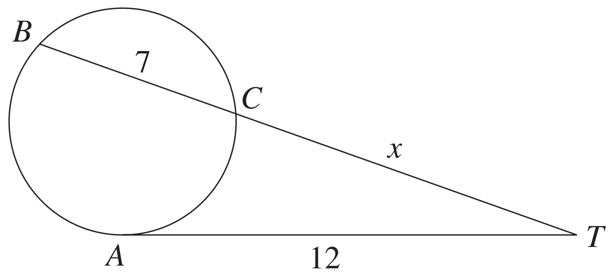
Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\lim_{x \to 0} \dfrac{\sin\left(\dfrac{x}{5}\right)}{2x}$. **2**

(b) Find $\dfrac{d}{dx}\cos^{-1}\left(3x^2\right)$. **2**

(c) **2**



The line AT is the tangent to the circle at A, and BT is a secant meeting the circle at B and C.

Given that $AT = 12$, $BC = 7$ and $CT = x$, find the value of x.

(d) (i) Write $8\cos x + 6\sin x$ in the form $A\cos(x - \alpha)$, where $A > 0$ and $0 \le \alpha \le \dfrac{\pi}{2}$. **2**

(ii) Hence, or otherwise, solve the equation $8\cos x + 6\sin x = 5$ for $0 \le x \le 2\pi$. Give your answers correct to three decimal places. **2**

(e) A four-person team is to be chosen at random from nine women and seven men.

(i) In how many ways can this team be chosen? **1**

(ii) What is the probability that the team will consist of four women? **1**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int \cos^2 4x \, dx$. **2**

(b) Let $P(x) = (x+1)(x-3)Q(x) + a(x+1) + b$, where $Q(x)$ is a polynomial and a and b are real numbers.

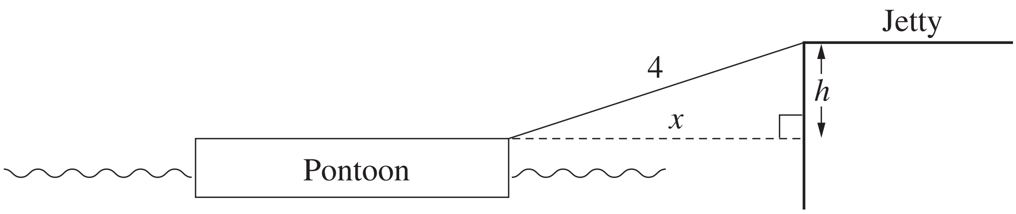
When $P(x)$ is divided by $(x+1)$ the remainder is -11.

When $P(x)$ is divided by $(x-3)$ the remainder is 1.

(i) What is the value of b? **1**

(ii) What is the remainder when $P(x)$ is divided by $(x+1)(x-3)$? **2**

(c) A ferry wharf consists of a floating pontoon linked to a jetty by a 4 metre long walkway. Let h metres be the difference in height between the top of the pontoon and the top of the jetty and let x metres be the horizontal distance between the pontoon and the jetty.



(i) Find an expression for x in terms of h. **1**

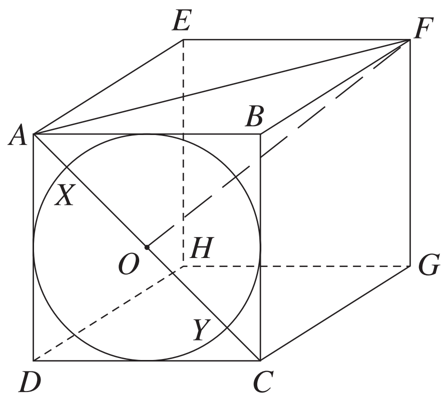
(ii) When the top of the pontoon is 1 metre lower than the top of the jetty, the tide is rising at a rate of 0.3 metres per hour. **3**

At what rate is the pontoon moving away from the jetty?

Question 3 continues on page 5

Question 3 (continued)

(d)



The length of each edge of the cube *ABCDEFGH* is 2 metres. A circle is drawn on the face *ABCD* so that it touches all four edges of the face. The centre of the circle is *O* and the diagonal *AC* meets the circle at *X* and *Y*.

(i) Explain why $\angle FAC = 60°$. **1**

(ii) Show that $FO = \sqrt{6}$ metres. **1**

(iii) Calculate the size of $\angle XFY$ to the nearest degree. **1**

End of Question 3

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that for all integers $n \geq 3$, **3**

$$\left(1-\frac{2}{3}\right)\left(1-\frac{2}{4}\right)\left(1-\frac{2}{5}\right)\cdots\left(1-\frac{2}{n}\right)=\frac{2}{n(n-1)}.$$

(b) The two points $P\left(2ap,\ ap^2\right)$ and $Q\left(2aq,\ aq^2\right)$ are on the parabola $x^2=4ay$.

(i) The equation of the tangent to $x^2=4ay$ at an arbitrary point $\left(2at,\ at^2\right)$ on the parabola is $y=tx-at^2$. (Do not prove this.) **2**

Show that the tangents at the points P and Q meet at R, where R is the point $\left(a(p+q),\ apq\right)$.

(ii) As P varies, the point Q is always chosen so that $\angle POQ$ is a right angle, where O is the origin. **2**

Find the locus of R.

(c) Katie is one of ten members of a social club. Each week one member is selected at random to win a prize.

(i) What is the probability that in the first 7 weeks Katie will win at least 1 prize? **1**

(ii) Show that in the first 20 weeks Katie has a greater chance of winning exactly 2 prizes than of winning exactly 1 prize. **2**

(iii) For how many weeks must Katie participate in the prize drawing so that she has a greater chance of winning exactly 3 prizes than of winning exactly 2 prizes? **2**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

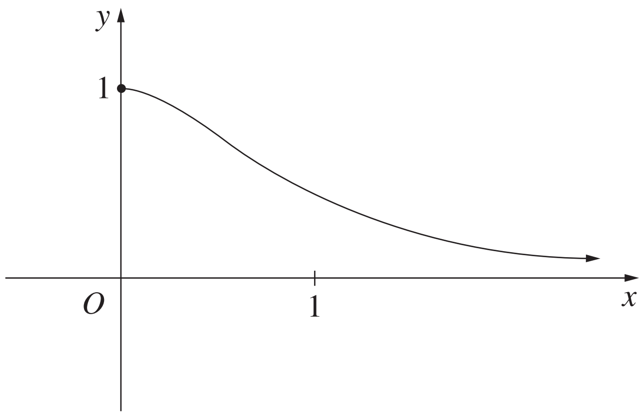
(a) A particle is moving along the x-axis, starting from a position 2 metres to the right of the origin (that is, $x = 2$ when $t = 0$) with an initial velocity of $5\ \mathrm{m\,s^{-1}}$ and an acceleration given by

$$\ddot{x} = 2x^3 + 2x.$$

(i) Show that $\dot{x} = x^2 + 1$. **2**

(ii) Hence find an expression for x in terms of t. **3**

(b) The diagram below shows a sketch of the graph of $y = f(x)$, where $f(x) = \dfrac{1}{1+x^2}$ for $x \geq 0$.



(i) Copy or trace this diagram into your writing booklet. **1**

On the same set of axes, sketch the graph of the inverse function, $y = f^{-1}(x)$.

(ii) State the domain of $f^{-1}(x)$. **1**

(iii) Find an expression for $y = f^{-1}(x)$ in terms of x. **2**

(iv) The graphs of $y = f(x)$ and $y = f^{-1}(x)$ meet at exactly one point P. **1**
Let α be the x-coordinate of P. Explain why α is a root of the equation

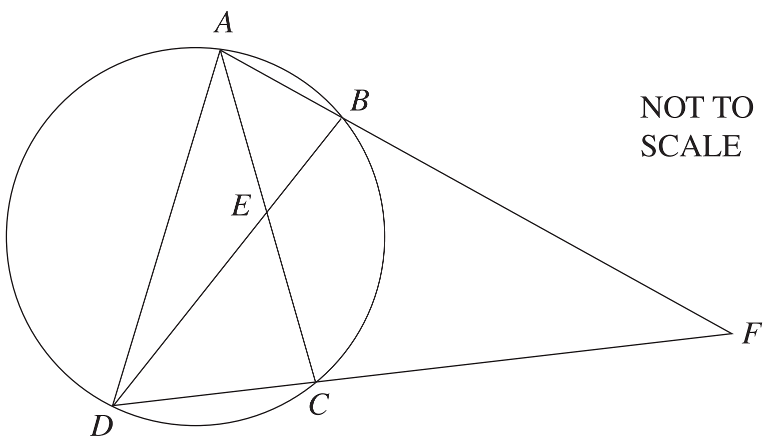
$$x^3 + x - 1 = 0\ .$$

(v) Take 0.5 as a first approximation for α. Use one application of Newton's method to find a second approximation for α. **2**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a)



The points A, B, C and D are placed on a circle of radius r such that AC and BD meet at E. The lines AB and DC are produced to meet at F, and $BECF$ is a cyclic quadrilateral.

Copy or trace this diagram into your writing booklet.

(i) Find the size of $\angle DBF$, giving reasons for your answer. **2**

(ii) Find an expression for the length of AD in terms of r. **1**

Question 6 continues on page 9

Marks

Question 6 (continued)

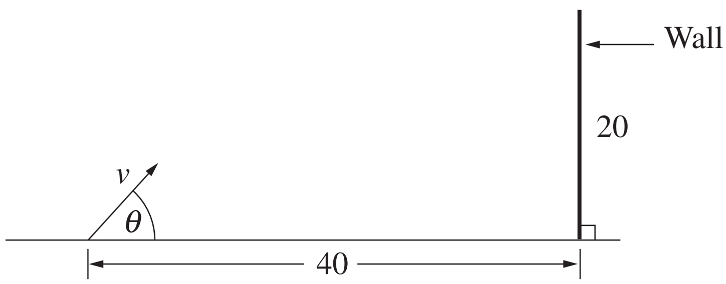
(b) A fire hose is at ground level on a horizontal plane. Water is projected from the hose. The angle of projection, θ, is allowed to vary. The speed of the water as it leaves the hose, v metres per second, remains constant. You may assume that if the origin is taken to be the point of projection, the path of the water is given by the parametric equations

$$x = vt\cos\theta$$
$$y = vt\sin\theta - \frac{1}{2}gt^2$$

where $g\ \mathrm{m\,s^{-2}}$ is the acceleration due to gravity. (Do NOT prove this.)

(i) Show that the water returns to ground level at a distance $\dfrac{v^2\sin 2\theta}{g}$ metres from the point of projection. **2**

This fire hose is now aimed at a 20 metre high thin wall from a point of projection at ground level 40 metres from the base of the wall. It is known that when the angle θ is 15°, the water just reaches the base of the wall.



(ii) Show that $v^2 = 80g$. **1**

(iii) Show that the cartesian equation of the path of the water is given by **2**

$$y = x\tan\theta - \frac{x^2\sec^2\theta}{160}.$$

(iv) Show that the water just clears the top of the wall if **2**

$$\tan^2\theta - 4\tan\theta + 3 = 0.$$

(v) Find all values of θ for which the water hits the front of the wall. **2**

End of Question 6

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) The rise and fall of the tide is assumed to be simple harmonic, with the time between successive high tides being 12.5 hours. A ship is to sail from a wharf to the harbour entrance and then out to sea. On the morning the ship is to sail, high tide at the wharf occurs at 2 am. The water depths at the wharf at high tide and low tide are 10 metres and 4 metres respectively.

(i) Show that the water depth, y metres, at the wharf is given by **2**

$y = 7 + 3\cos\left(\dfrac{4\pi t}{25}\right)$, where t is the number of hours after high tide.

(ii) An overhead power cable obstructs the ship's exit from the wharf. The ship can only leave if the water depth at the wharf is 8.5 metres or less. **2**

Show that the earliest possible time that the ship can leave the wharf is 4:05 am.

(iii) At the harbour entrance, the difference between the water level at high tide and low tide is also 6 metres. However, tides at the harbour entrance occur 1 hour earlier than at the wharf. In order for the ship to be able to sail through the shallow harbour entrance, the water level must be at least 2 metres above the low tide level. **2**

The ship takes 20 minutes to sail from the wharf to the harbour entrance and it must be out to sea by 7 am. What is the latest time the ship can leave the wharf?

(b) (i) Show that for all positive integers n, **1**

$$x\left[(1+x)^{n-1} + (1+x)^{n-2} + \cdots + (1+x)^2 + (1+x) + 1\right] = (1+x)^n - 1.$$

(ii) Hence show that for $1 \le k \le n$, **1**

$$\binom{n-1}{k-1} + \binom{n-2}{k-1} + \binom{n-3}{k-1} + \cdots + \binom{k-1}{k-1} = \binom{n}{k}.$$

(iii) Show that $n\binom{n-1}{k} = (k+1)\binom{n}{k+1}$. **1**

(iv) By differentiating both sides of the identity in (i), show that for $1 \le k < n$, **3**

$$(n-1)\binom{n-2}{k-1} + (n-2)\binom{n-3}{k-1} + \cdots + k\binom{k-1}{k-1} = k\binom{n}{k+1}.$$

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$