BOARD OF STUDIES
NEW SOUTH WALES

2004
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120
- Attempt Questions 1–8
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Use integration by parts to find $\int xe^{3x}\,dx$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin x}{\cos^3 x}\,dx$. **3**

(c) By completing the square, find $\int \frac{dx}{\sqrt{5+4x-x^2}}$. **2**

(d) (i) Find real numbers a and b such that **2**

$$\frac{x^2-7x+4}{(x+1)(x-1)^2} \equiv \frac{a}{x+1} + \frac{b}{x-1} - \frac{1}{(x-1)^2}.$$

(ii) Hence find $\int \frac{x^2-7x+4}{(x+1)(x-1)^2}\,dx$. **2**

(e) Use the substitution $x = 2\sin\theta$ to find $\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$. **4**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 1 + 2i$ and $w = 3 - i$.

Find, in the form $x + iy$,

(i) zw **1**

(ii) $\overline{\left(\frac{10}{z}\right)}$. **1**

(b) Let $\alpha = 1 + i\sqrt{3}$ and $\beta = 1 + i$.

(i) Find $\frac{\alpha}{\beta}$, in the form $x + iy$. **1**

(ii) Express α in modulus-argument form. **2**

(iii) Given that β has the modulus-argument form **1**

$$\beta = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right),$$

find the modulus-argument form of $\frac{\alpha}{\beta}$.

(iv) Hence find the exact value of $\sin\frac{\pi}{12}$. **1**

(c) Sketch the region in the complex plane where the inequalities **3**

$$|z + \bar{z}| \leq 1 \text{ and } |z - i| \leq 1$$

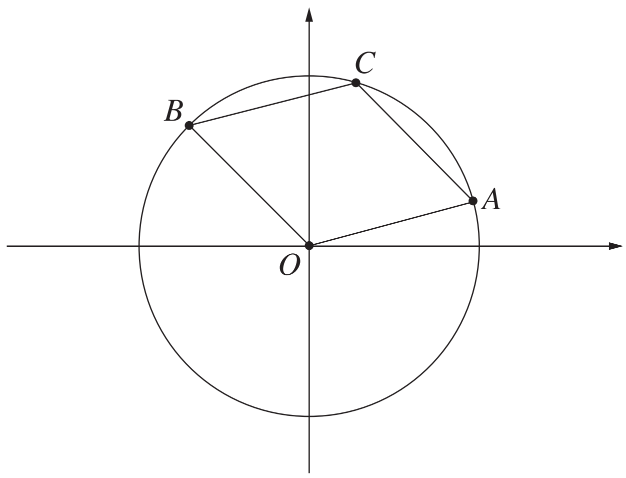
hold simultaneously.

Question 2 continues on page 4

Marks

Question 2 (continued)

(d) The diagram shows two distinct points A and B that represent the complex numbers z and w respectively. The points A and B lie on the circle of radius r centred at O. The point C representing the complex number $z + w$ also lies on this circle.



Copy the diagram into your writing booklet.

(i) Using the fact that C lies on the circle, show geometrically that **2**

$$\angle AOB = \frac{2\pi}{3}.$$

(ii) Hence show that $z^3 = w^3$. **2**

(iii) Show that $z^2 + w^2 + zw = 0$. **1**

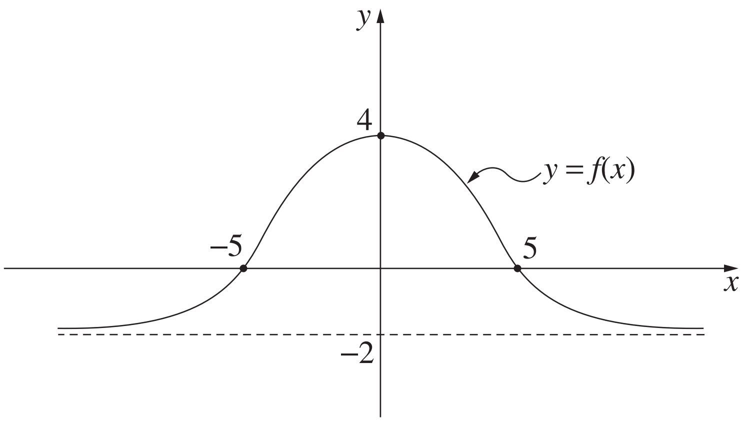
End of Question 2

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) Sketch the curve $y = \dfrac{4x^2}{x^2 - 9}$ showing all asymptotes. **3**

(b) The diagram shows the graph of $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$ **2**

(ii) $y = \left(f(x)\right)^2$ **2**

(iii) $y = \dfrac{1}{\sqrt{f(x)}}$. **2**

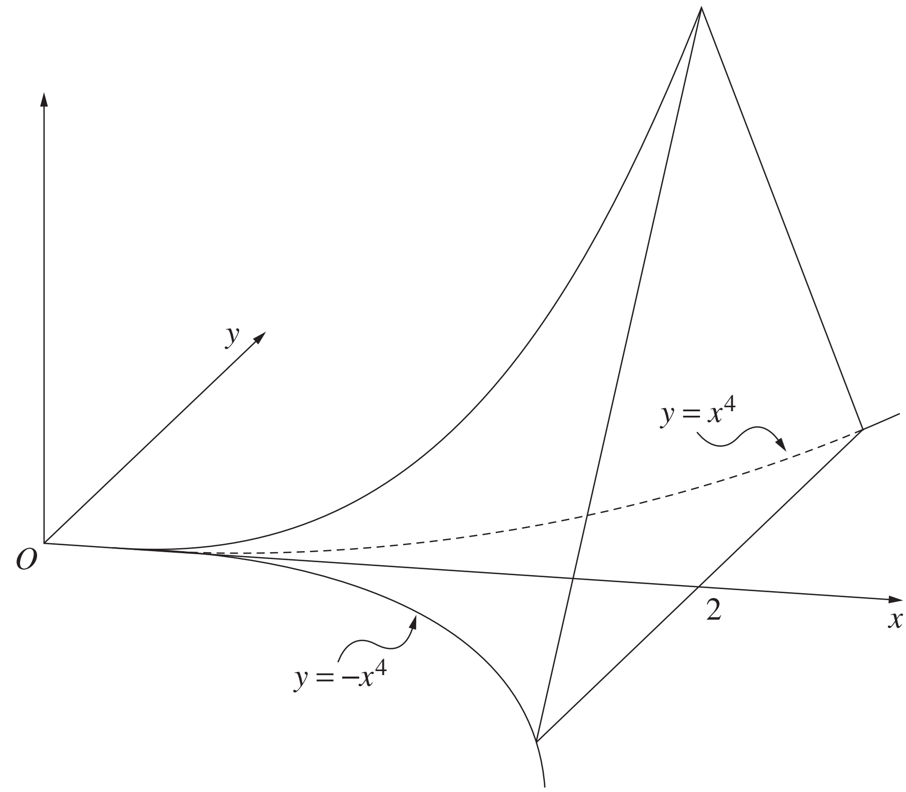
(c) Find the equation of the tangent to the curve defined by $x^2 - xy + y^3 = 5$ at the point $(2, -1)$. **3**

Question 3 continues on page 6

Marks

Question 3 (continued)

(d) The base of a solid is the region in the xy plane enclosed by the curves $y = x^4$, $y = -x^4$ and the line $x = 2$. Each cross-section perpendicular to the x-axis is an equilateral triangle.



(i) Show that the area of the triangular cross-section at $x = h$ is $\sqrt{3}\, h^8$. **1**

(ii) Hence find the volume of the solid. **2**

End of Question 3

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) Let α, β, and γ be the zeros of the polynomial $p(x) = 3x^3 + 7x^2 + 11x + 51$.

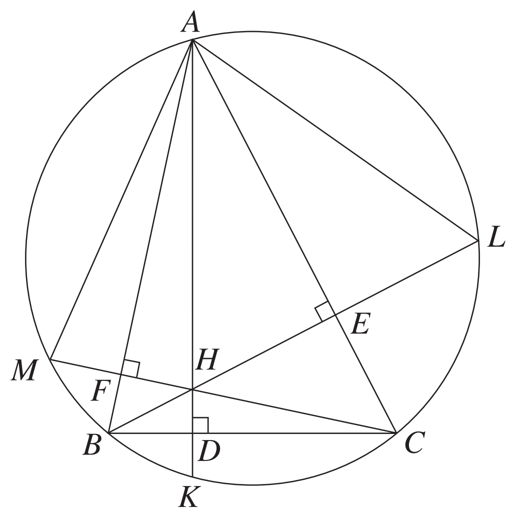
(i) Find $\alpha^2\beta\gamma + \alpha\beta^2\gamma + \alpha\beta\gamma^2$. **1**

(ii) Find $\alpha^2 + \beta^2 + \gamma^2$. **2**

(iii) Using part (ii), or otherwise, determine how many of the zeros of $p(x)$ are real. Justify your answer. **1**

(b) The vertices of an acute-angled triangle ABC lie on a circle. The perpendiculars from A, B and C meet BC, AC and AB at D, E and F respectively. These perpendiculars meet at H.

The perpendiculars AD, BE and CF are produced to meet the circle at K, L and M respectively.



(i) Prove that $\angle AHE = \angle DCE$. **2**

(ii) Deduce that $AH = AL$. **1**

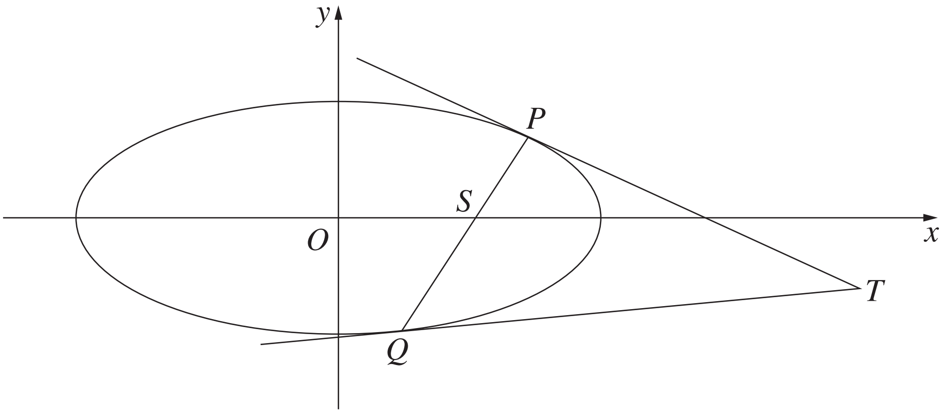
(iii) State a similar result for triangle AMH. **1**

(iv) Show that the length of the arc BKC is half the length of the arc MKL. **2**

Question 4 continues on page 8

Question 4 (continued)

(c)



The point P lies on the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. The chord through P and the focus $S(ae, 0)$ meets the ellipse at Q. The tangents to the ellipse at P and Q meet at the point $T(x_0, y_0)$, so the equation of PQ is $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$. (Do NOT prove this.)

(i) Using the equation of PQ, show that T lies on the directrix. **1**

The point P is now chosen so that T also lies on the x-axis.

(ii) What is the value of the ratio $\dfrac{PS}{ST}$? **2**

(iii) Show that $\angle PTQ$ is less than a right angle. **1**

(iv) Show that the area of triangle PQT is $b^2\left(\dfrac{1}{e} - e\right)$. **1**

End of Question 4

Marks

Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Let $a > 0$. Find the points where the line $y = ax$ and the curve $y = x(x - a)$ intersect. **1**

(ii) Let R be the region in the plane for which $x(x - a) \leq y \leq ax$. Sketch R. **1**

(iii) A solid is formed by rotating the region R about the line $x = -2a$. Use the method of cylindrical shells to find the volume of the solid. **4**

(b) (i) In how many ways can n students be placed in two distinct rooms so that neither room is empty? **1**

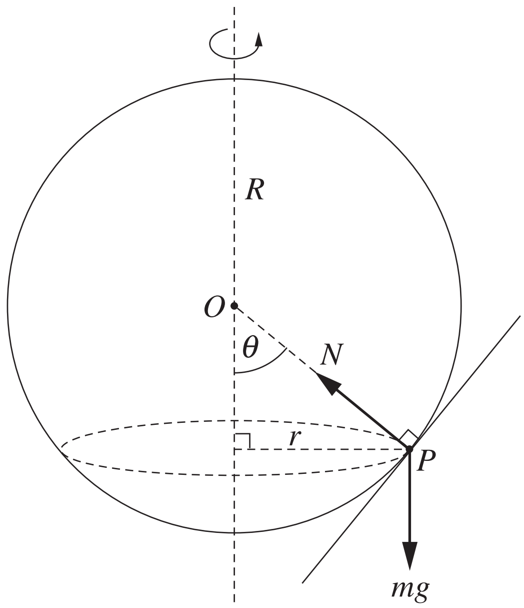
(ii) In how many ways can five students be placed in three distinct rooms so that no room is empty? **2**

Question 5 continues on page 10

Marks

Question 5 (continued)

(c) A smooth sphere with centre O and radius R is rotating about its vertical diameter at a uniform angular velocity, ω radians per second. A marble is free to roll around the inside of the sphere.



Assume that the marble can be considered as a point P which is acted upon by gravity and the normal reaction force N from the sphere. The marble describes a horizontal circle of radius r with the same uniform angular velocity, ω radians per second. Let the angle between OP and the vertical diameter be θ.

(i) Explain why $mr\omega^2 = N\sin\theta$ and $mg = N\cos\theta$. **2**

(ii) Show that either $\cos\theta = \dfrac{g}{R\omega^2}$ or $\theta = 0$. **3**

(iii) Hence, or otherwise, show that if $\theta \neq 0$ then $\omega > \sqrt{\dfrac{g}{R}}$. **1**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that **2**

$$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx = \frac{\pi}{2}\,.$$

(ii) By making the substitution $x = \pi - u$, find **3**

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx\,.$$

(b) A particle is released from the origin O with an initial velocity of $A\,\mathrm{m\,s^{-1}}$ directed vertically downward. The particle is subject to a constant gravitational force and a resistance which is proportional to the velocity, $v\,\mathrm{m\,s^{-1}}$, of the particle.

Let x be the displacement in metres of the particle below O at time t seconds after the release of the particle, so that the equation of motion is

$$\ddot{x} = g - kv\,,$$

where $g\,\mathrm{m\,s^{-2}}$ is the acceleration due to gravity.

(i) The terminal velocity of the particle is $B\,\mathrm{m\,s^{-1}}$. Show that $k = \frac{g}{B}$. **1**

(ii) Verify that v satisfies the equation $\frac{d}{dt}\left(ve^{kt}\right) = ge^{kt}$. **2**

(iii) Hence show that the velocity of the particle is given by **2**

$$v = B - (B-A)e^{-\frac{gt}{B}}.$$

(iv) Deduce that $x = Bt - \frac{B}{g}(B-A)\left(1 - e^{-\frac{gt}{B}}\right)$. **2**

Question 6 continues on page 12

Marks

Question 6 (continued)

At the same time as the particle is released from O, an identical particle is released from the point P which is h metres below O. The second particle has an initial velocity of $A\ \mathrm{m\,s^{-1}}$ directed vertically upward.

Its displacement below O is given by $x = h + Bt - \frac{B}{g}(B + A)\left(1 - e^{-\frac{gt}{B}}\right)$.
(Do NOT prove this.)

(v) Suppose that the two particles meet after T seconds. Show that **2**

$$T = \frac{B}{g}\log_e\left(\frac{2AB}{2AB - gh}\right).$$

(vi) The value of A can be varied. What condition must A satisfy so that the two particles can meet? **1**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Let a be a positive real number. Show that $a+\dfrac{1}{a}\geq 2$. **2**

(ii) Let n be a positive integer and $a_1, a_2, \ldots, a_n$ be n positive real numbers. **4**

Prove by induction that $\left(a_1+a_2+\ldots+a_n\right)\left(\dfrac{1}{a_1}+\dfrac{1}{a_2}+\ldots+\dfrac{1}{a_n}\right)\geq n^2$.

(iii) Hence show that $\operatorname{cosec}^2\theta+\sec^2\theta+\cot^2\theta\geq 9\cos^2\theta$. **1**

(b) Let α be a real number and suppose that z is a complex number such that

$$z+\frac{1}{z}=2\cos\alpha.$$

(i) By reducing the above equation to a quadratic equation in z, solve for z and use de Moivre's theorem to show that **3**

$$z^n+\frac{1}{z^n}=2\cos n\alpha.$$

(ii) Let $w=z+\dfrac{1}{z}$. Prove that **2**

$$w^3+w^2-2w-2=\left(z+\frac{1}{z}\right)+\left(z^2+\frac{1}{z^2}\right)+\left(z^3+\frac{1}{z^3}\right).$$

(iii) Hence, or otherwise, find all solutions of **3**

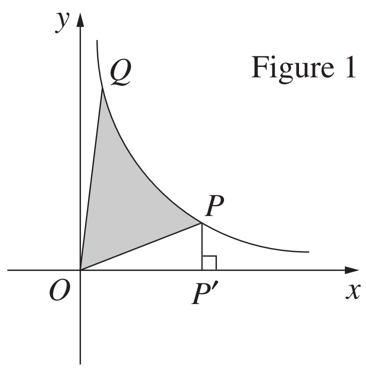
$$\cos\alpha+\cos 2\alpha+\cos 3\alpha=0,$$

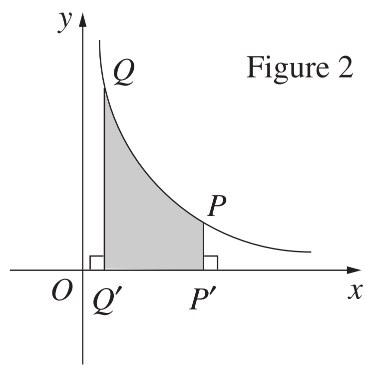
in the range $0\leq\alpha\leq 2\pi$.

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Let $P\left(p, \dfrac{1}{p}\right)$ and $Q\left(q, \dfrac{1}{q}\right)$ be points on the hyperbola $y = \dfrac{1}{x}$ with $p > q > 0$. Let P' be the point $(p, 0)$ and Q' be the point $(q, 0)$. The shaded region OPQ in Figure 1 is bounded by the lines OP, OQ and the hyperbola. The shaded region $Q'QPP'$ in Figure 2 is bounded by the lines QQ', PP', $P'Q'$ and the hyperbola.

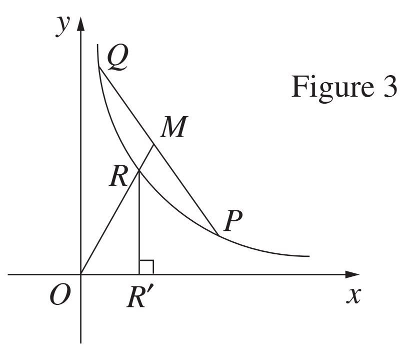




(i) Find the area of triangle OPP'. **1**

(ii) Prove that the area of the shaded region OPQ is equal to the area of the shaded region $Q'QPP'$. **1**

Let M be the midpoint of the chord PQ and $R\left(r, \dfrac{1}{r}\right)$ be the intersection of the line OM with the hyperbola. Let R' be the point $(r, 0)$, as shown in Figure 3.



(iii) By using similar triangles, or otherwise, prove that $r^2 = pq$. **2**

(iv) By using integration, or otherwise, show that the line RR' divides the shaded region $Q'QPP'$ into two pieces of equal area. **2**

(v) Deduce that the line OR divides the shaded region OPQ into two pieces of equal area. **1**

Question 8 continues on page 15

Marks

Question 8 (continued)

(b) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x \, dx$ and let $J_n = (-1)^n I_{2n}$ for $n = 0, 1, 2, \ldots$

(i) Show that $I_n + I_{n+2} = \dfrac{1}{n+1}$. **2**

(ii) Deduce that $J_n - J_{n-1} = \dfrac{(-1)^n}{2n-1}$ for $n \geq 1$. **1**

(iii) Show that $J_m = \dfrac{\pi}{4} + \sum_{n=1}^{m} \dfrac{(-1)^n}{2n-1}$. **2**

(iv) Use the substitution $u = \tan x$ to show that $I_n = \int_0^1 \dfrac{u^n}{1+u^2} \, du$. **1**

(v) Deduce that $0 \leq I_n \leq \dfrac{1}{n+1}$ and conclude that $J_n \to 0$ as $n \to \infty$. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$