2009 HSC Notes from the Marking Centre Mathematics Extension 2

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GPO Box 5300
Sydney 2001
Australia

Tel: (02) 9367 8111
Fax: (02) 9367 8484
Internet: www.boardofstudies.nsw.edu.au

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2009 HSC NOTES FROM THE MARKING CENTRE
MATHEMATICS EXTENSION 2

Introduction

This document has been produced for the teachers and candidates of the Stage 6 Mathematics Extension 2 course. It contains comments on candidate responses to the 2009 Higher School Certificate examination, indicating the quality of the responses and highlighting their relative strengths and weaknesses.

This document should be read along with the relevant syllabus, the 2009 Higher School Certificate examination, the marking guidelines and other support documents which have been developed by the Board of Studies to assist in the teaching and learning of Mathematics Extension 2.

Many parts in the Extension 2 paper require candidates to prove, show or deduce a result. Candidates are reminded of the need to give clear, concise reasons in their answers.

Teachers and students are advised that, in December 2008, the Board of Studies approved changes to the examination specifications and assessment requirements for a number of courses. These changes will be implemented for the 2010 HSC cohort. Information on a course-by-course basis is available on the Board's website at www.boardofstudies.nsw.edu.au/syllabus_hsc

Question 1

(a) In better responses, candidates recognised the substitution $u = \ln x$ and successfully found the primitive. A significant number of candidates confused $(\ln x)^2$ with $\ln(x^2)$ and incorrectly simplified $(\ln x)^2$ to obtain $2\ln x$. Some candidates successfully used the method of integration by parts to find the primitive.

(b) In most responses, candidates recognised that the method of integration by parts could be used. The most common errors involved the incorrect application of integration by parts, with many candidates differentiating, rather than integrating, the e^{2x} term.

(c) Responses that rewrote $\dfrac{x^2}{1+4x^2}$ in the form $a + \dfrac{b}{1+4x^2}$ were generally successful, although many candidates experienced some difficulty evaluating the coefficient of $\tan^{-1} 2x$ in the primitive of $\dfrac{1}{1+4x^2}$. A small number of candidates successfully found the primitive by using one of the substitutions $x = \dfrac{1}{2}\tan\theta$ or $u = 1 + 4x^2$.

(d) Responses that rewrote the integrand in the form $\dfrac{a}{x+4} + \dfrac{b}{x-1}$ were generally successful. A significant number of candidates chose instead to rewrite the integrand as $\dfrac{1}{2}\left(\dfrac{2x+3}{x^2+3x-4}\right) - \dfrac{15}{2}\left(\dfrac{1}{x^2+3x-4}\right)$ before proceeding to integrate the latter term using partial fractions. Common errors included incorrect substitution of the limits of integration in expressions for the primitive, the incorrect factorisation of $x^2 + 3x - 4$, and incorrectly obtaining an inverse tangent function as the primitive of $\dfrac{1}{\left(x+\frac{3}{2}\right)^2-\frac{5}{2}^2}$ after completing the square in x^2+3x-4.

(e) Responses that used the substitution $x = \tan\theta$ were often successful. The most common error in this approach was an incorrect simplification of the expression $\dfrac{\sec\theta}{\tan^2\theta}$. A small number of candidates successfully completed the integral using the substitution $u=\dfrac{1}{x}$. Candidates who used other substitutions, most commonly $u=1+x^2$ and $u^2=1+x^2$, generally only obtained part marks. A significant number of candidates using the substitution $u^2=1+x^2$ could not successfully differentiate $u=\sqrt{1+x^2}$. A small number of candidates unsuccessfully attempted the integral using the method of integration by parts.

Question 2

(a) Some candidates who used the modulus-argument form and De Moivre's Theorem neglected to convert their answer back to Cartesian form.

(b) Algebraic errors when expanding the numerator or denominator, particularly involving negative signs, were the only difficulties encountered by candidates here.

(c) In the better responses, candidates accurately showed the correct positions of $R(iz)$, $S(\overline{w})$, $Q(w)$ and $T(z+w)$ in a diagram, including labelling of the points, using right angle, equal length and parallel line notations.

(d) In the better responses, candidates had well-labelled diagrams, identifying the centre of the circle (the apex of the shaded sector), x-intercepts of the circle, radius of the circle and indicated the angle of $\dfrac{\pi}{4}$ in the sector. Apart from insufficient labelling, the most common errors were: the sector had its apex at the origin, or the sector was rotated from its correct position. A smaller number had the incorrect circle with some centred at $(0,1)$. Few displayed the answer by multiple shadings of the separate regions, most shading only the final region. Very few shaded the wrong regions in the correct curves.

(e) (i) Many candidates went directly to the formula $\cos\left(\dfrac{\pi+2k\pi}{5}\right)+i\sin\left(\dfrac{\pi+2k\pi}{5}\right)$ for the 5th roots of -1, rather than using De Moivre's Theorem, or did the equivalent by writing down the root $\cos\left(\dfrac{\pi}{5}\right)+i\sin\left(\dfrac{\pi}{5}\right)$ and then listing the rest by incrementing the angle in steps of $\dfrac{2\pi}{5}$. Candidates are reminded that they should take care to use the correct formula and correct argument. A number of candidates had roots $z=r(\cos(\theta)+i\sin(\theta))$ with $r=-1$, some with roots $z=-\left(\cos\left(\dfrac{2k\pi}{5}\right)+i\sin\left(\dfrac{2k\pi}{5}\right)\right)$, with $k=-2$, -1, 0, 1, 2, which are correct roots but were not expressed in polar form. This approach led to difficulties when plotting these points in the next question.

(ii) Candidates were required to label their points from part (i). Often $z_1, z_2, \cdots, z_5$ were used as labels but these had not been defined in parts (i) or (ii). Often those roots obtained with $r = -1$ in part (i) were plotted as if $r = 1$ in part (ii). The most common errors were inaccurate diagrams with different moduli on the roots and failure to vertically align the pairs of conjugate roots.

(f) (i) An extended solution to this part was required. Candidates are reminded that bald answers, of $\pm(2+i)$ in this case, are not sufficient for a part worth 3 marks. Errors in the algebraic solution included incorrect expansion of $(a+ib)^2$, or an error in going from the equation $2ab = 4$ to get the incorrect equation $b = \frac{a}{2}$ instead of the correct $b = \frac{2}{a}$. This error avoided the quartic equation in *a* and the requirement to factorise it, thus simplifying the work. Other errors were incorrect factorisation of the quartic in *a*, or failure to reject imaginary solutions for *a* or *b*. Some candidates left their answer to part (i) as $a = \pm 2$, $b = \pm 1$ and so did not indicate the relationship between signs, or write the roots in the form $a + ib$. Writing the final answer as $\pm 2 \pm i$ was ambiguous. Inappropriate use of the $\pm$ symbol in this way sometimes resulted in four square roots of $3 + 4i$ being found and then used in part (ii). Candidates should also check that one root is the negative of the other.

An alternative method, that equated the moduli of both sides of $z^2 = 3 + 4i$ to obtain $a^2 + b^2 = 5$ and so avoided the need to factorise a quartic, was used by several candidates, although a few did not find a second equation that would show that *a* and *b* have the same sign. Some candidates approached this problem by considering the modular-argument form of the given complex numbers: $z = \pm\sqrt{5}\left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)$, $\theta = \tan^{-1}\left(\frac{4}{3}\right)$, but few of these attempts managed to continue to completion.

(ii) A very common error made by those using the quadratic formula was that the '$-b$' term in the numerator, which should be $-i$, became -1 when the roots were simplified. Some had that term incorrect from the start using i or i^2. Inappropriate use of the $\pm$ symbol resulted in 4 roots being obtained. Some candidates found only one root.

Using the quadratic formula led to greater success than other methods tried by candidates, such as equating real and imaginary parts or attempting to use the sum of roots and product of roots formulae.

Question 3

(a) (i) The two vertical asymptotes, $x = 0$ and $x = 4$, were included in most responses. The better responses included a horizontal asymptote of $y = -\frac{1}{3}$ on the right and a horizontal asymptote of $y = 0$ on the left. A common error was to have only one horizontal asymptote, usually $y = 0$, on both left and right.

(ii) The best responses included the horizontal asymptote of $y = 9$ as $x \to \infty$, minimum turning points at $(0,0)$ and $(4,0)$ and $y \to \infty$ as $x \to -\infty$.

Weaker responses displayed graphs without a horizontal asymptote on the right, or with the asymptote on both right and left, or with the asymptote incorrectly positioned (most often at $y = 3$). Cusps, instead of minimum turning points, were drawn by a significant number of candidates and intercepts were not clearly labelled.

(iii) The best responses displayed an even function, a local minimum turning point at the origin, x-intercepts of 2, 0 and –2, maximum turning points above the x-axis between the x-intercepts and a horizontal asymptote of $y = -3$.

Common errors included: not showing an even function; an x-intercept of 16 or 8 instead of 2; reflecting the left side of the original graph so that the final answer was no longer a function; and not having a horizontal asymptote of $y = -3$.

(b) In weaker responses, candidates found implicit differentiation difficult. Very few candidates who attempted to make y the subject of an equation were successful. Overall, the more successful candidates chose to use $\frac{dy}{dx} = 0$ without first expressing it as a rational function of x after the implicit differentiation step.

Several candidates successfully solved the problem without using calculus, by letting $y = c$ in the equation of the curve and then using the fact that the resulting quadratic equation must have equal roots if the line is a tangent.
A significant number of candidates used $\frac{dy}{dx} = 0$ but often made algebraic errors, deducing that $y = x$ rather than $y = -x$ being the most common. Another common error was using $x + 3y = 0$, that is, setting the denominator equal to zero rather than the numerator.

(c) Most successful candidates recognised that $P(1) = 0$ and $P'(1) = 0$ thereby obtaining the two equations $a + b = -6$ and $2a + b = -3$.

The other common method was to use the sum and product of roots, namely 1, 1 and α, leading to $\alpha = -5$, then find the values of a and b.

A common error was to use $P(-1) = 0$ or $P'(-1) = 0$. Other candidates incorrectly substituted $x = 1$ into $P(x)$ or $P'(x)$, or decided that the third root was 5.

(d) Most candidates demonstrated an understanding of the method of cylindrical shells.

The most common mistake occurred when finding the height of a typical shell, that is, candidates made careless errors when subtracting $(x - 1)^2$ from $x + 1$. Other errors included finding the incorrect radius of a typical shell or using the incorrect limits for the definite integral.

Question 4

Where a question asks candidates to show a result, candidates are reminded that they need to display enough information to fully justify the result. In part (b), some candidates who could not do part (i) were unable to proceed successfully with later parts.

(a) (i) A few candidates found the equation of the tangent instead of the equation of the normal. Nearly all candidates correctly differentiated implicitly. Of the small number who attempted to make y the subject before differentiating, most struggled to differentiate correctly.

(ii) Most errors were due to incorrect handling of minus signs, mistakes made when moving terms from one side of an equation to the other, or by choosing incorrect expressions connecting a, b and e.

(iii) Although most candidates were able to state the focus-directrix definition, many were unable to apply the definition to the question. Some candidates found it difficult to find $NS' = ae + x_0e^2$ and $PM' = \frac{a}{e} + x_0$.

(iv) In weaker responses, candidates were often unable to apply the sine rule. Some candidates did not recognise that $\sin \angle PNS' = \sin(\pi - \angle PNS)$ while others thought that $\angle PNS' = \angle PNS = \frac{\pi}{2}$. In some instances, the notation used for $\sin \angle PNS'$ was unconventional.

(b) (i) Most candidates were able to show a knowledge of the resolution of forces. Some confused signs.

(ii) Most candidates obtained marks either by using $T\sin\theta + N\cos\theta$ or by making T or N the subject and then substituting into one of the equations. Some candidates lost marks after making errors in part (i) but still managed to find a correct expression for T. Some candidates correctly resolved forces in the direction of the string and the normal rather than the horizontal and vertical directions. Some who tried to resolve in those directions did not justify their answer sufficiently to be awarded many marks.

(iii) This was well done by candidates who attempted the part.

(iv) Most candidates did not realise that $\omega^2 > 0$. Many candidates who incorrectly attempted this part used algebraic manipulation to make α the subject.

Question 5

(a) Candidates are reminded that solutions to questions in geometry should be set out clearly with geometric justification provided.

(i) Successful candidates noted correctly that $\angle ADB = 90°$, the angle in the semicircle with diameter AB. Less successful responses failed to provide geometric justification for assertions such as $\angle ADB$ being a right angle.

(ii) Candidates who failed to consider $\angle DCA$ in the circle $DCBA$ did not make significant progress.

(iii) This part required candidates to show that angles $\angle KCA$ and $\angle BCA$ were supplementary. Using circle properties, many candidates correctly showed that these angles are both right angles. Some candidates proved the result by showing that $\angle BCA$ is an external angle of the cyclic quadrilateral $CKDX$.

(b) (i) There were several different approaches from candidates for this part. The best responses were from candidates who obtained the result directly by considering the integrand as a product of the two functions x^{2n} and xe^{x^2}. An alternative solution wrote the integrand as a product of x^{2n+1} and e^{x^2} leading to the result $I_n = \dfrac{e}{2(n+1)} - \dfrac{I_{n+1}}{n+1}$. Successful responses then proceeded to make I_{n+1} the subject leading to a correct solution once $n+1$ was replaced by n. Some candidates made use of the integral substitution $u = x^2$ to simplify the integral before integrating by parts. A substantial number of candidates included fallacious claims, including the claim that $\dfrac{e^{x^2}}{2x}$ is the primitive of e^{x^2}.

(ii) Successful responses calculated $I_2 = \int_0^1 x^5 e^{x^2}\,dx$ directly or used the recurrence relation from part (i) to reduce the integral to one involving the calculation of I_0.

(c) (i) Weaker responses made basic errors in differentiation. Successful responses used the fact that, for all $x > 0$, $e^x > e^{-x}$ in showing the required result.

(ii) Some responses answered this part without resorting to the result given in part (i) by first showing that $f'(x) = \frac{1}{2}\left(\sqrt{e^x} - \sqrt{e^{-x}}\right)^2$. However, responses that clearly explained their reasoning with reference to the result in part (i) were most successful.

(iii) The better responses made use of the result from (ii) and explained that, since $f'(x) > 0$ for all $x > 0$, then $f(x)$ is an increasing function for all $x > 0$ and thus $f(x) > f(0)$ for all $x > 0$. But $f(0) = 0$ and hence the result followed.

Question 6

(a) Most candidates realised that the width of the rectangle was $2y$, but many mistakenly thought that the height was x or y. Most who found the appropriate integral for this volume successfully evaluated it. Many candidates failed to recognise that $(4-x)\sqrt{4-x} = (4-x)^{\frac{3}{2}}$ and so resorted to quite difficult integration techniques.

(b) This part was difficult for a considerable number of candidates but there was quite a large variety of successful methods.

(i) Many responses verified that $x = -1$ is a root of the given polynomial. This was unnecessary as this fact was given. The simplest way to answer this part of the question was to note that the product of the three roots was -1 and since one root was equal to -1, the other two must be reciprocals. Some responses unfortunately assumed that two roots were reciprocals to begin with and this led to a faulty logical argument.

Another successful method was to show that $P\left(\dfrac{1}{\alpha}\right) = 0$. Unfortunately some responses incorrectly wrote that $\dfrac{1}{\alpha^3} + \dfrac{q}{\alpha^2} + \dfrac{q}{\alpha} + 1 = 1 + q\alpha + q\alpha^2 + \alpha^3$.

(ii)(1) Many candidates noted that since α was a not a real root then $\bar{\alpha}$ must also be a root, but failed to explain that this is true because the coefficients of the polynomial are real. Also many mistakenly assumed that $\alpha\bar{\alpha} = |\alpha|$ instead of the correct fact that $\alpha\bar{\alpha} = |\alpha|^2$.

(2) Most candidates who realised that $\bar{\alpha} = \frac{1}{\alpha}$ successfully completed this part.

(c) (i) This part was completed successfully by most candidates.

(ii) This part was completed successfully by most candidates.

(iii) This part caused a lot of difficulty. Only the few best responses noted that this was a parabola with its axis of symmetry being the x-axis, with the focus and the vertex on the x-axis and whose equation could be written in the form $y^2 = 4a(x - x_0)$.

(iv) Only a very few responses were successful in this part. Not many used the focus-directrix definition of a parabola to find the distance PS. Also quite a few incorrectly tried to show that $PS^2 - PQ^2$ was independent of x, while others erroneously thought that an expression in y is independent of x.

Question 7

(a) (i)(1) Candidates were familiar with the various expressions for acceleration. Those who chose $v\frac{dv}{dx}$ could then use one of three strategies: the substitution $u = g - rv$; carry out a long division; or use algebra to add and subtract terms in the numerator. The latter method was the most efficient. The next step then required the use of definite integrals or integration that led to finding the constant of integration. Both of these techniques were equally successful. Candidates should be aware of the use of parentheses and to avoid overwriting plus and minus signs, thus leaving their final meaning unclear. Those who chose to use $\frac{dv}{dt}$ for acceleration were generally unsuccessful in reaching the required result.

(2) This calculator work was well done.

(ii) The most successful responses found the maximum value of x by solving $v = 0$. Candidates who chose this method demonstrated a good understanding of the product rule or transformed either x or v into $R\sin(t + \alpha)$. However the solution of the resulting trigonometric equation was poorly done because candidates did not use radians or did not recognise the appropriate quadrant for t.

In the best responses, candidates showed that $x = \sqrt{941}\sin(t + \alpha)e^{-\frac{t}{10}} + 92 < 123$ by clear logical arguments. However many assumed the maximum of both $\sin(t + \alpha)$ and $e^{-\frac{t}{10}}$ occurred at the same value of t.

Candidates who attempted to show that the equation $x + 2 = 125$ does not have a solution, rarely managed to finish their calculation. A rare few demonstrated this fact using two non-intersecting graphs.

A significant number of candidates wrongly argued that as $t \to \infty$, $x \to 92$. Some candidates tried to use the value from part (a)(i)(2).

(b) (i) Most candidates were able to apply De Moivre's Theorem successfully.

(ii) The binomial theorem was well known. In the majority of responses, candidates were unable to explain the symmetry and grouping arguments that were needed to gain full marks.

(iii) Many candidates failed to account for the 2m or were careless in their manipulation of it.

Question 8

(a) (i) This part was very well done by candidates, who were generally successful in manipulating the left hand side successfully to obtain the right hand side.

(ii) Many candidates did not gain the first mark as they did not show any working when demonstrating that the proposition is true in the initial case, that is, showing that $\cot\frac{x}{2} - 2\cot x$ was equal to $\tan\frac{x}{2}$.

(iii) Many candidates had trouble showing clearly how the part of the expression: $\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{2^{r-1}}\tan\frac{x}{2^r} = \lim_{n\to\infty}\left(\frac{1}{2^{n-1}}\cot\frac{x}{2^n}\right)$ could be evaluated to give the answer $\frac{2}{x}$.

The presentation of most solutions was vague. Most candidates had little facility with trigonometric limits that involve $\frac{\theta}{\tan\theta}$ or $\frac{\tan\theta}{\theta}$, for example.

(iv) This part was reasonably well done in responses that used part (iii) and $x = \frac{\pi}{2}$ to obtain $\frac{4}{\pi}$, while others used $x = \frac{\pi}{4}$. Less successful responses used $x = \pi$ and then wrote: $\frac{2}{\pi} - 2\cot\pi = \frac{4}{\pi}$ or $2\left(\frac{2}{\pi} - 2\cot\pi\right) = \frac{4}{\pi}$.

(b) Most candidates successfully showed that $\frac{1}{n} < \ln\left(\frac{n}{n-1}\right) < \frac{1}{n-1}$.

(c) (i) This part was not well done by most candidates, as many failed to read the information carefully. Many did attempt this part but simply wrote a story in an attempt to explain why the result was $W = p + q^n W$.

(ii) Many candidates did not read or interpret the data correctly.