BOARD OF STUDIES

NEW SOUTH WALES

2005

HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 120

- Attempt Questions 1–10
- All questions are of equal value

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Total marks – 120
Attempt Questions 1–10
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\sqrt{\dfrac{275.4}{5.2\times 3.9}}$ correct to two significant figures. **2**

(b) Factorise $x^3 - 27$. **2**

(c) Find a primitive of $4 + \sec^2 x$. **2**

(d) Express $\dfrac{(2x-3)}{2} - \dfrac{(x-1)}{5}$ as a single fraction in its simplest form. **2**

(e) Find the values of x for which $|x-3| \le 1$. **2**

(f) Find the coordinates of the focus of the parabola $x^2 = 8(y-1)$. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Solve $\cos\theta = \dfrac{1}{\sqrt{2}}$ for $0 \le \theta \le 2\pi$. **2**

(b) Differentiate with respect to x:

(i) $x \sin x$ **2**

(ii) $\dfrac{x^2}{x-1}$. **2**

(c) (i) Find $\displaystyle\int \frac{6x^2}{x^3+1}\,dx$. **2**

(ii) Evaluate $\displaystyle\int_0^{\frac{\pi}{6}} \cos 3x \, dx$. **2**

(d) Find the equation of the tangent to $y = \log_e x$ at the point $(e, 1)$. **2**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

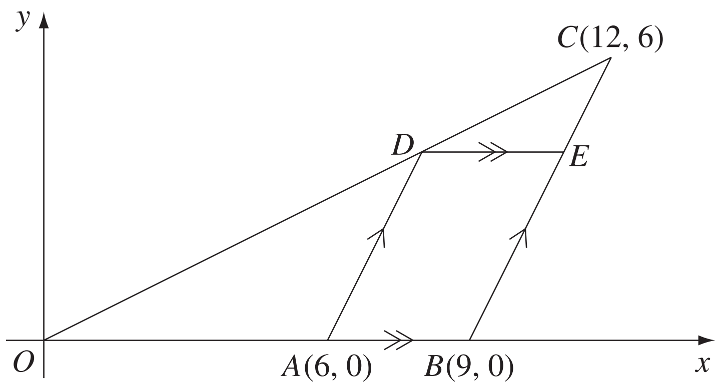
(a) Evaluate $\sum_{n=3}^{5}(2n+1)$. **1**

(b) The lengths of the sides of a triangle are 7 cm, 8 cm and 13 cm.

(i) Find the size of the angle opposite the longest side. **2**

(ii) Find the area of the triangle. **1**

(c)



NOT TO SCALE

In the diagram, A, B and C are the points $(6,0)$, $(9,0)$ and $(12,6)$ respectively. The equation of the line OC is $x-2y=0$. The point D on OC is chosen so that AD is parallel to BC. The point E on BC is chosen so that DE is parallel to the x-axis.

(i) Show that the equation of the line AD is $y=2x-12$. **2**

(ii) Find the coordinates of the point D. **2**

(iii) Find the coordinates of the point E. **1**

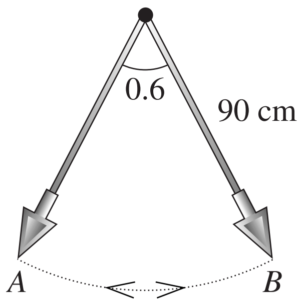
(iv) Prove that $\Delta OAD \,|||\, \Delta DEC$. **2**

(v) Hence, or otherwise, find the ratio of the lengths AD and EC. **1**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a)



NOT
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A pendulum is 90 cm long and swings through an angle of 0.6 radians. The extreme positions of the pendulum are indicated by the points *A* and *B* in the diagram.

(i) Find the length of the arc *AB*. **1**

(ii) Find the straight-line distance between the extreme positions of the pendulum. **2**

(iii) Find the area of the sector swept out by the pendulum. **1**

(b) A function $f(x)$ is defined by $f(x) = (x+3)(x^2-9)$.

(i) Find all solutions of $f(x) = 0$. **2**

(ii) Find the coordinates of the turning points of the graph of $y = f(x)$, and determine their nature. **3**

(iii) Hence sketch the graph of $y = f(x)$, showing the turning points and the points where the curve meets the x-axis. **2**

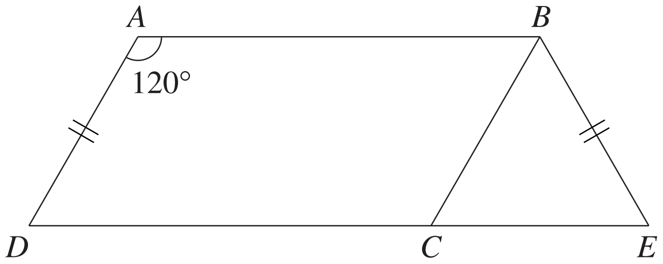
(iv) For what values of x is the graph of $y = f(x)$ concave down? **1**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Use the change of base formula to evaluate $\log_3 7$, correct to two decimal places. **1**

(b) **3**



The diagram shows a parallelogram $ABCD$ with $\angle DAB = 120°$. The side DC is produced to E so that $AD = BE$.

Copy or trace the diagram into your writing booklet.

Prove that ΔBCE is equilateral.

(c) Find the coordinates of the point P on the curve $y = 2e^x + 3x$ at which the tangent to the curve is parallel to the line $y = 5x - 3$. **3**

(d) A total of 300 tickets are sold in a raffle which has three prizes. There are 100 red, 100 green and 100 blue tickets.

At the drawing of the raffle, winning tickets are NOT replaced before the next draw.

(i) What is the probability that each of the three winning tickets is red? **2**

(ii) What is the probability that at least one of the winning tickets is not red? **1**

(iii) What is the probability that there is one winning ticket of each colour? **2**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Five values of the function $f(x)$ are shown in the table. **3**

x	0	5	10	15	20
$f(x)$	15	25	22	18	10

Use Simpson's rule with the five values given in the table to estimate

$$\int_0^{20} f(x)\,dx\,.$$

(b) A tank initially holds 3600 litres of water. The water drains from the bottom of the tank. The tank takes 60 minutes to empty.

A mathematical model predicts that the volume, V litres, of water that will remain in the tank after t minutes is given by

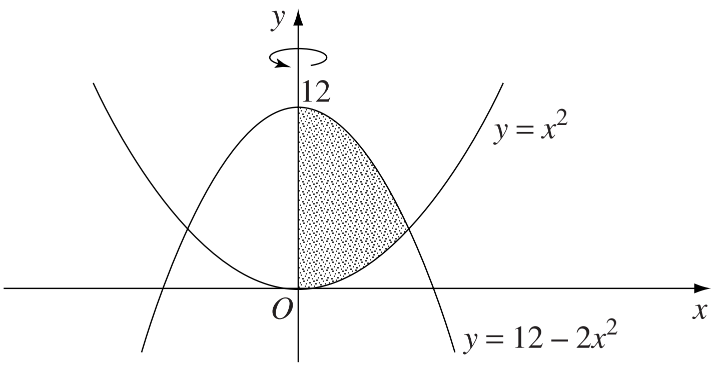
$$V = 3600\left(1 - \frac{t}{60}\right)^2, \quad \text{where} \quad 0 \le t \le 60.$$

(i) What volume does the model predict will remain after ten minutes? **1**

(ii) At what rate does the model predict that the water will drain from the tank after twenty minutes? **2**

(iii) At what time does the model predict that the water will drain from the tank at its fastest rate? **2**

(c)



The graphs of the curves $y = x^2$ and $y = 12 - 2x^2$ are shown in the diagram.

(i) Find the points of intersection of the two curves. **1**

(ii) The shaded region between the curves and the y-axis is rotated about the y-axis. By splitting the shaded region into two parts, or otherwise, find the volume of the solid formed. **3**

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Anne and Kay are employed by an accounting firm.

Anne accepts employment with an initial annual salary of \$50 000. In each of the following years her annual salary is increased by \$2500.

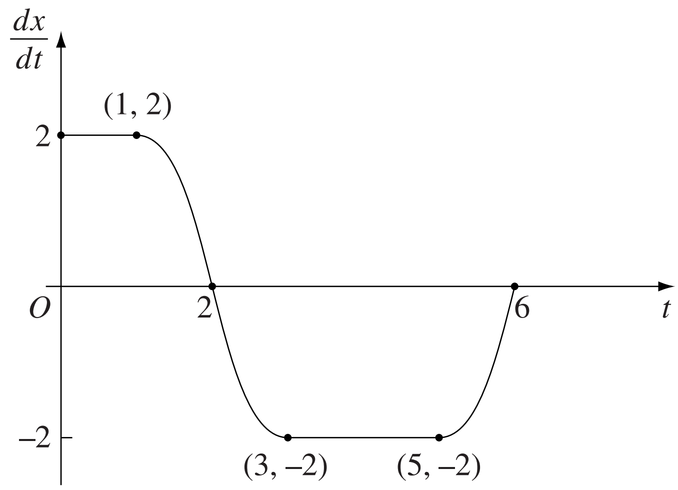
Kay accepts employment with an initial annual salary of \$50 000. In each of the following years her annual salary is increased by 4%.

(i) What is Anne's annual salary in her thirteenth year? **2**

(ii) What is Kay's annual salary in her thirteenth year? **2**

(iii) By what amount does the total amount paid to Kay in her first twenty years exceed that paid to Anne in her first twenty years? **3**

(b)



The graph shows the velocity, $\frac{dx}{dt}$, of a particle as a function of time. Initially the particle is at the origin.

(i) At what time is the displacement, x, from the origin a maximum? **1**

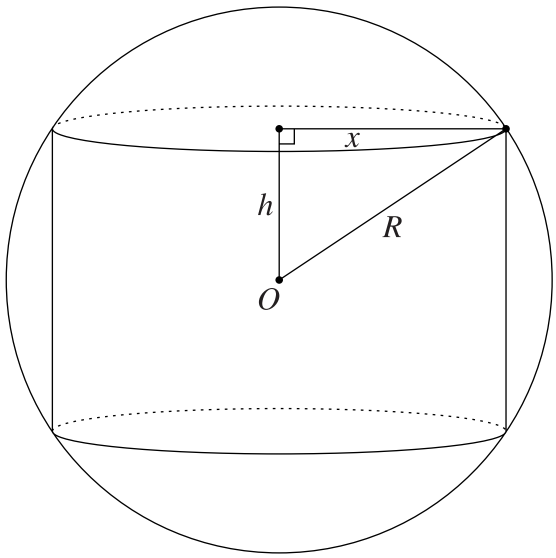
(ii) At what time does the particle return to the origin? Justify your answer. **2**

(iii) Draw a sketch of the acceleration, $\frac{d^2x}{dt^2}$, as a function of time for $0 \le t \le 6$. **2**

Marks

Question 8 (12 marks) Use a SEPARATE writing booklet.

(a)



A cylinder of radius x and height $2h$ is to be inscribed in a sphere of radius R centred at O as shown.

(i) Show that the volume of the cylinder is given by **1**

$$V = 2\pi h\left(R^2 - h^2\right).$$

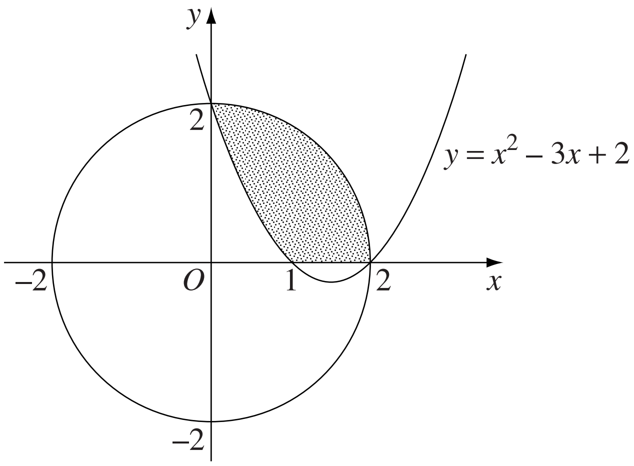
(ii) Hence, or otherwise, show that the cylinder has a maximum volume when $h = \dfrac{R}{\sqrt{3}}$. **3**

Question 8 continues on page 11

Marks

Question 8 (continued)

(b) **3**



The shaded region in the diagram is bounded by the circle of radius 2 centred at the origin, the parabola $y = x^2 - 3x + 2$, and the x-axis.

By considering the difference of two areas, find the area of the shaded region.

(c) Weelabarrabak Shire Council borrowed \$3 000 000 at the beginning of 2005. The annual interest rate is 12%. Each year, interest is calculated on the balance at the beginning of the year and added to the balance owing. The debt is to be repaid by equal annual repayments of \$480 000, with the first repayment being made at the end of 2005.

Let A_n be the balance owing after the n-th repayment.

(i) Show that $A_2 = (3 \times 10^6)(1.12)^2 - (4.8 \times 10^5)(1 + 1.12)$. **1**

(ii) Show that $A_n = 10^6\left[4 - (1.12)^n\right]$. **2**

(iii) In which year will Weelabarrabak Shire Council make the final repayment? **2**

End of Question 8

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Marks

Question 9 (12 marks) Use a SEPARATE writing booklet.

(a) A particle is initially at rest at the origin. Its acceleration as a function of time, t, is given by

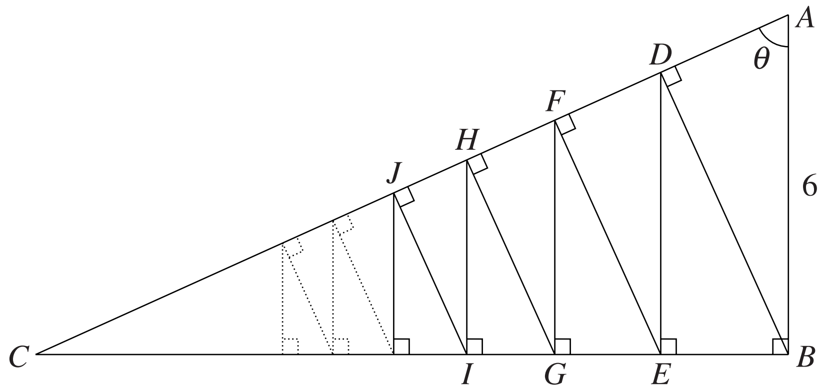
$$\ddot{x} = 4\sin 2t.$$

(i) Show that the velocity of the particle is given by $\dot{x} = 2 - 2\cos 2t$. **2**

(ii) Sketch the graph of the velocity for $0 \le t \le 2\pi$ AND determine the time at which the particle first comes to rest after $t = 0$. **3**

(iii) Find the distance travelled by the particle between $t = 0$ and the time at which the particle first comes to rest after $t = 0$. **2**

(b)



The triangle ABC has a right angle at B, $\angle BAC = \theta$ and $AB = 6$. The line BD is drawn perpendicular to AC. The line DE is then drawn perpendicular to BC. This process continues indefinitely as shown in the diagram.

(i) Find the length of the interval BD, and hence show that the length of the interval EF is $6\sin^3\theta$. **2**

(ii) Show that the limiting sum **3**

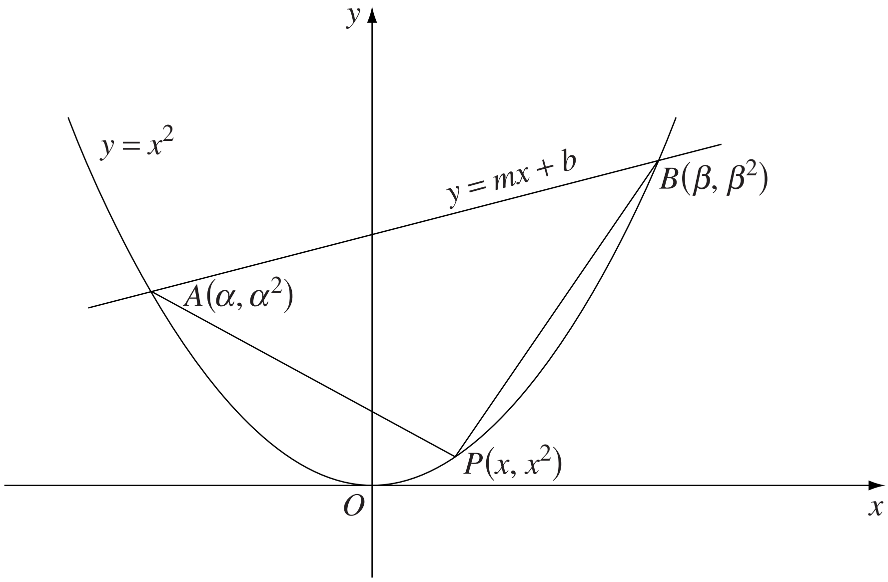
$$BD + EF + GH + \cdots$$

is given by $6\sec\theta\tan\theta$.

Marks

Question 10 (12 marks) Use a SEPARATE writing booklet.

(a)



The parabola $y = x^2$ and the line $y = mx + b$ intersect at the points $A(\alpha, \alpha^2)$ and $B(\beta, \beta^2)$ as shown in the diagram.

(i) Explain why $\alpha + \beta = m$ and $\alpha\beta = -b$. **1**

(ii) Given that $(\alpha - \beta)^2 + \left(\alpha^2 - \beta^2\right)^2 = (\alpha - \beta)^2\left[1 + (\alpha + \beta)^2\right]$, show that the distance $AB = \sqrt{\left(m^2 + 4b\right)\left(1 + m^2\right)}$. **2**

(iii) The point $P(x, x^2)$ lies on the parabola between A and B. Show that the area of the triangle ABP is given by $\frac{1}{2}\left(mx - x^2 + b\right)\sqrt{m^2 + 4b}$. **2**

(iv) The point P in part (iii) is chosen so that the area of the triangle ABP is a maximum. **2**

Find the coordinates of P in terms of m.

Question 10 continues on page 15

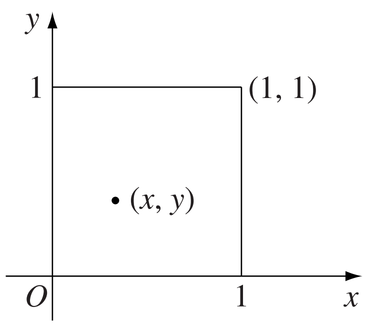
Marks

Question 10 (continued)

(b) Xuan and Yvette would like to meet at a cafe on Monday. They each agree to come to the cafe sometime between 12 noon and 1 pm, wait for 15 minutes, and then leave if they have not seen the other person.

Their arrival times can be represented by the point (x, y) in the Cartesian plane, where x represents the fraction of an hour after 12 noon that Xuan arrives, and y represents the fraction of an hour after 12 noon that Yvette arrives.

Thus $\left(\frac{1}{3}, \frac{2}{5}\right)$ represents Xuan arriving at 12:20 pm and Yvette arriving at 12:24 pm. Note that the point (x, y) lies somewhere in the unit square $0 \le x \le 1$ and $0 \le y \le 1$ as shown in the diagram.



(i) Explain why Xuan and Yvette will meet if $x - y \le \frac{1}{4}$ or $y - x \le \frac{1}{4}$. **1**

(ii) The probability that they will meet is equal to the area of the part of the region given by the inequalities in part (i) that lies within the unit square $0 \le x \le 1$ and $0 \le y \le 1$. **2**

Find the probability that they will meet.

(iii) Xuan and Yvette agree to try to meet again on Tuesday. They agree to arrive between 12 noon and 1 pm, but on this occasion they agree to wait for t minutes before leaving. **2**

For what value of t do they have a 50% chance of meeting?

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$