BOARD OF STUDIES
NEW SOUTH WALES

2011 Mathematics Extension 1 HSC Examination 'Sample Answers'

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This material is also provided to the Supervisor of Marking, to give some guidance about the nature and scope of the responses the committee expected students would produce. How sample answers are used at marking centres varies. Sample answers may be used extensively and even modified at the marking centre OR they may be considered only briefly at the beginning of marking. In a few cases, the sample answers may not be used at all at marking.

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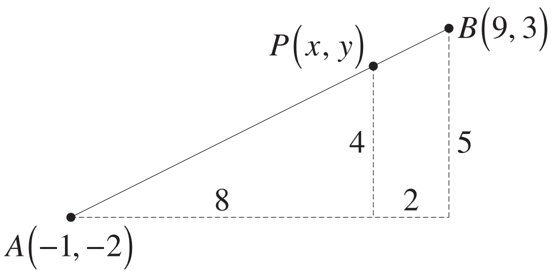
The 'sample answers' or similar advice contained in this document are not intended to be exemplary or even complete answers or responses. As they are part of the examination committee's 'working document', they may contain typographical errors, omissions, or only some of the possible correct answers.

Question 1 (a)



Using similarity, point P has coordinates $P(7, 2)$.

Question 1 (b)

$$y = \frac{\sin^2 x}{x}$$

$$y' = \frac{u'v - uv'}{v^2}$$

Let $u = \sin^2 x$
Let $v = x$
$u' = 2\sin x \cos x$
$v' = 1$

$$= \frac{2x\sin x\cos x - 1 \cdot \sin^2 x}{x^2}$$

$$= \frac{\sin x(2x\cos x - \sin x)}{x^2}$$

Question 1 (c)

$$\frac{4-x}{x} < 1$$

$$(4-x)x < x^2$$

$$4x - x^2 < x^2$$

$$4x < 2x^2$$

$$2x^2 - 4x > 0$$

$$2x(x-2) > 0$$

$\Rightarrow$ $x < 0$ OR $x > 2$

Question 1 (d)

$$2\int_1^4 \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx = 2\int_1^2 e^u \cdot du$$

$$= \left[2e^u\right]_1^2$$

$$= 2e^2 - 2e$$

$$= 2e\left(e-1\right)$$

Let $u = x^{\frac{1}{2}}$

$$du = \frac{1}{2}x^{-\frac{1}{2}}\,dx$$

$$= \frac{dx}{2\sqrt{x}}$$

When $x = 1,\ u = 1$

$x = 4,\ u = 2$

Question 1 (e)

$$\cos^{-1}\left(-\frac{1}{2}\right) = \pi - \cos^{-1}\left(\frac{1}{2}\right)$$

$$= \pi - \frac{\pi}{3}$$

$$= \frac{2\pi}{3}$$

Question 1 (f)

$f\left(x\right) = \ln\left(x^2 + e\right)$

Minimum value of $x^2 + e$ is e, so:

$f\left(x\right) \ge \ln\left(e\right) = 1$

Range is $f\left(x\right) \ge 1$

Question 2 (a)

$$\begin{aligned} P(x) &= x^3 - ax^2 + x \\ P(3) &= 27 - 9a + 3 = 30 - 9a = 12 \quad \text{(remainder theorem)} \\ 9a &= 18 \Rightarrow a = 2 \\ \text{So } P(-1) &= -1 - 2 - 1 \\ &= -4 \end{aligned}$$

$\therefore$ the remainder when $P(x)$ is divided by $x + 1$ is -4.

Question 2 (b)

$f(x) = \cos(2x) - x$; $x_0 = \dfrac{1}{2}$

Newton's method: $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$

Here, $f'(x) = -2\sin(2x) - 1$

$$x_1 = \frac{1}{2} - \frac{f\left(\frac{1}{2}\right)}{f'\left(\frac{1}{2}\right)}$$

$$= \frac{1}{2} - \frac{\cos(1) - \frac{1}{2}}{-2\sin(1) - 1}$$

$x_1 \approx 0.52$

Question 2 (c)

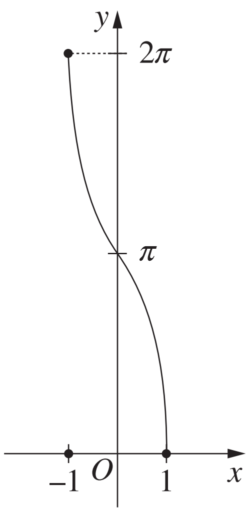
$$\left(3x - \frac{4}{x}\right)^8 = \sum_{k=0}^{8} \binom{8}{k} \left(3x\right)^k \left(\frac{-4}{x}\right)^{8-k}$$

General term is: $\binom{8}{k} 3^k x^k \left(-4\right)^{8-k} x^{k-8}$

Need $x^k \cdot x^{k-8} = x^2 \Rightarrow 2k - 8 = 2$, so k = 5

$\therefore$ coefficient is $\binom{8}{5} 3^5 \cdot \left(-4\right)^3 = -870\ 912$

Question 2 (d)



Domain: $-1 \le x \le 1$

Range: $0 \le y \le 2\pi$

Question 2 (e) (i)

40!

Question 2 (e) (ii)

$3! \times 37!$

Question 3 (a) (i)

$$x = A\cos nt + B\sin nt$$
$$\dot{x} = -An\sin nt + Bn\cos nt$$
$$\ddot{x} = -An^2\cos nt - Bn^2\sin nt$$
$$-n^2x = -n^2\left(A\cos nt + B\sin nt\right)$$
$$= -An^2\cos nt - Bn^2\sin nt = \ddot{x}, \text{ as required}$$

Question 3 (a) (ii)

$$x\left(0\right) = A = 0$$
$$\dot{x}\left(0\right) = Bn = 2n$$
$\therefore$ A = 0 and B = 2.

Question 3 (a) (iii)

$$x\left(t\right) = 2\sin nt$$

When $x\left(t\right) = 2$, $\sin nt = 1$

$$\therefore nt = \frac{\pi}{2}$$

$$\Rightarrow \qquad t = \frac{\pi}{2n}$$

Question 3 (a) (iv)

$t = \dfrac{2\pi}{n}$ represents one full period. The amplitude is 2.

$\therefore$ distance travelled is $4 \times 2 = 8$.

Question 3 (b) (i)

$$y = x^2$$
$$\frac{dy}{dx} = 2x$$

At $P\left(t, t^2\right)$, $\dfrac{dy}{dx} = 2t$

$$\therefore y - t^2 = 2t\left(x - t\right)$$
$$y = 2tx - t^2$$

Question 3 (b) (ii)

At Q, $y = 2\left(1-t\right)x - \left(1-t\right)^2$ is the equation of the tangent.

Question 3 (b) (iii)

Have $\quad 2tx - t^2 = 2\left(1-t\right)x - \left(1-t\right)^2 \quad$ (from (i) and (ii))

$$\Rightarrow x\left(2t - 2\left(1-t\right)\right) = t^2 - \left(1-t\right)^2 = 2t - 1$$
$$x\left(4t - 2\right) = 2t - 1$$
$$x = \frac{2t-1}{4t-2} = \frac{1}{2}$$

When $x = \dfrac{1}{2}$, $y = 2t\left(\dfrac{1}{2}\right) - t^2 = t - t^2$

$\therefore$ R has coordinates $R\left(\dfrac{1}{2},\ t - t^2\right)$

Question 3 (b) (iv)

Note that $x = \dfrac{1}{2}$ is constant.

$y = t - t^2 = t\left(1-t\right)$ has a maximum when $t = \dfrac{1}{2}$.

$\therefore$ locus of R is the vertical line $x = \dfrac{1}{2}$, for $y < \dfrac{1}{4}$.

Question 4 (a) (i)

$$f(x) = e^{-x} - 2e^{-2x}$$

$$f'(x) = -e^{-x} + 4e^{-2x}$$

Question 4 (a) (ii)

$$\begin{aligned} f'(x) = 0 \quad &\Leftrightarrow \quad e^{-x} = 4e^{-2x} \\ &\Leftrightarrow \quad 1 = 4e^{-x} \\ &\Leftrightarrow \quad e^{-x} = \frac{1}{4} \\ &\Leftrightarrow \quad -x = \ln\left(4^{-1}\right) \\ &\Leftrightarrow \quad x = \ln 4 = 2\ln 2 \end{aligned}$$

$\therefore$ coordinates are $x = \ln 4$,

$$y = f(\ln 4) = \frac{1}{4} - 2 \cdot e^{-2\ln 4} = \frac{1}{4} - 2 \cdot \frac{1}{16} = \frac{1}{8}$$

Question 4 (a) (iii)

$$\begin{aligned} f(\ln 2) &= e^{-\ln 2} - 2e^{-2\ln 2} \\ &= \frac{1}{2} - 2 \cdot \frac{1}{4} = 0 \end{aligned}$$

Question 4 (a) (iv)

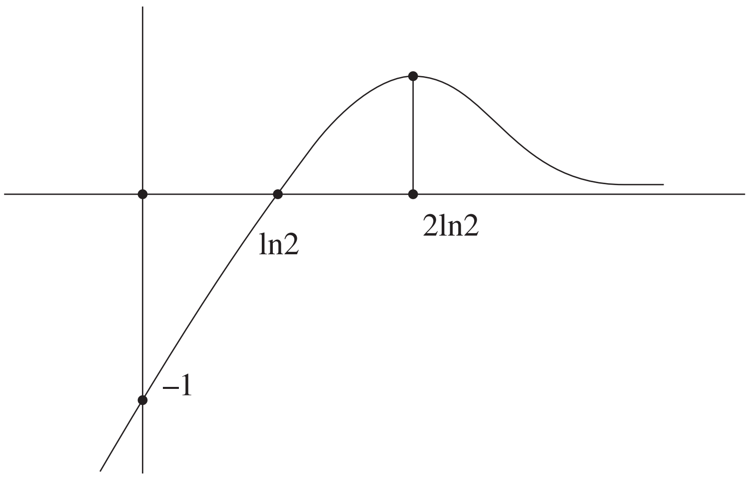
As $x \to \infty,\ f(x) \to 0$

Question 4 (a) (v)

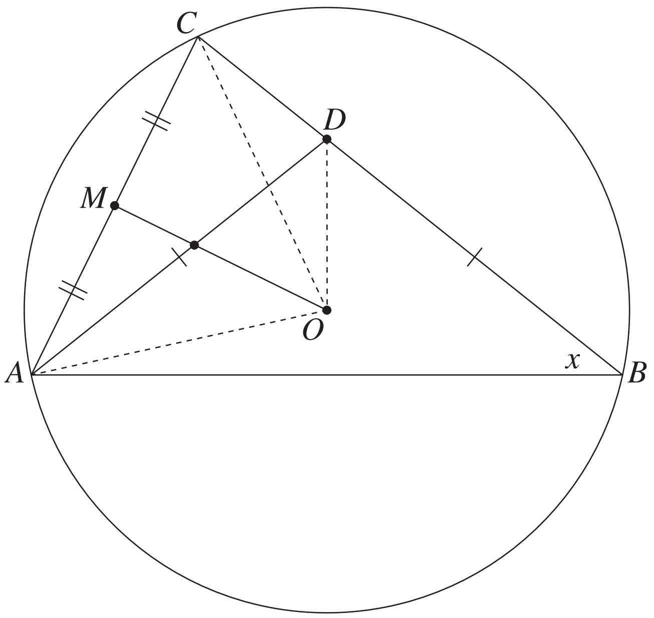
y-intercept is when $x = 0$ ie $y = f(0)$

$= -1$

Question 4 (a) (vi)



Question 4 (b)



Question 4 (b) (i)

The angle at the centre is twice the angle at the circumference subtended by the same arc.

Question 4 (b) (ii)

As $\triangle ADB$ is isosceles,

$\angle DAB = x$, so

$\angle ADB = 180 - 2x$

So $\angle CDA = 2x$

So $\angle CDA = 2x = \angle COA$. So D and O are on a circle with chord AC.

$\therefore$ A, C, D and O are on a circle and $ACDO$ is a cyclic quadrilateral.

Question 4 (b) (iii)

AC is the common chord of the intersection of circles ABC and $ACDO$.
Therefore the mid-point is collinear.

Question 5 (a) (i)

$$\frac{NT}{TQ} = \frac{NS}{SP} \qquad \text{(by similarity)}$$

$$\Rightarrow \frac{1-\sin\theta}{\cos\theta} = \frac{2}{SP} \qquad (\text{since } NT = 1-\sin\theta,\ TQ = \cos\theta)$$

$$SP = \frac{2\cos\theta}{1-\sin\theta}$$

Question 5 (a) (ii)

$$\frac{\cos\theta}{1-\sin\theta} \times \frac{1+\sin\theta}{1+\sin\theta} = \frac{\cos\theta(1+\sin\theta)}{1-\sin^2\theta}$$

$$= \frac{\cos\theta(1+\sin\theta)}{\cos^2\theta}$$

$$= \frac{1+\sin\theta}{\cos\theta}$$

$$= \sec\theta + \tan\theta$$

Question 5 (a) (iii)

$\triangle QNO$ is isosceles (since $ON = OQ$)

$$\angle NOQ = \frac{\pi}{2} - \theta, \text{ so}$$

$$\angle SNP = \frac{1}{2}\left(\pi - \left(\frac{\pi}{2} - \theta\right)\right) = \frac{\theta}{2} + \frac{\pi}{4}$$

Question 5 (a) (iv)

$$\tan(\angle SNP) \quad = \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \frac{SP}{2}$$

$$\therefore \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \sec\theta + \tan\theta \qquad \text{by (i), (ii), and (iii)}$$

Question 5 (a) (v)

$$\sec\theta + \tan\theta = \sqrt{3}$$

$$\Rightarrow \tan\left(\frac{\theta}{2} + \frac{\pi}{4}\right) = \sqrt{3} \qquad \text{by (iv)}$$

$$\frac{\theta}{2} + \frac{\pi}{4} = \frac{\pi}{3}, \frac{4\pi}{3}, \cdots$$

$$\frac{\theta}{2} = \frac{\pi}{12}, \frac{13\pi}{12}, \cdots$$

$$\theta = \frac{\pi}{6}, \frac{13\pi}{6}, \cdots$$

Required solution is $\theta = \dfrac{\pi}{6}$

Question 5 (b) (i)

$$T = 5 + 25e^{-kt}$$

$$T(1) = 5 + 25e^{-k} = 20$$

$$e^{-k} = \frac{15}{25} = \frac{3}{5}$$

$$-k = \ln\left(\frac{3}{5}\right) \Rightarrow k = \ln\left(\frac{5}{3}\right)$$

Question 5 (b) (ii)

Model is: $T(t) = 22 + 15e^{-kt}$

Need $22 + 15e^{-kt} = 30$

$$e^{-kt} = \frac{8}{15}$$

$$kt = \ln\left(\frac{15}{8}\right)$$

$$t = \ln\left(\frac{15}{8}\right) \Big/ \ln\left(\frac{5}{3}\right)$$

$t \approx 1.231$ hours $\approx$ 1 hour and 14 minutes.

$\therefore$ time when the object had a temperature of 37°C was 8:46 am.

Question 6 (a)

Let $P(n)$ be the proposition that:

$$1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + n(n+4) = \frac{1}{6}n(n+1)(2n+13)$$

RTP: $P(1)$ true

Proof: $LHS = 1 \times 5 = 5$ $\qquad RHS = \frac{1}{6}(1)(1+1)(2(1)+13) = 5$

$\therefore P(1)$ true.

Assume $P(k)$ true, that is

$$1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + k(k+4) = \frac{1}{6}k(k+1)(2k+13)$$

RTP: $P(k+1)$ true, that is

$$1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + k(k+4) + (k+1)(k+5) = \frac{1}{6}(k+1)(k+2)(2k+15)$$

Proof: Consider the *LHS* of $P(k+1)$:

$$1 \times 5 + 2 \times 6 + 3 \times 7 + \cdots + k(k+4) + (k+1)(k+5)$$

$$= \frac{1}{6}k(k+1)(2k+13) + (k+1)(k+5) \qquad \text{using } P(k)$$

$$= \frac{1}{6}k(k+1)(2k+13) + \frac{6(k+1)(k+5)}{6}$$

$$= \frac{(k+1)}{6}\left(k(2k+13) + 6(k+5)\right)$$

$$= \frac{(k+1)}{6}\left(2k^2 + 13k + 6k + 30\right)$$

$$= \frac{(k+1)}{6}\left(2k^2 + 19k + 30\right)$$

$$= \frac{(k+1)}{6}(2k+15)(k+2)$$

$$= RHS \text{ of } P(k+1)$$

$\therefore$ $P(n)$ is true for all integers $n \geq 1$ by mathematical induction.

Question 6 (b) (i)

The ball strikes the ground when $y = 0$, so

$$0 = h - \frac{1}{2}gt^2$$

$$\therefore\ t^2 = \frac{2h}{g}$$

As t must be positive when the ball strikes the ground,

$$t = \sqrt{\frac{2h}{g}}\ .$$

Question 6 (b) (ii)

When the ball strikes the ground $x = d$ and so $vt = d$.

At this time $\dfrac{dx}{dt} = -\dfrac{dy}{dt}$

Therefore $\quad v = -(-gt)$

And so $\quad v = gt$

$$\begin{aligned} \therefore \quad d &= vt \\ &= gt^2 \\ &= g \times \frac{2h}{g} \\ &= 2h \end{aligned}$$

Question 6 (c) (i)

Probability is

$1-(1-p)^2$

$= 1-1+2p-p^2$

$= 2p-p^2$

Question 6 (c) (ii)

Probability is

$1-(1-p)^3-3(1-p)^2\,p$

$= \cancel{1}-\cancel{1}+\cancel{3p}-3p^2+p^3-\cancel{3p}+6p^2-3p^3$

$= 3p^2-2p^3$

Question 6 (c) (iii)

Consider

$(2p-p^2)-(3p^2-2p^3)$

$= p(2-p-3p+2p^2)$

$= p(2-4p+2p^2)$

$= 2p(p-1)^2$

> 0, as $0 < p < 1$.

Question 6 (c) (iv)

Value of p is solution of

$$2p - p^2 = 2\left(3p^2 - 2p^3\right)$$

$\Leftrightarrow \quad 2 - p = 6p - 4p^2 \qquad$ as $p > 0$

$\Leftrightarrow \quad 4p^2 - 7p + 2 = 0$

$\Leftrightarrow \quad p = \dfrac{7 \pm \sqrt{49 - 32}}{8}$

$\Leftrightarrow \quad p = \dfrac{7 \pm \sqrt{17}}{8} \qquad$ as $0 < p < 1$

$\therefore \quad p = \dfrac{7 - \sqrt{17}}{8}$

Question 7 (a) (i)

The radius of the cones is h cm. The volume of water in the lower cone at time t is:

$$V = \frac{\pi h^2 \cdot h}{3} - \frac{\pi \ell^2 \cdot \ell}{3}$$

$$= \frac{\pi}{3}\left(h^3 - \ell^3\right)$$

Question 7 (a) (ii)

We are given $\frac{d\ell}{dt}$ and can calculate $\frac{dV}{d\ell}$, so

$$\frac{dV}{dt} = \frac{dV}{d\ell} \times \frac{d\ell}{dt}$$

$$= -\pi\ell^2 \times 10$$

$$= -10\pi\ell^2$$

When $\ell = 2$ cm this is -40π cm^3 s^{-1}.

Question 7 (a) (iii)

The cone has lost $\frac{1}{8}$ of the water when

$$\frac{\pi\ell^3}{3} = \frac{1}{8}\frac{\pi h^3}{3},$$

that is, when $h = 2\ell$.

The rate of change of volume is $\frac{dV}{dt} = -10\pi\ell^2 = -\frac{10}{4}\pi h^2 = -\frac{5}{2}\pi h^2$ cm^3 s^{-1}.

Question 7 (b) (i)

Differentiate $\left(1+x\right)^n = \sum_{r=0}^{n} \binom{n}{r} x^r$ with respect to x, then multiply by x:

$$nx\left(1+x\right)^{n-1} = \sum_{r=1}^{n} \binom{n}{r} r x^r \ .$$

(Note that the constant term is zero.)

Question 7 (b) (ii)

Differentiate the identity in part (i) with respect to x:

$$n\left(1+x\right)^{n-1} + n\left(n-1\right)x\left(1+x\right)^{n-2} = \sum_{r=1}^{n} \binom{n}{r} r^2 x^{r-1} \qquad (*)$$

Substitute $x = 1$:

$$n2^{n-1} + n\left(n-1\right)2^{n-2} = \sum_{r=1}^{n} \binom{n}{r} r^2$$

which is:

$$n\left(n+1\right)2^{n-2} = \sum_{r=1}^{n} \binom{n}{r} r^2 \qquad (**)$$

Question 7 (b) (iii)

In equation (*), part (ii), substitute $x = -1$:

$$0 = \sum_{r=1}^{n} \binom{n}{r} r^2 (-1)^{r-1}.$$

Rearranging gives

$$\binom{n}{2}2^2 + \binom{n}{4}4^2 + \cdots + \binom{n}{n}n^2 = \binom{n}{1}1^2 + \binom{n}{3}3^2 + \cdots + \binom{n}{n-1}(n-1)^2.$$

The sum of the *LHS* and *RHS* of this equation is $\sum_{r=1}^{n} \binom{n}{r} r^2$, ie the *RHS* of (**).

Dividing (**) by 2 one obtains

$$n(n+1)2^{n-3} = \binom{n}{2}2^2 + \binom{n}{4}4^2 + \cdots + \binom{n}{n}n^2.$$