BOARD OF STUDIES
NEW SOUTH WALES

2005
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value

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Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\int \frac{1}{x^2 + 49}\, dx$. **1**

(b) Sketch the region in the plane defined by $y \le |2x + 3|$. **2**

(c) State the domain and range of $y = \cos^{-1}\left(\frac{x}{4}\right)$. **2**

(d) Using the substitution $u = 2x^2 + 1$, or otherwise, find $\int x\left(2x^2 + 1\right)^{\frac{5}{4}} dx$. **3**

(e) The point $P(1, 4)$ divides the line segment joining $A(-1, 8)$ and $B(x, y)$ internally in the ratio $2 : 3$. Find the coordinates of the point B. **2**

(f) The acute angle between the lines $y = 3x + 5$ and $y = mx + 4$ is $45°$. Find the two possible values of m. **2**

Marks

Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Find $\frac{d}{dx}\left(2\sin^{-1}5x\right)$. **2**

(b) Use the binomial theorem to find the term independent of x in the expansion of $\left(2x-\frac{1}{x^2}\right)^{12}$. **3**

(c) (i) Differentiate $e^{3x}(\cos x - 3\sin x)$. **2**

(ii) Hence, or otherwise, find $\int e^{3x}\sin x\,dx$. **1**

(d) A salad, which is initially at a temperature of 25°C, is placed in a refrigerator that has a constant temperature of 3°C. The cooling rate of the salad is proportional to the difference between the temperature of the refrigerator and the temperature, T, of the salad. That is, T satisfies the equation

$$\frac{dT}{dt} = -k(T-3),$$

where t is the number of minutes after the salad is placed in the refrigerator.

(i) Show that $T = 3 + Ae^{-kt}$ satisfies this equation. **1**

(ii) The temperature of the salad is 11°C after 10 minutes. Find the temperature of the salad after 15 minutes. **3**

Marks

Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) (i) Show that the function $g(x) = x^2 - \log_e(x+1)$ has a zero between 0.7 and 0.9. **1**

(ii) Use the method of halving the interval to find an approximation to this zero of $g(x)$, correct to one decimal place. **2**

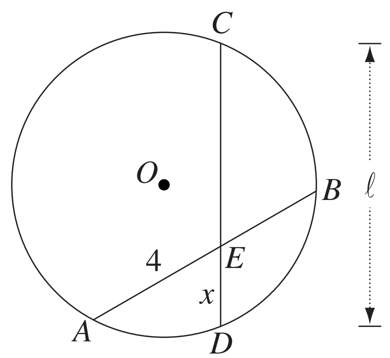
(b) (i) By expanding the left-hand side, show that **1**

$$\sin(5x+4x) + \sin(5x-4x) = 2\sin 5x \cos 4x.$$

(ii) Hence find $\int \sin 5x \cos 4x \, dx$. **2**

(c) Use the definition of the derivative, $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$, to find $f'(x)$ when $f(x) = x^2 + 5x$. **2**

(d)



NOT TO SCALE

In the circle centred at O the chord AB has length 7. The point E lies on AB and AE has length 4. The chord CD passes through E.

Let the length of CD be ℓ and the length of DE be x.

(i) Show that $x^2 - \ell x + 12 = 0$. **2**

(ii) Find the length of the shortest chord that passes through E. **2**

Marks

Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^{\frac{\pi}{4}} \cos x \sin^2 x \, dx$. **2**

(b) By making the substitution $t = \tan\frac{\theta}{2}$, or otherwise, show that **2**

$$\operatorname{cosec}\theta + \cot\theta = \cot\frac{\theta}{2}.$$

(c) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The equation of the normal to the parabola at P is $x + py = 2ap + ap^3$ and the equation of the normal at Q is similarly given by $x + qy = 2aq + aq^3$.

(i) Show that the normals at P and Q intersect at the point R whose coordinates are **2**

$$\left(-apq[p+q],\ a\left[p^2 + pq + q^2 + 2\right]\right).$$

(ii) The equation of the chord PQ is $y = \frac{1}{2}(p+q)x - apq$. (Do NOT show this.) **1**

If the chord PQ passes through $(0, a)$, show that $pq = -1$.

(iii) Find the equation of the locus of R if the chord PQ passes through $(0, a)$. **2**

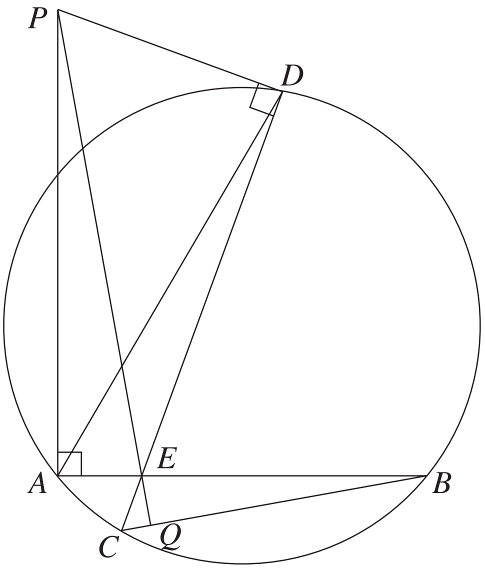
(d) Use the principle of mathematical induction to show that $4^n - 1 - 7n > 0$ for all integers $n \geq 2$. **3**

Marks

Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of the volume of the solid of revolution formed when the region bounded by the curve $y = \sin 2x$, the x-axis and the line $x = \dfrac{\pi}{8}$ is rotated about the x-axis. **3**

(b) Two chords of a circle, AB and CD, intersect at E. The perpendiculars to AB at A and CD at D intersect at P. The line PE meets BC at Q, as shown in the diagram.



(i) Explain why $DPAE$ is a cyclic quadrilateral. **1**

(ii) Prove that $\angle APE = \angle ABC$. **2**

(iii) Deduce that PQ is perpendicular to BC. **1**

(c) A particle moves in a straight line and its position at time t is given by

$$x = 5 + \sqrt{3}\sin 3t - \cos 3t \,.$$

(i) Express $\sqrt{3}\sin 3t - \cos 3t$ in the form $R\sin(3t - \alpha)$, where α is in radians. **2**

(ii) The particle is undergoing simple harmonic motion. Find the amplitude and the centre of the motion. **2**

(iii) When does the particle first reach its maximum speed after time $t = 0$? **1**

Marks

Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) There are five matches on each weekend of a football season. Megan takes part in a competition in which she earns one point if she picks more than half of the winning teams for a weekend, and zero points otherwise. The probability that Megan correctly picks the team that wins any given match is $\frac{2}{3}$.

(i) Show that the probability that Megan earns one point for a given weekend is 0.7901, correct to four decimal places. **2**

(ii) Hence find the probability that Megan earns one point every week of the eighteen-week season. Give your answer correct to two decimal places. **1**

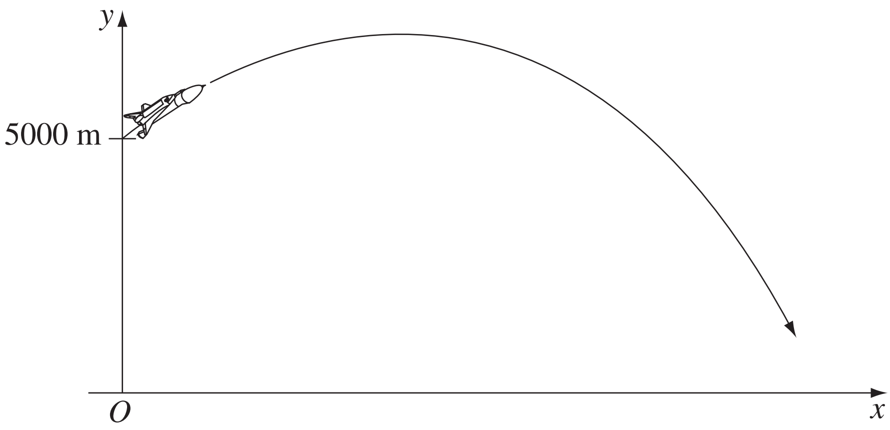
(iii) Find the probability that Megan earns at most 16 points during the eighteen-week season. Give your answer correct to two decimal places. **2**

Question 6 continues on page 9

Marks

Question 6 (continued)

(b) An experimental rocket is at a height of 5000 m, ascending with a velocity of $200\sqrt{2}$ m s^{-1} at an angle of 45° to the horizontal, when its engine stops.



After this time, the equations of motion of the rocket are:

$$x = 200t$$
$$y = -4.9t^2 + 200t + 5000,$$

where t is measured in seconds after the engine stops. (Do NOT show this.)

(i) What is the maximum height the rocket will reach, and when will it reach this height? **2**

(ii) The pilot can only operate the ejection seat when the rocket is descending at an angle between 45° and 60° to the horizontal. What are the earliest and latest times that the pilot can operate the ejection seat? **3**

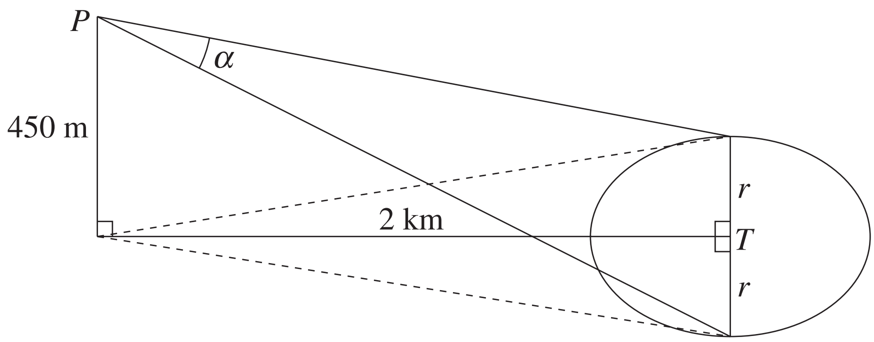
(iii) For the parachute to open safely, the pilot must eject when the speed of the rocket is no more than 350 m s^{-1}. What is the latest time at which the pilot can eject safely? **2**

End of Question 6

Marks

Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) An oil tanker at T is leaking oil which forms a circular oil slick. An observer is measuring the oil slick from a position P, 450 metres above sea level and 2 kilometres horizontally from the centre of the oil slick.



(i) At a certain time the observer measures the angle, α, subtended by the diameter of the oil slick, to be 0.1 radians. What is the radius, r, at this time? **2**

(ii) At this time, $\frac{d\alpha}{dt} = 0.02$ radians per hour. Find the rate at which the radius of the oil slick is growing. **2**

(b) Let $f(x) = Ax^3 - Ax + 1$, where $A > 0$.

(i) Show that $f(x)$ has stationary points at $x = \pm\frac{\sqrt{3}}{3}$. **1**

(ii) Show that $f(x)$ has exactly one zero when $A < \frac{3\sqrt{3}}{2}$. **2**

(iii) By observing that $f(-1) = 1$, deduce that $f(x)$ does not have a zero in the interval $-1 \le x \le 1$ when $0 < A < \frac{3\sqrt{3}}{2}$. **1**

(iv) Let $g(\theta) = 2\cos\theta + \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. **3**

By calculating $g'(\theta)$ and applying the result in part (iii), or otherwise, show that $g(\theta)$ does not have any stationary points.

(v) Hence, or otherwise, deduce that $g(\theta)$ has an inverse function. **1**

End of paper

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$