BOARD OF STUDIES
NEW SOUTH WALES

2009 HSC Mathematics Extension 1
Sample Answers

This document contains ‘sample answers’. These are developed by the examination committee for two purposes. The committee does this:

(a) as part of the development of the examination paper to ensure the questions will effectively assess students’ knowledge and skills, and

(b) in order to provide some advice to the Supervisor of Marking about the nature and scope of the responses expected of students.

The ‘sample answers’ or similar advice, are not intended to be exemplary or even complete responses. They have been reproduced in their original form as part of the examination committee’s ‘working document’.

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Mathematics Extension 1 - Sample answers.

Question 1

(a) $(2x+3)(4x^2-6x+9)$

(b) $x > 3$

(c) $\lim_{x \to 0} \frac{\sin 2x}{x} = 2 \lim_{x \to 0} \frac{\sin 2x}{2x}$

$= 2$

(d) Method 1:

$4x^2 \frac{(x+3)}{2x} > 4x^2$

$2x^2 + 6x > 4x^2$

$2x^2 - 6x < 0$

$x(x-3) < 0$

$0 < x < 3$

Method 2:

If $x > 0$: $x+3 > 2x$, $x < 3$ ie. $0 < x < 3$

If $x < 0$: $x+3 < 2x$, $x > 3$: No solution

(e) $x \times 2\cos x \times (-\sin x) + \cos^2 x$

$= -2x\cos x \sin x + \cos^2 x$

(f) $\int_0^2 x^2 . e^{x^3+1}\, dx = \left[\frac{e^{x^3+1}}{3} \right]_0^2$

$= \frac{1}{3}(e^9 - e)$

Alternately,

If $u = x^3+1$, $du = 3x^2 dx$,

$\int_0^2 x^2 e^{x^3+1} dx = \int_1^9 \frac{e^u . du}{3} = \left.\frac{e^u}{3}\right]_1^9 = \frac{1}{3}(e^9 - e)$

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Question 2

(a) $p(x) = x^3 - ax + b$

$p(1) = 2, \quad p(-2) = 5.$

$$\left.\begin{aligned} p(1) &= 1 - a + b = 2 \Rightarrow -a + b = 1 \\ p(-2) &= -8 + 2a + b = 5 \Rightarrow 2a + b = 13 \end{aligned}\right\} \begin{aligned} a &= 4 \\ b &= 5 \end{aligned}$$

(b) (i) $A\sin(x+\alpha) = A\sin x\cos\alpha + A\cos x\sin\alpha$

So, $A\cos\alpha = 3$ and $A\sin\alpha = 4$

$\Rightarrow A = 5, \quad \cos\alpha = \frac{3}{5}$

$A = 5, \quad \alpha = \cos^{-1}\left(\frac{3}{5}\right) \approx 0.927$

(ii) $5\sin(x+\alpha) = 5$

$\Rightarrow \sin(x+\alpha) = 1$

$x + \alpha = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$

$x = \frac{\pi}{2} - \alpha, \ldots$

$x \approx 0.64$

(c) (i) $y = \frac{x^2}{4}$

$y' = \frac{x}{2}$

$y'(2t) = t$

$y - t^2 = t(x - 2t)$

$y = tx - t^2$

(ii) tangent at Q:

$y - 4t^2 = 2t(x - 4t)$

$y = 2tx - 4t^2$

$2tx - 4t^2 = tx - t^2$

$tx = 3t^2$

$x = 3t \quad y = 2t^2$

(iii) $t = \frac{x}{3} \Rightarrow y = 2\left(\frac{x^2}{9}\right)$

$y = \frac{2x^2}{9}$

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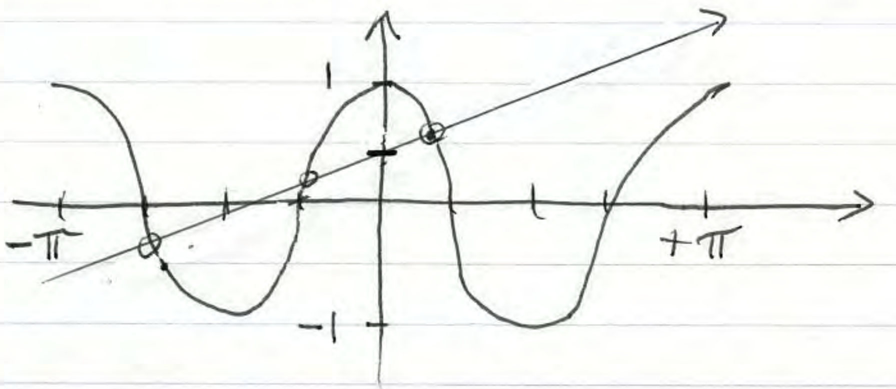
Question 3

(a) $f(x) = \dfrac{3 + e^{2x}}{4}$

(i) Range: $y > \frac{3}{4}$

(ii) $4y = e^{2x} + 3$
$4y - 3 = e^{2x}$
$2x = \ln(4y-3)$
$x = \ln\sqrt{4y-3}$ or $x = \frac{1}{2}\ln(4y-3)$
$\Rightarrow f^{-1}(x) = \ln\sqrt{4x-3}$ or $f^{-1}(x) = \frac{1}{2}\ln(4x-3)$

(b) (i)



(ii) 3

(iii) $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$f(x) = 2\cos 2x - x - 1 = 0$
$f'(x) = -4\sin 2x - 1$
$f'(0.4) = -3.869$

$x_1 = 0.4 - \dfrac{f(0.4)}{f'(0.4)} \approx 0.398$

Question 3 (continued)

(c)(i) $\tan^2\theta = \dfrac{1-\cos 2\theta}{1+\cos 2\theta}$

$$\frac{1-\cos 2\theta}{1+\cos 2\theta} = \frac{1-(\cos^2\theta - \sin^2\theta)}{1+\cos^2\theta - \sin^2\theta}$$

$$= \frac{\sin^2\theta + \cancel{\cos^2\theta} - \cancel{\cos^2\theta} + \sin^2\theta}{\cancel{\sin^2\theta} + \cos^2\theta + \cos^2\theta - \cancel{\sin^2\theta}}$$

$$= \frac{2\sin^2\theta}{2\cos^2\theta}$$

$$= \tan^2\theta$$

(ii) $\tan^2\frac{\pi}{8} = \dfrac{1-\cos\frac{\pi}{4}}{1+\cos\frac{\pi}{4}}$

$$= \frac{1-\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}-1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$$

$$= 3-2\sqrt{2}$$

$$\tan\frac{\pi}{8} = \sqrt{3-2\sqrt{2}}$$

End of Question 3

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Question 4

(a) (i) $^5C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 = \frac{90}{1024} = \frac{45}{512}$

(ii) $^5C_3\left(\frac{1}{4}\right)^3\left(\frac{3}{4}\right)^2 + {^5C_4}\left(\frac{1}{4}\right)^4\left(\frac{3}{4}\right) + {^5C_5}\left(\frac{1}{4}\right)^5$

$= \frac{90+15+1}{1024} = \frac{106}{1024} = \frac{53}{512}$

(iii) $1 - {^5C_5}\left(\frac{1}{4}\right)^5 = \frac{1023}{1024}$

(b) $f(x) = \frac{x^4+3x^2}{x^4+3}$

(i) $f(-x) = \frac{(-x)^4+3(-x)^2}{(-x)^4+3} = \frac{x^4+3x^2}{x^4+3} = f(x)$

so $f(x)$ is even.

(ii) $y = 1$

(iii) $y' = \frac{(x^4+3)(4x^3+6x)-(x^4+3x^2)(4x^3)}{(x^4+3)^2}$

$= \frac{-6x^5+12x^3+18x}{(x^4+3)^2}$

$y'=0 \quad x(6x^4-12x^2-18)=0$

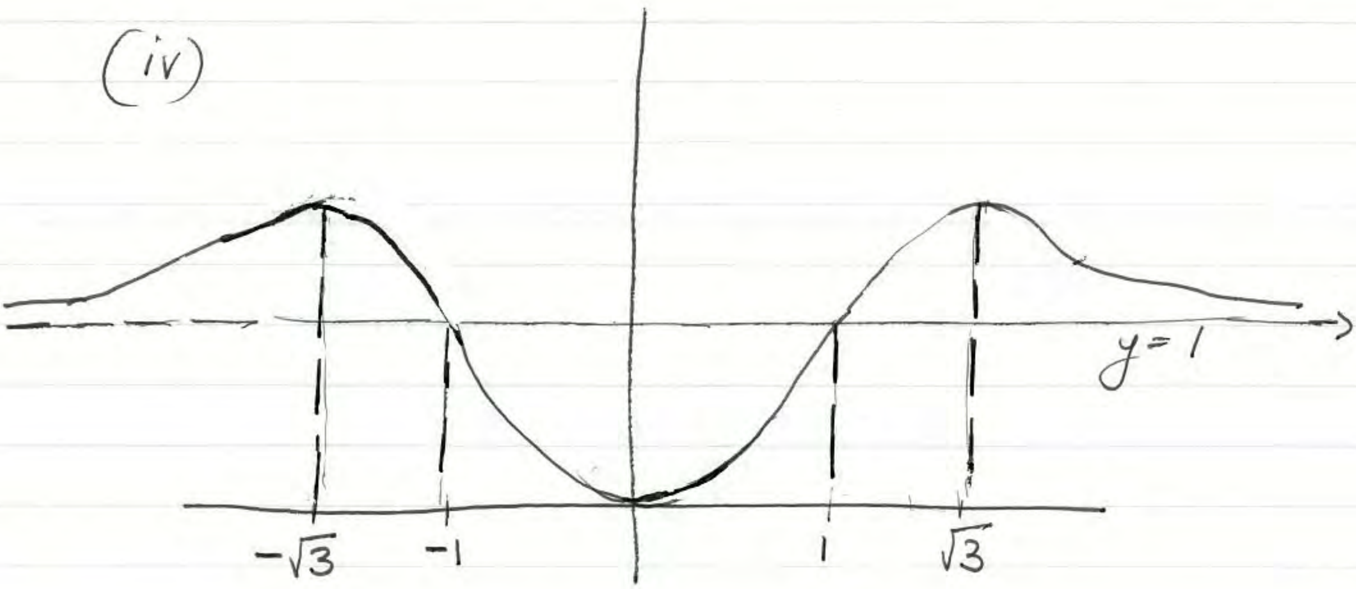
$6x(x^4-2x^2-3)=0$

$6x(x^2-3)(x^2+1)=0$

Question 4 (continued)

(b) (iii) x coordinates at stationary points are $0, \pm\sqrt{3}$.

(iv)



End of Question 4

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Question 5

(a) (i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2 x$

$$\frac{1}{2}v^2 = -\frac{n^2x^2}{2} + c$$

When $x = a$, $v = 0$, so $C = \frac{1}{2}n^2a^2$

$$v^2 = n^2(a^2 - x^2)$$

(ii) $x = 0$, $v = na$ — maximum speed

(iii) $x = a$ $\ddot{x} = -n^2 a$

$x = -a$ $\ddot{x} = n^2 a$ — maximum acceleration.

(iv) $x = a \sin nt$

$\dot{x} = an \cos nt$

$= \frac{1}{2}an$

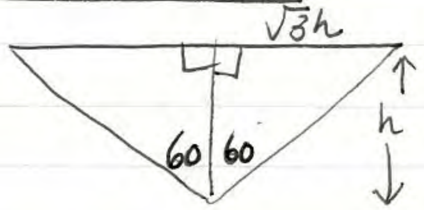
$\cos nt = \frac{1}{2}$

$nt = \frac{\pi}{3}$

$t = \frac{\pi}{3n}$

Question 5 (continued)

(b) (i)



Volume $= 10 \times \sqrt{3}\, h^2$

(ii) $A = 10 \times 2\sqrt{3}\, h$

(iii) $\dfrac{dV}{dt} = -k\, 20\sqrt{3}\, h$

$V = 10\sqrt{3}\, h^2$

$\dfrac{dV}{dh} = 20\sqrt{3}\, h$

$$\frac{dh}{dt} = \frac{dV/dt}{dV/dh} = \frac{-k\, 20\sqrt{3}\, h}{20\sqrt{3}\, h}$$

$$= -k$$

(iv) 100 days

End of Question 5

Question 6

(a) (i) $x_1 = x_2$ at $t = T$

$\therefore UT\cos\theta = R - Vt\cos\theta$

$\therefore (U+V)T\cos\theta = R$

$\therefore T = \dfrac{R}{(U+V)\cos\theta}$

(ii) At time T,

$y_2 - y_1 = h - VT\sin\theta - \frac{1}{2}gT^2 - UT\sin\theta + \frac{1}{2}gT^2$

$= h - (U+V)T\sin\theta$

$= h - (U+V)\dfrac{R\sin\theta}{(U+V)\cos\theta}$

$= h - R\tan\theta$

$= h - R\left(\dfrac{h}{R}\right) = 0$

$\therefore y_1 = y_2$ at time $t = T$.

$\therefore$ projectiles collide.

(iii) Let $x_1 = \lambda R$ when $t = T$

$\therefore UT\cos\theta = \lambda R$

$\therefore U\left(\dfrac{R}{(U+V)\cos\theta}\right)\cos\theta = \lambda R$

$\therefore \dfrac{U}{U+V} = \lambda$

$\therefore U = \lambda U + \lambda V$

$\lambda V = (1-\lambda)U$

$\therefore V = \left(\dfrac{1}{\lambda} - 1\right)U$

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Question 6 (continued)

(b) (i) $(1+x)^r + (1+x)^{r+1} + \dots + (1+x)^n$

Geometric series $a = (1+x)^r$
common ratio $= (1+x)$
no of terms $= n-r+1$

$$\therefore \text{Sum} = \frac{(1+x)^r\left[(1+x)^{n-r+1} - 1\right]}{(1+x)-1}$$

$$= \frac{(1+x)^{n+1} - (1+x)^r}{x}$$

Coefficient of x^r on LHS
$$= \binom{r}{r} + \binom{r+1}{r} + \dots + \binom{n}{r}$$

Coefficient of x^r on RHS
= coeff of x^{r+1} on numerator
$$= \binom{n+1}{r+1}$$

Equating these gives the result

Question 6 (continued)

(b) (ii) (1) The number of points on the diagonals of gradient 1 increase from 1 to n and decrease to 1.

(2) If a diagonal contains k points (where $2 \leq k \leq n$) there are $\binom{k}{2}$ such intervals

$$\therefore S_n = \binom{2}{2} + \binom{3}{2} + \dots + \binom{n-1}{2} + \binom{n}{2} + \dots + \binom{2}{2}.$$

(iii) $$S_n = \left\{ \binom{2}{2} + \dots \binom{3}{2} + \dots + \binom{n}{2} \right\} + \left\{ \binom{n-1}{2} + \dots + \binom{2}{2} \right\}$$

$$= \binom{n+1}{3} + \binom{n}{3} \quad \text{from (ii)}$$

$$= \frac{(n+1)n(n-1)}{1.2.3} + \frac{n(n-1)(n-2)}{1.2.3.}$$

$$= \frac{n(n-1)}{6}\left[(n+1) + (n-2)\right]$$

$$= \frac{n(n-1)(2n-1)}{6}$$

End of Question 6

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Question 7

(a) (i) $\frac{d}{dx}(x) = \lim_{h \to 0} \frac{x+h-x}{h} = 1$

(ii) when $n = 1$
we have $\frac{d}{dx}(x) = 1$.

Assume true for $n = k$
ie. $\frac{d}{dx}(x^k) = kx^{k-1}$

$$\frac{d}{dx}(x^{k+1}) = \frac{d}{dx}(x \cdot x^k)$$
$$= x \cdot \frac{d}{dx}(x^k) + x^k \cdot 1 \text{ (Product rule)}$$
$$= x \cdot kx^{k-1} + x^k \text{ (by hypothesis)}$$
$$= kx^k + x^k$$
$$= (k+1)x^k \text{ as required.}$$

$\therefore$ statement is true for $n = k+1$
$\therefore$ statement is true for all integers $n \geq 1$
by mathematical induction.

Question 7 (continued)

(b) (i)
$$\tan\theta = \frac{\frac{a+h}{x} - \frac{h}{x}}{1+\left(\frac{a+h}{x}\right)\frac{h}{x}} \times \frac{x^2}{x^2}$$

$$= \frac{x(a+h)-hx}{x^2+h(a+h)} = \frac{ax}{x^2+h(a+h)}$$

$$\therefore\ \theta = \tan^{-1}\left[\frac{ax}{x^2+h(a+h)}\right]$$

(ii)
$$\frac{d\theta}{dx} = \frac{1}{1+\left[\frac{ax}{x^2+h(a+h)}\right]^2} \times \frac{(x^2+h(a+h))a - ax.2x}{\left[x^2+h(a+h)\right]^2}$$

$$\frac{d\theta}{dx} = 0 \Rightarrow ax^2 + ah(a+h) - 2ax^2 = 0$$

$$\therefore\ x^2 = h(a+h) \qquad a \neq 0$$

$$x = \sqrt{h(a+h)}$$

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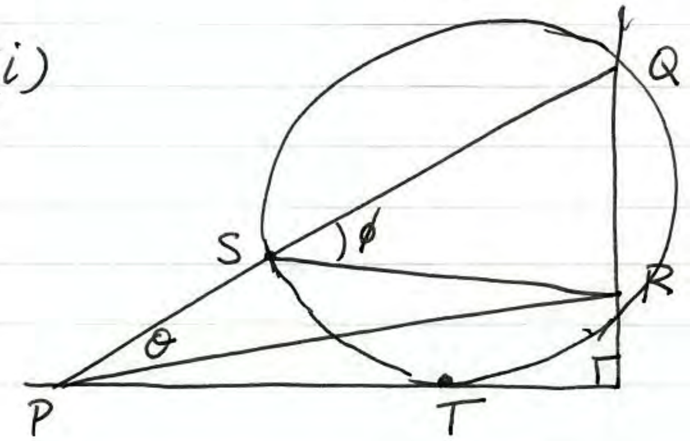
Question 7 (continued)

(c) (i)



$\phi = \theta + \angle SRP$ (exterior $\angle$ ΔPSR)

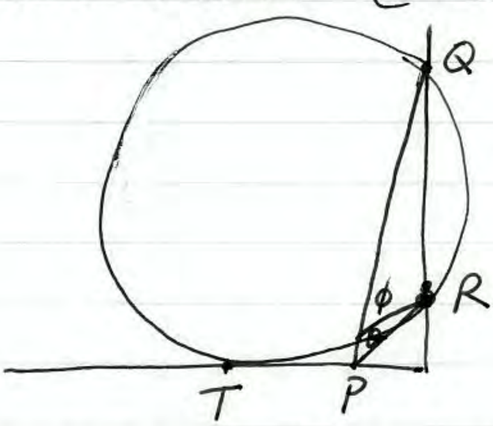
$\therefore \phi > \theta$

When $P = T$ Then $S = P = T$

so that $\phi = \theta$

If P is on the right of T, diagram is as shown

ϕ remains constant ($\angle$ in same segment)



Similar argument shows $\phi > \theta$

Hence θ is a maximum

when $P = T$

in which case $\theta = \phi$

Question 7 (continued)

(c) (ii) Using circle results
(the square of the tangent = the product of the intercepts from an external point)

$$OT^2 = OR \cdot OQ$$

ie. $OT^2 = h(a+h)$

$$OT = \sqrt{h(a+h)}.$$

End of Question 7