BOARD OF STUDIES
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2009 HSC Mathematics Extension 2
Sample Answers

This document contains ‘sample answers’. These are developed by the examination committee for two purposes. The committee does this:

(a) as part of the development of the examination paper to ensure the questions will effectively assess students’ knowledge and skills, and

(b) in order to provide some advice to the Supervisor of Marking about the nature and scope of the responses expected of students.

The ‘sample answers’ or similar advice, are not intended to be exemplary or even complete responses. They have been reproduced in their original form as part of the examination committee’s ‘working document’.

2009 HSC

Mathematics Extension 2 Sample Answers

Question 1

(a) $\int \frac{\ln x}{x} dx = \int x\, dx = \frac{x^2}{2} + c = \frac{1}{2}(\ln x)^2 + c$

(b) $\int x e^{2x} dx = \frac{1}{2} x e^{2x} - \frac{1}{4} e^{2x} + c$

(c) $\int \frac{x^2}{1+4x^2} dx = \frac{1}{4}\int 1 - \frac{1}{1+4x^2} dx$

$= \frac{1}{4} x - \frac{1}{16}\int \frac{1}{x^2 + \frac{1}{4}} dx$

$= \frac{1}{4} x - \frac{1}{8} \tan^{-1}(2x) + c$

(d) $\int_2^5 \frac{x-6}{x^2+3x-4} dx = \int_2^5 \frac{x-6}{(x+4)(x-1)} dx$

$= \int_2^5 \frac{A}{x+4} + \frac{B}{x-1} dx$

$A + B = 1 \qquad A - 4B = 6 \qquad A = 2 \quad B = -1$

$= \int_2^5 \frac{2}{x+4} - \frac{1}{x-1} dx$

$= \left[2\ln(x+4) - \ln(x-1) \right]_2^5$

$= 2\ln\left(\frac{9}{6}\right) - \ln\left(\frac{4}{1}\right)$

$= 2\ln\frac{3}{2} - 2\ln 2 = 2\ln\frac{3}{4}$

Question 1 (continued)

(e) $$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx \qquad x = \tan\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta\ \sec\theta}\ \sec^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\cos\theta}{\sin^2\theta}\,d\theta$$

$$= \left[-\frac{1}{\sin\theta}\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \sqrt{2} - \frac{2}{\sqrt{3}}$$

$$= \frac{\sqrt{6}-2}{\sqrt{3}}$$

$$= \frac{3\sqrt{2}-2\sqrt{3}}{3}$$

End of Question 1

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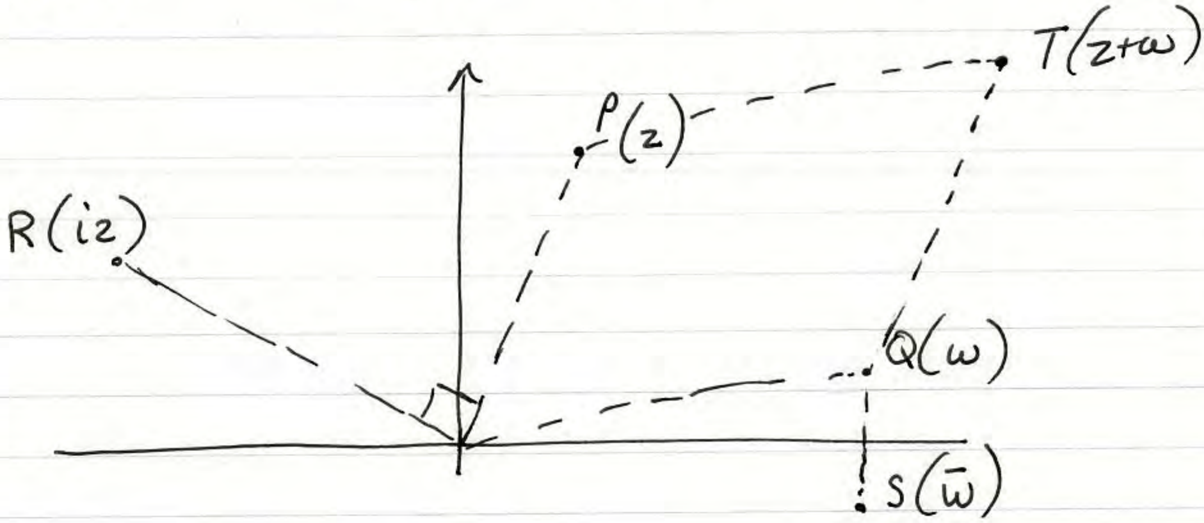
Question 2

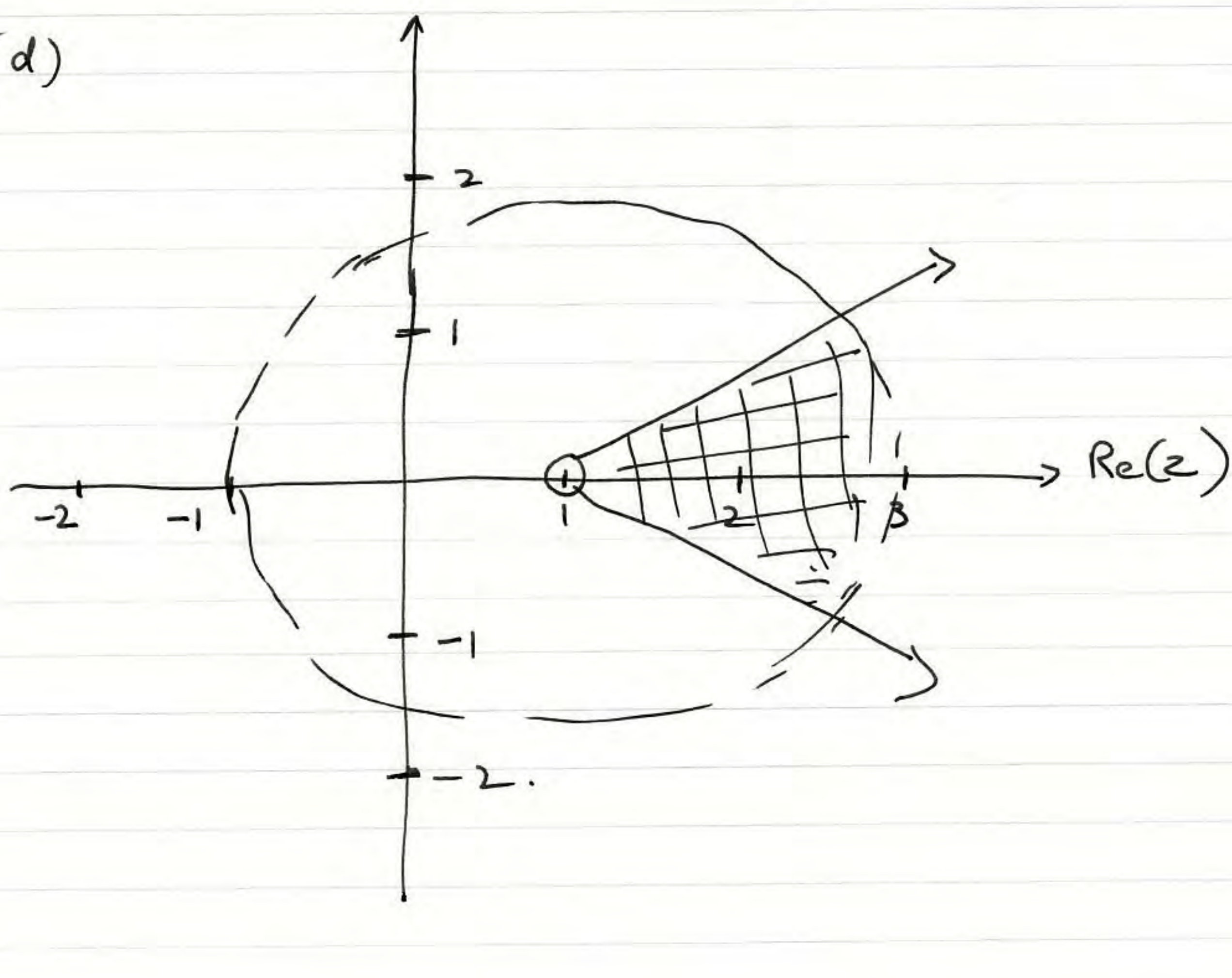
(a) $i^9 = i^8 . i = i = 0 + 1i$

(b) $\dfrac{-2+3i}{2+i} = \dfrac{-2+3i}{2+i} \times \dfrac{2-i}{2-i} = \dfrac{-1+8i}{5} = -\frac{1}{2} + \frac{8i}{5}$

(c)



(d)



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Question 2 (continued)

(e) $z^5 = -1 = 1(\cos\pi + i\sin\pi)$

$$z = 1\left(\cos\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2k\pi}{5}\right)\right)$$

(i) $z = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$,

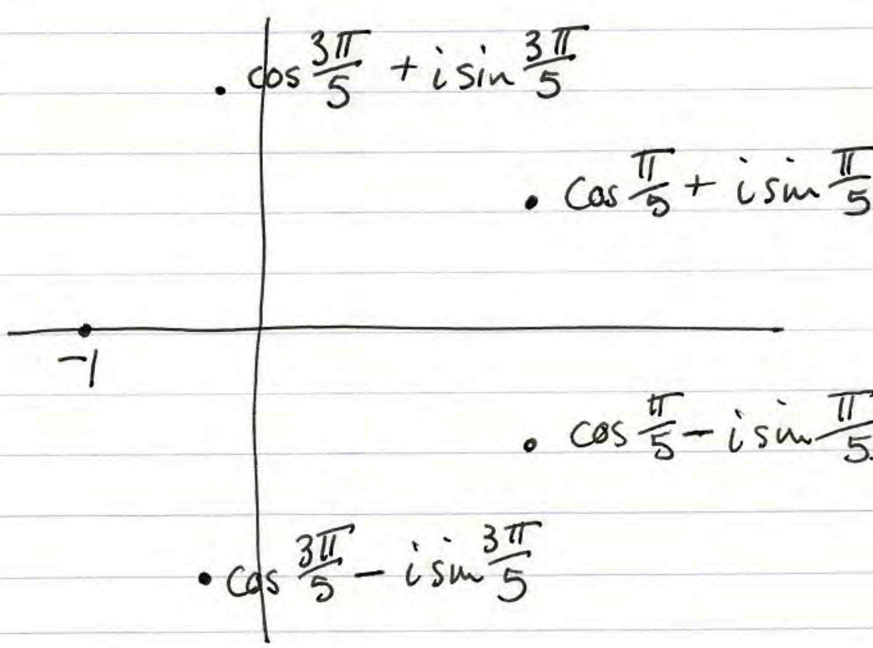
$z = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$,

$z = -1$,

$z = \cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$,

$z = \cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$

(ii)



Question 2 (continued)

(f) (i) $(a+bi)^2 = 3+4i$

$a^2-b^2=3 \qquad 2ab=4$

$a=2, b=1$ or $a=-2, b=-1$

$\sqrt{3+4i} = \pm(2+i)$

(ii) $z^2+(-2+i)z-2i=0$

$$z = \frac{2-i \pm \sqrt{(-2+i)^2+8i}}{2}$$

$$= \frac{2-i \pm \sqrt{3+4i}}{2}$$

$$= \frac{2-i \pm (2+i)}{2}$$

$= 2$ or $-i$

End of Question 2

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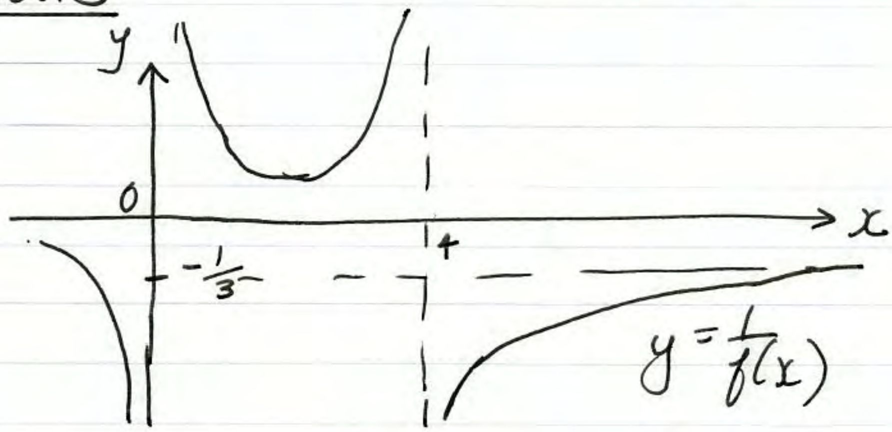
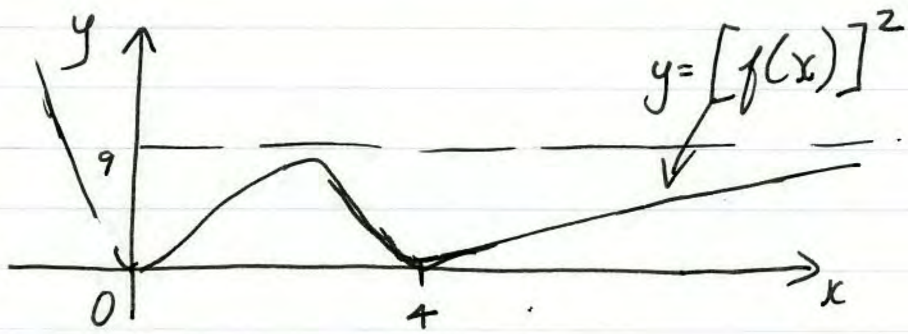
Question 3

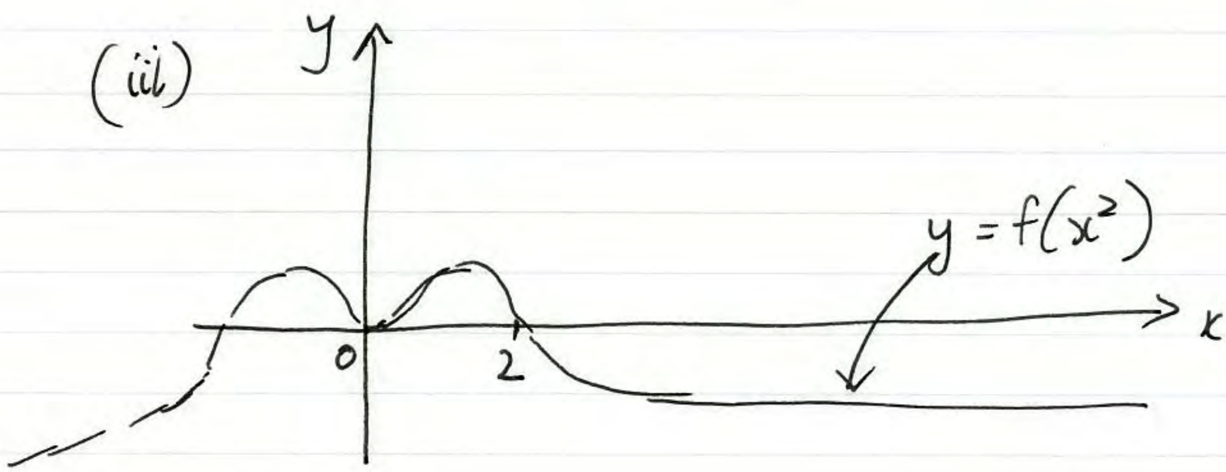
(a) (i)



(ii)



(iii)



Question 3 (continued)

(b) $x^2 + 2xy + 3y^2 = 18$

$2x + 2y + 2x\dfrac{dy}{dx} + 6y\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(2x + 6y) = -2x - 2y$

$\dfrac{dy}{dx} = -\left(\dfrac{x+y}{x+3y}\right)$

Need $\dfrac{dy}{dx} = 0$ So, $x = -y$

Sub $y^2 - 2y^2 + 3y^2 = 18$

$\Rightarrow y^2 = 9$

$y = \pm 3, \quad x = \mp 3$

Reqd points are $(-3,3)$ and $(3,-3)$

(c) $P(x) = x^3 + ax^2 + bx + 5$

$P(1) = 1 + a + b + 5 = 0 \qquad a + b = -6$

$P'(x) = 3x^2 + 2ax + b$

$P'(1) = 3 + 2a + b = 0 \qquad 2a + b = -3$

$a = 3, \; b = -9$

$$\left(\begin{aligned} P(x) &= x^3 + 3x^2 - 9x + 5 \\ &= (x-1)(x^2 + 4x - 5) \\ &= (x-1)^2(x+5) \end{aligned}\right)$$

Question 3 (continued)

(d) $V = \int_0^3 2\pi x \left[(x+1) - (x-1)^2 \right] dx$

$= 2\pi \int_0^3 x^2 + x - x^3 + 2x^2 - x \, dx$

$= 2\pi \left[-\frac{x^4}{4} + x^3 \right]_0^3$

$= 2\pi \left(27 - \frac{81}{4} \right)$

$= \frac{27\pi}{2}$

End of Question 3

Question 4

(a) (i) If we differentiate $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

we get $\frac{2x_0}{a^2} + \frac{2y_0}{b^2} y' = 0$

so $y' = -\frac{b^2 x_0}{a^2 y_0}$, which is the slope of the tangent. Hence the slope of the normal is $\frac{a^2 y_0}{b^2 x_0}$. The equation of the normal passing through (x_0, y_0) is

$$\frac{y - y_0}{x - x_0} = \frac{a^2 y_0}{b^2 x_0}$$

so

$$y - y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0)$$

as claimed.

(ii) Set $y = 0$ in the formula in (i):

$$-y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0)$$

$$-b^2 x_0 = a^2 (x - x_0)$$

$$a^2 x = (a^2 - b^2) x_0$$

$$x = \frac{a^2 - b^2}{a^2} x_0$$

$$= \frac{a^2 - a^2(1 - e^2)}{a^2} x_0 = e^2 x_0$$

Question 4 (continued)

(a)(iii) The focus directrix definition asserts that

$$PS = e\,PM \text{ and } PS' = e\,PM'$$

Also, $PM = \frac{a}{e} - x_0$ and $PM' = \frac{a}{e} + x_0$

Hence
$$\frac{PS}{PS'} = \frac{\cancel{e}PM}{\cancel{e}PM'} = \frac{a/e - x_0}{a/e + x_0} = \frac{ae - e^2x_0}{ae + e^2x_0}$$
$$= \frac{NS}{NS'}$$

since N has x-coordinate e^2x_0

and S, S' have x-coordinates $\pm ea$.

(iv) Let $\gamma = \angle SNP$. By the sine rule
$$\frac{\sin\beta}{SN} = \frac{\sin\gamma}{PS}$$

$$\frac{\sin\alpha}{S'N} = \frac{\sin(\pi - \gamma)}{PS'} = \frac{\sin\gamma}{PS'}$$

Hence
$$\sin\beta = \frac{SN}{PS}\sin\gamma,$$
$$\sin\alpha = \frac{S'N}{PS'}\sin\gamma,$$

so
$$\frac{\sin\beta}{\sin\alpha} = \frac{SN}{PS}\,\frac{PS'}{S'N}$$
$$= \frac{\left(\frac{SN}{S'N}\right)}{\left(\frac{PS}{PS'}\right)} = 1$$

by using (iii). Thus $\sin\alpha = \sin\beta$,

and so $\alpha = \beta$ since $0 \le \alpha, \beta \le \frac{\pi}{2}$

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Question 4 (continued)

(b) (i) vertical : $mg = N\sin\alpha + T\cos\alpha$
horizontal : $mr\omega^2 = T\sin\alpha - N\cos\alpha$

(ii) From (i):

$$\begin{aligned} &mg\cos\alpha + mr\omega^2\sin\alpha \\ &= \cancel{N\sin\alpha\cos\alpha} + T\cos^2\alpha \\ &\quad \cancel{- N\cos\alpha\sin\alpha} + T\sin^2\alpha \\ &= T(\cos^2\alpha + \sin^2\alpha) = T \end{aligned}$$

Similarly,

$$\begin{aligned} &mg\sin\alpha - mr\omega^2\cos\alpha \\ &= N\sin^2\alpha + \cancel{T\cos\alpha\sin\alpha} \\ &\quad + N\cos^2\alpha \cancel{- T\cos\alpha\sin\alpha} \\ &= N(\sin^2\alpha + \cos^2\alpha) = N \end{aligned}$$

So $N = m(g\sin\alpha - r\omega^2\cos\alpha)$

(iii) By (ii),

$$m(g\cos\alpha + r\omega^2\sin\alpha) = m(g\sin\alpha - r\omega^2\cos\alpha)$$

divide by $m\cos\alpha$:

$$g + r\omega^2\tan\alpha = g\tan\alpha - r\omega^2$$

$$r\omega^2(1 + \tan\alpha) = g(\tan\alpha - 1)$$

$$\omega^2 = \frac{g}{r}\,\frac{\tan\alpha - 1}{\tan\alpha + 1}$$

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Question 4 (continued)

(b) (iv) We need

$$\omega^2 = \frac{g}{r}\,\frac{\tan\alpha - 1}{\tan\alpha + 1} > 0,$$

so $\tan\alpha - 1$ must be positive.

Hence $\tan\alpha > 1$, so $\alpha > \frac{\pi}{4}$.

End of Question 4.

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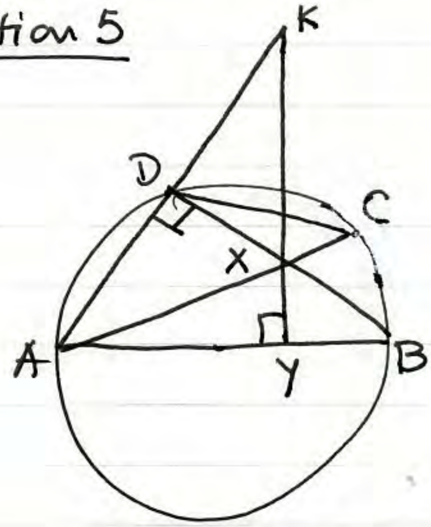
Question 5

(a)



(i) $\angle AKY =$ complement of $\angle KAY$

$=$ complement of $\angle DAB$

$= \angle ABD$ since $\angle ADB = 90°$ (angle in a semi circle)

(ii) $\angle AKY = \angle ABD = \angle ACD$ (angles on same arc AD)

$\therefore$ $\angle DKX = \angle DCX$

$\therefore$ CKDX is cyclic

(iii) From (ii), $\angle XCK = \angle XDK = 90°$

(opposite angles in a cyclic quadrilateral are supplementary.)

Also $\angle ACB = 90°$ (angle in semicircle)

$\therefore$ KCB is a straight angle

$\therefore$ B, C, K are collinear.

Question 5 (continued)

(b)(i)

$$I_n = \int_0^1 x^{2n+1} e^{x^2}\,dx$$

$$= \frac{1}{2}\int_0^1 x^{2n} \cdot 2x e^{x^2}\,dx$$

$$= \frac{1}{2}\int_0^1 u^n \cdot e^u\,du$$

$$= \frac{1}{2}\left\{\left[u^n e^u\right]_0^1 - \int_0^1 n u^{n-1} e^u\,du\right\}$$

$$= \frac{1}{2}e - n \cdot \frac{1}{2}\int_0^1 u^{n-1} e^u\,du$$

$$= \frac{1}{2}e - n \cdot I_{n-1}$$

(ii)

$$I_2 = \frac{1}{2}e - 2I_1$$

$$I_1 = \frac{1}{2}e - I_0$$

$$I_0 = \int_0^1 x e^{x^2}\,dx = \frac{1}{2}e - \frac{1}{2}$$

$$I_1 = \frac{1}{2}$$

$$I_2 = \frac{1}{2}e - 1$$

Question 5 (continued)

(c) (i) $f(x) = \dfrac{e^x - e^{-x}}{2} - x$

$f'(x) = \dfrac{e^x + e^{-x}}{2} - 1$

$f''(x) = \dfrac{e^x - e^{-x}}{2}$

$x > 0 \quad e^x > 1 > e^{-x} \quad e^x - e^{-x} > 0$

$f''(x) > 0$

(ii) $f'(0) = 0, \quad f''(x) > 0$ for $x > 0$

$\therefore f'(x) > 0$ for $x > 0$

(iii) $f(0) = 0, \quad f'(x) > 0$ for $x > 0$

$\therefore f(x) > 0$ for $x > 0$

$\therefore \dfrac{e^x - e^{-x}}{2} > x$ for $x > 0$

End of Question 5

Question 6

(a) $V = \int_0^4 (4-x)\, 2y \, dx$

$= 2\int_0^4 (4-x)\sqrt{4-x}\, dx$

$= 2\left[-\frac{(4-x)^{5/2}}{5/2}\right]_0^4 = \frac{4}{5} \times 32 = \frac{128}{5}$

(b) (i) Let roots be $-1, \alpha, \beta$

$-1 \times \alpha \times \beta = -1$

$\beta = \frac{1}{\alpha}$

(ii) roots are $-1, \alpha, \bar{\alpha}$

1. $\bar{\alpha} = \frac{1}{\alpha}$

$|\alpha|^2 = 1$

$|\alpha| = 1$

2. Let $\alpha = a + ib$

$\bar{\alpha} = a - ib$

$-1 + (a+ib) + (a-ib) = -a$

$2a = 1 - q$

$a = \frac{1-q}{2}$

$Re(\alpha) = \frac{1-q}{2}$

Question 6 (continued)

(c) (i) $PQ = \sqrt{x^2+y^2-r^2}$

(ii) $\sqrt{x^2+y^2-r^2} = d-x$

$$x^2+y^2-r^2 = d^2-2dx+x^2$$
$$y^2 = r^2+d^2-2dx$$

(iii) $y^2 = -4 \times \frac{1}{2}d\left(x - \frac{r^2+d^2}{2d}\right)$

vertex is $\left(\frac{r^2+d^2}{2d}, 0\right)$

focus is $\left(\frac{r^2+d^2}{2d} - \frac{1}{2}d, 0\right)$

$$= \left(\frac{r^2}{2d}, 0\right)$$

(iv) directrix is $x = \frac{r^2+d^2}{2d} + \frac{1}{2}d$

$$= \frac{r^2}{2d} + d$$

$PS - PQ = \left(\frac{r^2}{2d} + d - x\right) - (d-x)$

$= \frac{r^2}{2d}$ is independent of x.

End of Question 6.

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Question 7

(a) (i) 1. $\ddot{x} = g - rv$

$$v\frac{dv}{dx} = g - rv$$

$$\frac{dx}{dv} = \frac{v}{g - rv}$$

$$= \frac{-\frac{1}{r}(g - rv) + \frac{g}{r}}{g - rv}$$

$$= -\frac{1}{r} + \frac{g}{r} \cdot \frac{1}{g - rv}$$

$$x = -\frac{1}{r}v + \frac{g}{r} \cdot -\frac{1}{r} \ln(g - rv) + c$$

$$= -\frac{v}{r} - \frac{g}{r^2} \ln(g - rv) + c$$

When $x = 0$, $v = 0$ so $c = \frac{g}{r^2} \ln g$,

$$x = -\frac{v}{r} - \frac{g}{r^2} \ln(g - rv) + \frac{g}{r^2} \ln g$$

$$x = \frac{g}{r^2} \ln \frac{g}{g - rv} - \frac{v}{r}$$

2. $g = 9.8$, $r = 0.2$ $x = L$, $v = 30$

$$L = \frac{9.8}{0.2^2} \ln \frac{9.8}{9.8 - 0.2 \times 30} - \frac{30}{0.2}$$

$= 82$ to 2 sig. figs.

Question 7 (continued)

ALTERNATELY.

(a) (i) 1.

For $v = 0$

$$\frac{g}{r^2}\ln\left(\frac{g}{g - r.0}\right) - \frac{0}{r} = \frac{g}{r^2}\ln 1 = 0 = x$$

Differentiate the given identity:

$$v = \dot{x} = \frac{g}{r^2}\,\frac{g-rv}{g}\left(-\frac{-r}{(g-rv)^2}\right)\dot{v} - \frac{\dot{v}}{r}$$

$$= \frac{\dot{v}}{r}\left(\frac{g}{g-rv} - 1\right)$$

$$= \frac{\dot{v}}{r}\;\frac{g-g+rv}{g-rv}$$

$$= \dot{v}\,\frac{v}{g-rv}$$

Since $\dot{v} = \ddot{x}$ we get

$\ddot{x} = g - rv$ as required.

2. Use calculator with given numbers and the formula in (i):

$L = 82{\cdot}11\text{ m}$

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Question 7 (continued)

(a)(ii) $x = e^{-t/10}(29\sin t - 10\cos t) + 92$

$$\dot{x} = e^{-t/10}(29\cos t + 10\sin t) + (29\sin t - 10\cos t) \times \frac{1}{10}e^{-t/10}$$

$$= \frac{e^{-t/10}}{10}\left(300\cos t + 71\sin t\right)$$

$$= 0$$

$$\tan t = -\frac{300}{71}$$

$$t \cong -1.3384 + \pi$$

$$t \cong 1.8032$$

$$x \cong 25.5 + 92 = 117.5$$

$117.5 + 2 = 119.5 < 125$

so bungee jumper doesn't get wet!

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Question 7 (continued)

(b)(i) $z = \cos\theta + i\sin\theta$

$z^n = \cos n\theta + i\sin n\theta$

$z^{-n} = \cos n\theta - i\sin n\theta$

$z^n + z^{-n} = 2\cos n\theta$

(ii) $(2\cos\theta)^{2m} = \left(z + \frac{1}{z}\right)^{2m}$

$z^{2m} + \binom{2m}{1} z^{2m-2} + \ldots + \binom{2m}{m} + \ldots + \binom{2m}{2m-1} z^{-2m+2} + z^{-2m}$

$= \left(z^{2m} + z^{-2m}\right) + \binom{2m}{1}\left(z^{2m-2} + z^{-2m+2}\right) + \ldots + \binom{2m}{m}$

$= 2\cos 2m\theta + \binom{2m}{1}.2\cos(2m-2)\theta + \ldots + \binom{2m}{m-1}2\cos 2\theta + \binom{2m}{m}$

as required.

(iii) $\int_0^{\pi/2} \cos 2m\,\theta\, d\theta = \frac{1}{2^{2m}}\binom{2m}{m}.\int_0^{\frac{\pi}{2}} 1\, d\theta$

(all other integrals $= 0$)

$= \frac{\pi}{2^{2m+1}}\binom{2m}{m}$

Question 7 (continued)

ALTERNATELY:

(b)(i) By de Moivre's formula

$$z^n = \cos n\theta + i \sin n\theta$$
$$z^{-n} = \cos n\theta - i \sin n\theta$$

Hence, $z^n + z^{-n} = \cos n\theta + i \sin n\theta$
$+ \cos n\theta - i \sin n\theta$
$= 2 \cos n\theta$

(ii) From (i) $2\cos\theta = z + z^{-1}$ if we set $n = 1$
Then, using the binomial theorem,

$$(2\cos\theta)^{2m} = (z + z^{-1})^{2m}$$
$$= \sum_{k=0}^{2m} \binom{2m}{k} z^k z^{-(2m-k)}$$
$$= \sum_{k=0}^{2m} \binom{2m}{k} z^{2k} z^{-2m} = z^{-2m} \sum_{k=0}^{2m} \binom{2m}{k} z^{2k}$$

Since $2m$ is even, the middle term of the sum is

$$\binom{2m}{m} z^{2m} \cdot z^{-2m} = \binom{2m}{m}$$

We then pair the k-th and the $(n-k)$-th terms.
Using $\binom{2m}{k} = \binom{2m}{2m-k}$ and (ii)

$$\binom{2m}{k} z^{2k} z^{-2m} + \binom{2m}{2m-k} z^{2(2m-k)} z^{(-2m)}$$
$$= \binom{2m}{k} \left(z^{-(2m-2k)} + z^{2m-2k} \right)$$
$$= 2\binom{2m}{k} \cos(2m-2k)\theta$$

Question 7 (continued)

(ii) cont.

Putting everything together,

$$(2\cos\theta)^{2m} = \binom{2m}{m} + 2\sum_{k=0}^{m-1}\binom{2m}{k}\cos(2m-2k)\theta$$

as required.

(iii) $$\int_0^{\frac{\pi}{2}} \cos(2m-2k)\theta . d\theta = \frac{1}{2m-2k}\sin(2m-2k)\theta\Big]_0^{\frac{\pi}{2}} = 0$$

Hence from (ii)

$$\int_0^{\frac{\pi}{2}} \cos^{2m}\theta . d\theta = 2^{\frac{1}{2m}}\left(\int_0^{\frac{\pi}{2}}\binom{2m}{m}d\theta + 2\sum_{k=0}^{m-1}\binom{2m}{k}\int_0^{\frac{\pi}{2}}\cos(2m-2k)\theta . d\theta\right)$$

$$= \frac{\pi}{2^{2m+1}}\binom{2m}{m}$$

End of Question 7

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Question 8.

(a) (i) $\cot\theta + \frac{1}{2}\tan\frac{\theta}{2} = \frac{1-t^2}{2t} + \frac{t}{2} = \frac{1}{2t} = \frac{1}{2}\cot\frac{\theta}{2}$

(ii) $n=1$

$LHS = \tan\frac{x}{2} = \cot\frac{x}{2} - 2\cot x = RHS$

by (i) with $\theta = x$.

Suppose

$$\sum_{r=1}^{k} \frac{1}{2^{r-1}} \tan\frac{x}{2^r} = \frac{1}{2^{k-1}}\cot\frac{x}{2^k} - 2\cot x$$

Add $\frac{1}{2^k}\tan\frac{x}{2^{k+1}}$

$$\sum_{r=1}^{k+1} \frac{1}{2^{r-1}} \tan\frac{x}{2^r} = \frac{1}{2^{k-1}}\cot\frac{x}{2^k} + \frac{1}{2^k}\tan\frac{x}{2^{k+1}} - 2\cot x$$

$$= \frac{1}{2^k}\cot\frac{x}{2^{k+1}} - 2\cot x$$

by (i) with $\theta = \frac{x}{2^k}$

The result follows by mathematical induction.

Question 8 (continued)

(a) (iii)

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{2^{r-1}} \tan \frac{x}{2^r} = \lim_{n\to\infty} \frac{1}{2^{n-1}} \cot \frac{x}{2^n} - 2\cot x$$

$$= \lim_{n\to\infty} \frac{2}{x} \cdot \cos \frac{x}{2^n} \cdot \frac{\frac{x}{2^n}}{\sin \frac{x}{2^n}} - 2\cot x$$

$$= \frac{2}{x} \cdot 1 \cdot 1 - 2\cot x$$

$$= \frac{2}{x} - 2\cot x$$

(iv) Put $x = \frac{\pi}{2}$

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{2^{r-1}} \tan \frac{\pi}{2^{r+1}} = \frac{4}{\pi} - 2\times 0 = \frac{4}{\pi}$$

Question 8 (continued)

(b) $\frac{1}{n} < \frac{1}{x} < \frac{1}{n-1}$ for $n-1 < x < n$

$$\int_{n-1}^{n} \frac{1}{n}\,dx < \int_{n-1}^{n} \frac{1}{x}\,dx < \int_{n-1}^{n} \frac{1}{n-1}\,dx$$

Integrate to get $\frac{1}{n} < \ln\frac{n}{n-1} < \frac{1}{n-1}$

Exponentiate to get $e^{\frac{1}{n}} < \frac{n}{n-1} < e^{\frac{1}{n-1}}$

Take with power and invert to get $e^{-\frac{n}{n-1}} < \left(1-\frac{1}{n}\right)^n < e^{-1}$

(c) (i) $W = p + q^n p + q^{2n} p + \ldots\ldots$
$= p + q^n\left(p + q^n p + q^{2n} p + \ldots\right)$
$= p + q^n(W)$

[Explanation of first line: A_1 wins immediately, or after all n fail to win, or all n fail to win twice, or..]

(c)(ii) $W_m = p + q^n p + \ldots\ldots + q^{(m-1)n} p$
$= p\,\frac{1-q^{mn}}{1-q^n}$

$W = p \cdot \frac{1}{1-q^n}$

$\frac{W_m}{W} = 1 - q^{mn} = 1 - \left[\left(1-\frac{1}{n}\right)^n\right]^m$
$\rightarrow 1 - \left[e^{-1}\right]^m = 1 - e^{-m}$

End of Question 8.