BOARD OF STUDIES
NEW SOUTH WALES

2010 HSC Mathematics Extension 1
Sample Answers

This document contains ‘sample answers’, or, in the case of some questions, ‘answers could include’. These are developed by the examination committee for two purposes. The committee does this:

(a) as part of the development of the examination paper to ensure the questions will effectively assess students’ knowledge and skills, and

(b) in order to provide some advice to the Supervisor of Marking about the nature and scope of the responses expected of students.

The ‘sample answers’ or similar advice are not intended to be exemplary or even complete answers or responses. As they are part of the examination committee’s ‘working document’, they may contain typographical errors, omissions, or only some of the possible correct answers.

Question 1 (a)

$\sin^{-1}\left(\frac{x}{2}\right) + C$

Question 1 (b)

$-2 \le x \le 2$

Question 1 (c)

$$\ln(x+6) = \ln(x^2)$$
$$\therefore x^2 = x + 6$$
$$x^2 - x - 6 = 0$$
$$(x-3)(x+2) = 0$$
$$x = 3 \quad \text{or} \quad -2$$

Rejecting the negative solution, $x = 3$

Question 1 (d)

If $x > -2$ $\quad 3 < 4(x+2)$

$\therefore x > -\frac{5}{4}$

If $x < -2$ $\quad 3 > 4(x+2)$

$\therefore x < -\frac{5}{4}$ (true)

Solution: $x > -\frac{5}{4}$ or $x < -2$

Question 1 (e)

$u = 1 - x$ $\qquad \therefore du = -dx$

When $x = 0 \quad u = 1$ $\qquad$ When $x = 1 \quad u = 0$

$$\int_0^1 x\sqrt{1-x}\,dx = \int_1^0 (1-u)\sqrt{u}\,(-du)$$

$$= \int_0^1 u^{\frac{1}{2}} - u^{\frac{3}{2}}\,du = \left[\frac{2}{3}u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}}\right]_0^1 = \frac{4}{15}$$

Question 1 (f)

$$^5C_2\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^3$$

Question 2 (a)

$$f'(x) = \sin^2 x$$

$$= \frac{1-\cos 2x}{2}$$

$$f(x) = \frac{1}{2}\left(x - \frac{1}{2}\sin 2x\right) + C$$

$$2 = f(0) = \frac{1}{2}\left(0 - \frac{1}{2}\sin 0\right) + C$$

$$\therefore C = 2$$

$$\therefore f(x) = \frac{1}{2}x - \frac{1}{4}\sin 2x + 2$$

Question 2 (b) (i)

$$\frac{dM}{dt} = 35.5ke^{-kt}$$

$$k(36 - M) = k\left(36 - 36 + 35.5e^{-kt}\right)$$

$$= 35.5ke^{-kt}$$

$$= \frac{dM}{dt}$$

Question 2 (b) (ii)

$$20 = 36 - 35.5e^{-10k}$$

$$e^{-10k} = \frac{16}{35.5}$$

$$\therefore k = -\frac{1}{10}\ln\left(\frac{16}{35.5}\right) \doteqdot 0.080 \quad (3 \text{ dp})$$

Question 2 (b) (iii)

As $\quad t \to \infty \qquad e^{-kt} \to 0$

$\therefore M \to 36$

limiting mass $= 36$

Question 2 (c) (i)

$$P(3) = 0 \qquad \therefore 0 + 3a + b = 0$$

$$P(-1) = 8 \qquad \therefore 0 - a + b = 8$$

$$\therefore a = -2 \quad \text{and} \quad b = 6$$

Question 2 (c) (ii)

Remainder $= -2x + 6$

Question 2 (d)

$$r^2 = 6^2 + x^2$$

$$r = \sqrt{36 + x^2}$$

$$\frac{dr}{dx} = \frac{1}{2}\left(36 + x^2\right)^{-\frac{1}{2}} .2x = \frac{x}{\sqrt{36 + x^2}}$$

$$\frac{dr}{dx} = \frac{dr}{dx} \cdot \frac{dx}{dt}$$

$$= \frac{100x}{\sqrt{36 + x^2}} \quad \text{km/hr}$$

Question 3 (a) (i)

$$\frac{5!}{2!} = 60$$

Question 3 (a) (ii)

$4! = 24$

Question 3 (b) (i)

$$f'(x) = -2x.e^{-x^2}$$

$$f''(x) = (4x^2 - 2)e^{-x^2}$$

$$f''(x) = 0 \Rightarrow x = \pm\frac{1}{\sqrt{2}}$$

Question 3 (b) (ii)

On the domain as shown there are pairs of distinct x-values with the same y-value.

Question 3 (b) (iii)

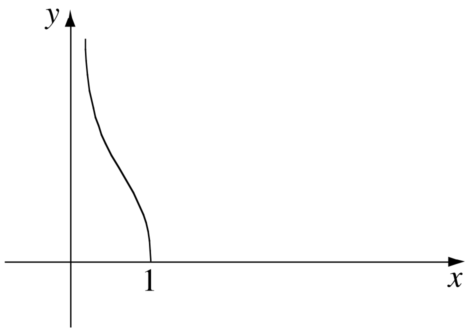
$$\begin{aligned} y &= e^{-x^2} \\ \ln y &= -x^2 \\ x^2 &= -\ln y \\ x &= \pm\sqrt{-\ln y} \quad \text{and} \quad x \ge 0 \end{aligned}$$

$$\therefore f^{-1}(x) = \sqrt{-\ln x} = \sqrt{\ln\left(\frac{1}{x}\right)}$$

Question 3 (b) (iv)

$0 < x \le 1$

Question 3 (b) (v)



Question 3 (b) (vi) (1)

Let $g(x) = x - e^{-x^2}$
$g(0.6) \doteqdot 0.6 - 0.6977 < 0$
$g(0.7) \doteqdot 0.7 - 0.6126 > 0$
$\therefore g(x)$ has a root between 0.6 and 0.7
ie $x = e^{-x^2}$ has a root between 0.6 and 0.7

Question 3 (b) (vi) (2)

$g(0.65) = 0.65 - 0.655 < 0$
Since root lies between 0.65 and 0.7
$x = 0.7$ to 1 decimal place

Question 4 (a) (i)

$$v = 0 \Rightarrow 24 - 8x - 2x^2 = 0$$
$$\therefore x^2 + 4x - 12 = 0$$
$$x = 2 \text{ or } -6$$

Question 4 (a) (ii)

$$\frac{1}{2}v^2 = 12 - 4x - x^2$$
$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -4 - 2x$$

Question 4 (a) (iii)

Maximum speed when $\ddot{x} = 0$
ie when $x = -2$
$v^2 = 32$
Maximum speed $= \sqrt{32} = 4\sqrt{2}$

Question 4 (b) (i)

$$\begin{aligned}
&2\cos\theta + 2\cos\left(\theta + \frac{\pi}{3}\right)\\
&= 2\cos\theta + 2\left[\cos\theta\cos\frac{\pi}{3} - \sin\theta\sin\frac{\pi}{3}\right]\\
&= 2\cos\theta + \cos\theta - \sqrt{3}\sin\theta\\
&= 3\cos\theta - \sqrt{3}\sin\theta
\end{aligned}$$

$$R\cos(\theta + \alpha) = R\cos\theta\cos\alpha - R\sin\theta\sin\alpha$$
$$\therefore R\cos\alpha = 3 \text{ and } R\sin\alpha = \sqrt{3}$$
$$\therefore \tan\alpha = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}}$$
$$\alpha = \frac{\pi}{6} \text{ and } R = 2\sqrt{3}$$

Question 4 (b) (ii)

$$2\sqrt{3}\cos\left(\theta + \frac{\pi}{6}\right) = 3$$

$$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\theta + \frac{\pi}{6} = \frac{\pi}{6} \quad \text{or} \quad \frac{11\pi}{6}$$

$$\therefore \theta = \frac{5\pi}{3} \text{ (only solution in given interval)}$$

Question 4 (c)

L is $\left(0, -ap^2\right)$
M is $\left(2ap, -a\right)$

$$M_{PS} = \frac{ap^2 - a}{2ap - 0} = \frac{p^2 - 1}{2p}$$

$$M_{LM} = \frac{-a - -ap^2}{2ap - 0} = \frac{p^2 - 1}{2p}$$

$\therefore PS \parallel LM$ (same gradient)
$OL \parallel PM$ (both are vertical)
$\therefore SLMP$ is a parallelogram
But $PM = PS$ (locus definition of a parallelogram)
$\therefore SLMP$ is a rhombus (adjacent sides equal)

Question 5 (a) (i)

$$\frac{PL}{PA} = \tan 20°$$

$$\therefore PA = \frac{PL}{\tan 20°} = \frac{1}{\tan 20°}$$

$$\frac{PT}{PA} = \tan 3°$$

$$\therefore PT = PA.\tan 3° = \frac{\tan 3°}{\tan 20°}$$

$$\frac{PT}{BP} = \tan 30° = \frac{1}{\sqrt{3}}$$

$$\therefore BP = \sqrt{3}.PT$$

$$= \frac{\sqrt{3}\tan 3°}{\tan 20°}$$

Question 5 (a) (ii)

$$AB = PA - BP$$

$$= \frac{1 - \sqrt{3}\tan 3°}{\tan 20°}$$

$$\doteqdot 2.498 \quad (3 \text{ dp})$$

Question 5 (b) (i)

$$f'(x) = \frac{1}{1+x^2} + \frac{1}{1+\left(\frac{1}{x}\right)^2} \times \frac{-1}{x^2}$$

$$= \frac{1}{1+x^2} - \frac{1}{x^2\left(1+\frac{1}{x^2}\right)}$$

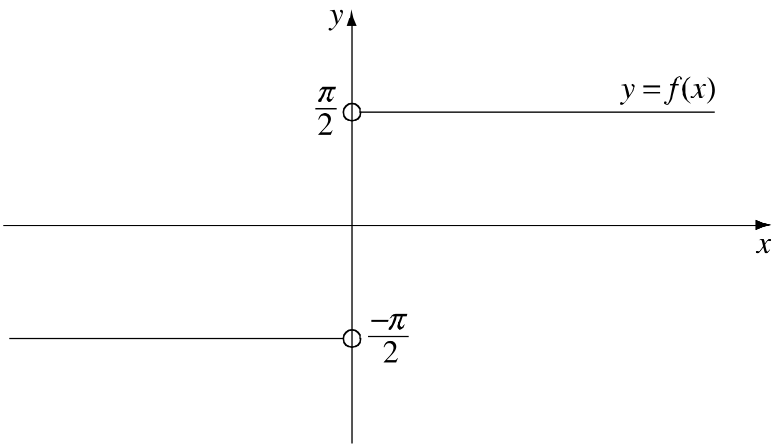
$$= \frac{1}{1+x^2} - \frac{1}{x^2+1} = 0$$

$$\therefore f'(x) = 0 \qquad \therefore f(x) = \text{constant}$$

Substitute $x = 1$ (say)

$$f(1) = \frac{\pi}{4} + \frac{\pi}{4} = \frac{\pi}{2} \quad \therefore \text{constant} = \frac{\pi}{2}$$

Question 5 (b) (ii)



Question 5 (c) (i)

$\angle AXD = \angle ABD + \angle XDB$
(exterior $\angle$ of $\Delta XBD =$ sum of opposite interior $\angle$s)

Question 5 (c) (ii)

$\angle AXD = \angle TAD$ ($\angle$ in alternate segment)
$= \angle TAC + \angle CAD$

Question 5 (c) (iii)

From (i) and (ii)
$\angle ABD = \angle XDB = \angle TAC + \angle CAD$
Now $\angle ABD = \angle TAC$ ($\angle$ in alternate segment)
$\therefore \angle XDB = \angle CAD$
Also $\angle XDB = \angle XAD$ ($\angle$ in alternate segment)
$\therefore \angle XAD = \angle CAD$

ie AD bisects $\angle BAC$

Question 6 (a) (i)

$$\begin{aligned}\cos(A-B) &= \cos A\cos B + \sin A\sin B\\ &= \cos A\cos B\left(1+\frac{\sin A\sin B}{\cos A\cos B}\right)\\ &= \cos A\cos B(1+\tan A\tan B)\end{aligned}$$

Question 6 (a) (ii)

Suppose $\tan A\tan B = -1$

Then from (i) $\cos(A-B) = 0$

$\therefore A - B = \dfrac{\pi}{2}$

Since $0 < A - B < \pi - B < \pi$

Question 6 (b) (i)

When $x = d\ \ y = h$

$$\therefore d = vt\cos\theta \text{ and } h = vt\sin\theta - 5t^2$$

$$t = \frac{d}{v\cos\theta}$$

$$\therefore h = v\left(\frac{d}{v\cos\theta}\right)\sin\theta - \frac{5d^2}{v^2\cos^2\theta}$$

$$\therefore d\tan\alpha = d\frac{\sin\theta}{\cos\theta} - \frac{5d^2}{v^2\cos^2\theta}$$

$$\therefore \frac{5d}{v^2\cos^2\theta} = \frac{\sin\theta}{\cos\theta} - \tan\alpha = \frac{\sin\theta - \tan\alpha\cos\theta}{\cos\theta}$$

$$\therefore v^2\cos^2\theta = \frac{5d\cos\theta}{\sin\theta - \tan\alpha\cos\theta}$$

$$\therefore v^2 = \frac{5d}{\cos\theta\sin\theta - \cos^2\theta\tan\alpha}$$

Question 6 (b) (ii) (1)

As $\theta \to \alpha$

$$\cos\theta\sin\theta - \cos^2\theta\tan\alpha$$

$$\to \cos\theta\sin\theta - \cos^2\theta\frac{\sin\theta}{\cos\theta} = 0$$

$$\therefore v^2 \to \infty \quad \therefore v \to \infty$$

Question 6 (b) (ii) (2)

As $\theta \to \alpha$

$$\sin\theta\cos\theta - \cos^2\theta\tan\alpha$$
$$\to 0 - 0$$
$$\therefore v \to \infty$$

Question 6 (b) (iii)

$$F(\theta) = \cos\theta\sin\theta - \cos^2\theta\tan\alpha$$
$$F'(\theta) = \cos\theta\cos\theta + \sin\theta(-\sin\theta) - 2\cos\theta(-\sin\theta)\tan\alpha$$
$$= \cos^2\theta - \sin^2\theta + 2\cos\theta\sin\theta\tan\alpha$$
$$= \cos 2\theta + \sin 2\theta\tan\alpha$$
$$= \cos 2\theta(1 + \tan 2\theta\tan\alpha)$$

If $\tan 2\theta\tan\alpha = -1$

$$\frac{\sin 2\theta}{\cos 2\theta}\tan\alpha = -1$$
$$\therefore \sin 2\theta\tan\alpha = -\cos 2\theta$$
$$\therefore F'(\theta) = 0$$

Question 6 (b) (iv)

$F'(0) = 0$ when $\tan 2\theta\tan\alpha = -1$

From (a) (ii) since $0 < \alpha < 2\theta < \pi$

$$2\theta - \alpha = \frac{\pi}{2}$$
$$\therefore \theta = \frac{\alpha}{2} + \frac{\pi}{4}$$

Question 6 (b) (v)

v^2 is a minimum when $F(\theta)$ is a maximum

$v^2 > 0 \quad \therefore F(\theta) > 0$

$F(\theta)$ has a stationary point at $\theta = \dfrac{\alpha}{2} + \dfrac{\pi}{2}$

Since $F(\theta) \to 0$ as $\theta \to \alpha$ and as $\theta \to \dfrac{\pi}{2}$ (from (b) (ii))

This point gives a maximum for $F(\theta)$.

Question 7 (a)

When $n = 1$ we have $47^1 + 53 \times 147^0 = 100$ which is divisible by 100.

Assume that the statement is true for $n = k$ then $47^k + 53 \times 147^{k-1} = 100M$

where M is an integer.

$47^{k+1} + 53 \times 147^k$

$= 47 \times 47^k + 53 \times (100 + 47) \times 147^{k-1}$

$= 47 \times 47^k + 47 \times 53 \times 147^{k-1} + 53 \times 100 \times 147^{k-1}$

$= 47\left(47^k + 53 \times 147^{k-1}\right) + 100 \times 53 \times 147^{k-1}$

$= 47 \times 100M + 100 \times 53 \times 147^{k-1}$

$= 100\left(47M + 53 \times 147^{k-1}\right)$ which is a multiple of 100

$\therefore$ result is true for all integers $n \geq 1$ by induction.

Question 7 (b) (i)

Substitute $x = 1$. Result follows

Question 7 (b) (ii)

Use the result from (i) with $n = 100$

Answer $= 2^{100}$

Question 7 (b) (iii)

Differentiate both sides of (i)

$$n(1+x)^{n-1} = \binom{n}{1} + 2x\binom{n}{2} + 3x^2\binom{n}{3} + \ldots + nx^{n-1}\binom{n}{n}$$

Substitute $x = 1$

$$n2^{n-1} = \binom{n}{1} + 2\binom{n}{2} + \ldots + n\binom{n}{n} = \sum_{k=1}^{n} k\binom{n}{k}$$

Question 7 (c) (i)

The number of red balls could be 0, 1, 2, …, r.
The remaining balls will be blue
$\therefore$ The number of colour combinations is $r + 1$.

Question 7 (c) (ii)

The number of possible selections of $n - r$ white balls is $\binom{n}{n-r} = \binom{n}{r}$.

Question 7 (c) (iii)

The selection could contain r from the red and blue balls and $n - r$ from the labelled white ones where $r = 0, 1, \ldots, n$
For a particular r,

$$\begin{aligned} \text{Number of selections} &= (r+1)\binom{n}{r} \\ &= r\binom{n}{r} + \binom{n}{r} \end{aligned}$$

$$\begin{aligned} \therefore \text{ Total number of selections} &= \sum_{r=0}^{n} r\binom{n}{r} + \sum_{r=0}^{n}\binom{n}{r} \\ &= n2^{n-1} + 2^n \text{ from (b) (i) and (ii)} \\ &= (n+2)2^{n-1} \end{aligned}$$