

2010 HSC NOTES FROM THE MARKING CENTRE

MATHEMATICS EXTENSION 2

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Introduction

This document has been produced for the teachers and candidates of the Stage 6 Mathematics Extension 2 course. It contains comments on candidate responses to the 2010 Higher School Certificate examination, indicating the quality of the responses and highlighting their relative strengths and weaknesses.

This document should be read along with the relevant syllabus, the 2010 Higher School Certificate examination, the marking guidelines and other support documents which have been developed by the Board of Studies to assist in the teaching and learning of Mathematics Extension 2.

Many parts in the Extension 2 paper require candidates to prove, show or deduce a result. Candidates are reminded of the need to give clear, concise reasons in their answers.

Question 1

(a) This part was generally well answered with most candidates successfully using the substitution $u = 1 + 3x^2$. However, some left their response as a function of u , not x , that is, $\frac{\sqrt{u}}{3} + c$.

A small number of candidates successfully found the primitive by using the substitution

$$x = \frac{1}{\sqrt{3}} \sec \theta.$$

(b) In most responses, candidates realised they had to use $\frac{\sin x}{\cos x}$ and successfully found the correct primitive. However, a significant number of errors were made in the evaluation of $\log_e \left(\cos \left(\frac{\pi}{4} \right) \right)$ or by leaving out the negative sign in finding the primitive of $\tan x$. Those candidates who tried to use integration by parts were usually unsuccessful.

(c) In most responses, candidates recognised and successfully used partial fractions. Many, however, used an inappropriate decomposition such as $\frac{a}{x} + \frac{b}{x^2 + 1}$. Others had the correct

decomposition $\frac{a}{x} + \frac{bx+c}{x^2+1}$ but experienced difficulty in evaluating the constants a , b and c .

Those who found the correct values of the constants frequently substituted incorrectly, writing the new integrand as $\frac{1}{x} - \frac{1}{x^2+1}$ instead of $\frac{1}{x^2+1} - \frac{1}{x}$. Some candidates successfully used the trigonometric substitution $x = \tan \theta$ and arrived correctly at an answer in terms of θ , that is $\log_e|\sin \theta|$; however, they usually did not go on to express this in terms of x .

- (d) Most candidates quoted or deduced the relevant expressions $\sin \theta = \frac{2t}{1+t^2}$, $\frac{d\theta}{dt} = \frac{2}{1+t^2}$, changed the limits correctly and arrived at the correct answer. Common integration errors confused $\int_0^1 \frac{2}{(1+t)^2} dt$ with $\int_0^1 \frac{2}{1+t^2} dt$ or found an incorrect primitive due to the absence of the negative sign. Those who misquoted the expressions involving t or found the derivative to be $\frac{d\theta}{dt} = \frac{2t}{1+t^2}$ usually found a much more complicated expression resulting in an unsuccessful attempt.
- (e) Responses that used the substitutions $u = \sqrt{x}$ or $u = 1 + \sqrt{x}$ were most successful. The most common errors were the incorrect simplification of $2\int \frac{u}{1+u} du$ or $2\int \frac{u-1}{u} du$. A small number of candidates who used the substitution $x = \tan^4 \theta$ could not successfully evaluate $4\int \tan^3 \theta d\theta$. Those candidates who tried to rationalise the denominator were generally unsuccessful.

Question 2

- (a) (i) This part was well done by almost all candidates.
- (ii) This part was generally well done; however, the simplification required in collecting like terms was a problem for some candidates.
- (iii) The realisation of the denominator was well done by almost all candidates. Some, however, finished with a denominator of $5+1=6$ rather than $25+1=26$.
- (b)(i) In better responses, candidates located the complex number on the Argand diagram and this usually enabled them to find the appropriate argument. Failure to locate the complex number appropriately often resulted in an (incorrect) argument of $\frac{\pi}{6}$.
- (ii) The application of De Moivre's theorem to the modulus–argument form of $(-\sqrt{3}-i)^6$ demonstrated that most candidates have a solid grasp of this concept.
- (c) Candidates who successfully expressed the given relationship in the Cartesian form usually graphed the correct region. Some, however, had difficulty in correctly interpreting the correct Cartesian form equivalent to $0 \leq z + \bar{z} \leq 3$.

- (d) (i) The fact that a rhombus is a parallelogram with a pair of adjacent sides equal was used by candidates to show that $|z| = |z^2| = 1$. Many candidates stated the properties of a rhombus rather than actually proving that the parallelogram was a rhombus.

Many candidates simply said that $\overrightarrow{OA} = \overrightarrow{OB}$ and that $\overrightarrow{OB} = \overrightarrow{AC}$ and concluded that the figure was a rhombus without justification.

- (ii) The realisation that the diagonal of a rhombus bisects the angle through which it passes enabled candidates to find the argument of $|z + z^2|$.

- (iii) In better responses, candidates used a variety of approaches to show that $|z + z^2| = 2 \cos \frac{\theta}{2}$.

These approaches included the use of the sine rule, the cosine rule and the definition of the modulus. The most successful were those who used the property that the diagonals bisect at right angles and then used the cosine ratio to find half of OC .

- (iv) The ability to combine the results shown in parts (ii) and (iii) to successfully obtain

$z + z^2 = 2 \cos \frac{\theta}{2} \left(\cos \frac{3\theta}{2} + i \sin \frac{3\theta}{2} \right)$ was the most common approach in reaching the required conclusion. Other successful approaches included using right-angled triangle trigonometry to show that $\operatorname{Re}(z + z^2) = OC \times \cos \frac{3\theta}{2}$ which implies that $\operatorname{Re}(z + z^2) = 2 \cos \frac{\theta}{2} \cos \frac{3\theta}{2}$, or rewriting $\cos \theta + \cos 2\theta$ as $\cos\left(\frac{3\theta}{2} - \frac{\theta}{2}\right) + \cos\left(\frac{3\theta}{2} + \frac{\theta}{2}\right)$ and then expanding, or using the identity $\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$.

Question 3

Only a small number of candidates earned full marks in this question. Where a question asks candidates to show a result, they need to display enough information to justify the result.

- (a) (i) Almost all candidates were awarded the mark. A small number of candidates sketched either $y = 4x - x^2$, $y = x^2 + 4x$ or $y = x^2$. Some candidates sketched both $y = 4x$ and $y = x^2$ and added ordinates.
- (ii) Most candidates indicated asymptotes and correctly found the reciprocal of their graph from part (i).
- (b) Most candidates chose to use shells to find the volumes. Most correct responses came from rotating the function about $x = 4$ as asked in the question. However some were awarded full marks for realising they could achieve the answer by rotating about $x = -2$ or by translating axes and then rotating, for example rotating $y = 1 - x^2$ about $x = 3$. Some candidates did not achieve full marks due to careless errors in finding the primitive or not substituting correctly into the primitive. Candidates attempting the slicing method needed to work with roots, but then had difficulty completing the integral.

- (c) Many candidates had difficulty understanding the question in this part. A tree diagram may have assisted some candidates to understand the question, particularly if they recognised that complementary events were involved.
- (d) (i) Most candidates found the equation of l_1 . Candidates who didn't simplify the gradient of l_1 found it difficult to prove the result. In finding the equation of l_1 , candidates who used the point-gradient form of the equation of a line had more success than candidates who used the form $y = mx + b$.
- (ii) Some candidates found the correct answer by replacing t with $-t$ in the answer to part (i).
- (iii) Most candidates correctly solved simultaneous equations using their results in parts (i) and (ii). Candidates choosing to substitute the given point of intersection into their equations often forgot to substitute into both equations.
- (iv) Most candidates found that the locus was the same rectangular hyperbola. However, only a small number of candidates correctly indicated that it was only the branch of that hyperbola in the first quadrant.

Question 4

- (a) (i) Most candidates demonstrated an understanding of implicit differentiation and obtained the correct derivative. Some candidates who demonstrated an understanding of implicit differentiation made a subsequent error when attempting to make $\frac{dy}{dx}$ the subject. A few candidates ignored the instruction to use implicit differentiation, avoiding the use of this technique by rearranging the equation of the curve to obtain an expression for y . These candidates generally did not achieve full marks.
- (ii) Most candidates indicated the x - and y -intercepts of the curve, but many failed to see the link between parts (i) and (ii) and so failed to sketch the curve correctly.
- (iii) Many candidates understood the effect of including absolute values on x and y , and demonstrated this by correctly reflecting their graph from part (ii).
- (b) (i) Candidates who chose to resolve forces in the horizontal and vertical directions were usually successful in deriving the required expression for F . A few candidates made an error with the resolution of forces in one direction; many of these candidates made a second error in attempting to complete their calculation and obtain the correct expression for F . A small number of candidates did not supply sufficient detail in their working to show how they arrived at their answer from their resolution of forces.

A small number of candidates attempted to answer the question by resolving forces along the plane of the track. Some of these candidates did not justify their answer sufficiently to be awarded full marks.

- (ii) Most candidates obtained a correct expression for v , including those candidates who were not successful in obtaining full marks for part (i) but realised that they could make use of the result given on the paper. A significant number of candidates made algebraic errors in their attempt to make v (or v^2) the subject of the equation $F = 0$.

- (c) This part caused difficulty for the majority of candidates, most of whom incorrectly assumed that they were required to approach the problem using inequalities, with the large majority of these beginning with a statement such as $(a - b)^2 \geq 0$ which assumes from the outset that both a and b are real numbers. A small number of candidates realised that the question was asking them to show that a is a real, and positive, number and correctly began from the quadratic equation $(a + b)^2 = abk$ (or equivalent). These candidates typically proceeded to examine the discriminant; however, most of them did not correctly demonstrate that a is both real and positive.
- (d) (i) Most candidates correctly applied a combinatorial approach to this problem and obtained the correct answer.
- (ii) Very few candidates were successful in obtaining the correct answer for this part. Most realised that the answer involved $\binom{12}{4} \times \binom{8}{4}$ or $\frac{12!}{4! \times 4! \times 4!}$ but did not realise that the resulting equal-sized groups were interchangeable and so neglected to divide their result by $3!$.

Question 5

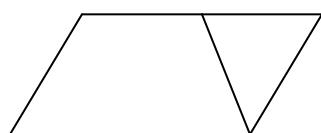
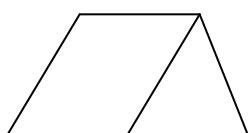
This question was reasonably well attempted but many marks were lost due to carelessness in setting out and algebraic manipulation.

- (a) (i) This was extremely well answered and most candidates were successful in gaining the mark.
- (ii) This was also very well attempted and most candidates were successful in gaining the mark.
- (iii) Candidates' responses to this part varied widely. Most showed they were experienced in the derivation of the equation of the tangent at $P(\cos \theta, b \sin \theta)$ efficiently.
- (iv) A variety of approaches were used for this part required considerable algebraic manipulation. The candidate's skills were often not sufficient to complete the calculation efficiently.
- (b) Candidates' responses to this part were very good and most gained full marks. Their algebraic manipulations were sound in this part. Most successful candidates used one of two approaches – partial fraction decomposition and the differentiation of the primitive function. Other approaches, although infrequently used, were by substitution: $y = u^{-1}$, $(1-u)^{-1}$, e^{-u} or vice versa, and $y = \sec^2 \theta$, $\sin^2 \theta$ or $\cos^2 \theta$.
- (c) (i) The candidates were challenged by this part, with few completing it. The most common approach was to use calculus; however, many failed to test their stationary point at $y = \frac{1}{2}$.

- (ii) Candidates' responses varied. Better candidates achieved the required result. The early introduction of a constant of integration, for example $x = \frac{1}{a} \ln\left(\frac{y}{1-y}\right) + k$ or $ax = \ln\left(\frac{y}{1-y}\right) + k$, induced algebraic errors while progressing to the required function $y = \frac{1}{ke^{-ax} + 1}$.
- (iii) Many candidates were successful in finding only that $k = 9$, but were challenged by the other parts of (c). Many interpreted the question to imply that $\frac{1}{2} \times \frac{1}{10} = \frac{1}{k+1}$ or $2 \times 10 = k + 1$.
- (iv) Many candidates interpreted this question correctly by using the result found in part (i) by stating it had a maximum gradient or a point of inflection at $y = \frac{1}{2}$.
- (v) The responses indicated that the candidates were very challenged by this part. Many candidates who had answered part (iv) correctly did not use the information to advantage in sketching the function. Many had their point of inflection at $x = 0$.

Question 6

- (a) (i) Many methods were used to show this result. Successful responses included a variety of similar triangle solutions (with diagrams to explain which triangles were being compared), areas of trapezia, or a consideration of the linear relationship between s and x . Partly successful responses approached the problem in a similar fashion did not complete the calculation. An example using similar triangles was to divide a vertical cross-section of the frustum into a parallelogram and a triangle, or to add on a triangle to create a parallelogram, for example using one of the following:



- (ii) Successful responses to this part presented an integral of the area of the cross-section from $x = 0$ to $x = h$. As part (i) supplied the cross-sectional area, this approach was quite common. Less than successful responses made a variety of errors when integrating or expanding $\left(a - \frac{a-b}{h}x\right)^2$, or by using incorrect limits, for example from a to b , or by integrating the side length s rather than the area s^2 .

- (b) Successful responses presented a first step to prove the result to be true for the two initial values of n . Assuming the result was true for $n = k - 1$ and $n = k$ provided the necessary link to prove the general result. Those who completed this were able to first factorise $2(1+\sqrt{2})^k + (1+\sqrt{2})^{k-1}$ and $2(1-\sqrt{2})^k + (1-\sqrt{2})^{k-1}$. Responses that started with $2(1+\sqrt{2})^k + (1-\sqrt{2})^{k-1}$ in an attempt to work towards $2a_k + a_{k-1}$ were generally less successful.

- (c) (i) The binomial expansion was generally well done.
- (ii) Most responses successfully presented an expansion using De Moivre's theorem and most of these went on to confirm the desired result by first equating the imaginary parts and then correctly replacing $\cos^2 \theta$ with $1 - \sin^2 \theta$.
- (iii) Successful responses recognised and used the result in part (ii) and noted that $\sin 5\theta = 1$ when $\theta = \frac{\pi}{10}$. Hence, by subtracting 1 from both sides, the given polynomial in x was found.
- (iv) This part was quite well done, usually by long division. The polynomial $p(x)$ was also found by other means, including by inspection. The long division process was, however, prone to errors.
- (v) Expanding the first few terms of $(4x^2 + ax + 1)^2$ and equating the coefficients of x^3 was the most common successful approach. Responses that equated the coefficients of x^2 found two possible values for a . Checking the coefficients of x^3 , the correct response was generally found.
- (vi) The responses that recognised the need to solve the equation $4x^2 + 2x + 1 = 0$, together with the fact that $\sin \frac{\pi}{10} > 0$, were mostly successful.

Question 7

- (a) (i) It should be noted that to prove that two triangles are similar, it is sufficient to prove two angles are equal in the relevant triangles (which most candidates did) but a significant number wasted time by trying to show that the third angle was also equal in all triangles.
- (ii) The best attempts found relevant equivalent ratios and substituted into the required expression. Many attempts were far less efficient and included lists of various ratios, not all of which were relevant.
- (iii) Most candidates who realised that all the diagonals of the regular pentagon were equal to x successfully answered this part. Many candidates referred to sides or diagonals in such terms AD or BC . However, as this notation was neither explained on the given diagram nor in the candidate's working, the markers were unsure which distances the candidates were referring to.
- (b) This part was successfully done by most candidates. Some attempted to answer this question without actually drawing the graph. This approach required a much more detailed and precise explanation in order to be rewarded.
- (c) (i) Some candidates made errors using index laws and others had trouble solving equations such as $x^{n-2} = x^{n-1}$ or $x^{n-2} = 0$.
- (ii) Although this part was successfully done by most candidates, some only showed that $P'(0) = 0$ without showing that $P(1) = 0$. Some substituted $n = 1$ instead of $x = 1$.

- (iii) Most candidates gave a sketch of $y = P(x)$ but many failed to realise that $(0,1)$ is a maximum stationary point. Many did not state that there is another real zero because the graph crosses the x -axis at a point other than $x = 1$. Many did not draw a graph that was consistent with all the facts found in the previous two parts of this question.
- (iv) Most candidates who attempted this part did not appear to realise that they needed to show that $P(-1) < 0$ and $P(-0.5) \geq 0$. Also, many did not use the fact that $(-1)^n = -1$ and $(-1)^{n-1} = 1$, since n is odd. Some of the few candidates who did make progress with this part failed to get an expression in a form that allowed them to use part (b).
- (v) Most candidates who attempted this part failed to realise that the polynomial had 5 zeros, three of which are $1, 1$ and a , and that the other two are complex conjugates. A significant number said that the product of the zeros was 0.25 , instead of the correct -0.25 .

Question 8

A large number of candidates did not attempt this question. What appeared to be rushed work indicates that some candidates left themselves little time for this question. Candidates who made their application of integration by parts very explicit fared best in getting the derivatives, primitives and algebra (including signs) correct.

- (a) Many candidates replaced a factor of $\cos^2 x$ with $1 - \sin^2 x$ immediately and then could not go further or had a more difficult solution. Some used the double angle formula for a $\cos^2 x$ or $\sin^2 x$ factor but in most cases they failed to reach a correct solution. Algebraic errors were common, as were subscript errors, typically A_{n-2} for A_{n-1} .
- (b) This part was generally well done.
- (c) A good number of attempts did not make a correct choice of u and v^1 using integration by parts as a way of introducing a factor of x^2 . Some candidates persisted with poor choices such as $v^1 = \cos 2x$ or $x \sin x$, that generally went nowhere. Poor setting out or rushed work often led to algebraic errors. A correct solution starting with B_n was very rare.
- (d) Many candidates tried this part, but with mixed results. Some quickly got the result in a line or two; others struggled with up to a page of algebra.
- (e) Some candidates did not see the telescoping sum.
- (f) This part was rarely attempted, but in those cases the candidate usually made substantial progress. Some candidates tried to argue that $\cos x = \sin\left(\frac{\pi}{2} - x\right) \geq \frac{2}{\pi}\left(\frac{\pi}{2} - x\right) = 1 - \frac{2x}{\pi}$, rather than $\cos^2 x = 1 - \sin^2 x \leq 1 - \left(\frac{2x}{\pi}\right)^2$, but this leads to a inequality different from the one desired.
- (g) This part was generally poorly done in the few cases it was attempted. The most common error was the calculation of the primitive in the attempt at integration by parts, incorrectly

finding the primitive $-\frac{\pi^2}{8x(n+1)}\left(1 - \frac{4x^2}{\pi^2}\right)^{n+1}$ from the chosen integrand $\left(1 - \frac{4x^2}{\pi^2}\right)^n$.

Some candidates tried expressing the factor x^2 in terms of $1 - \frac{4x^2}{\pi^2}$ but this did not lead to a solution. A small number took the easier route of correctly working from right to left.

- (h) Those candidates who attempted this part generally made a correct change of variables in this integral, but very few could provide the reason for the inequality with A_n .
- (i) This part was rarely attempted. It required using the formula from part (e) and the inequality from part (h).
- (j) Many candidates wrote down the correct answer $\frac{\pi^2}{6}$, with or without justification, even if the rest of this question was poorly done.