# 2012 HSC Notes from the Marking Centre – Mathematics Extension 2

## Contents



### Introduction

This document has been produced for the teachers and candidates of the Stage 6 Mathematics Extension 2 course. It contains comments on candidate responses to the 2012 Higher School Certificate examination, indicating the quality of the responses and highlighting their relative strengths and weaknesses.

This document should be read along with the relevant syllabus, the 2012 Higher School Certificate examination, the marking guidelines and other support documents developed by the Board of Studies to assist in the teaching and learning of Mathematics Extension 2.

### General comments

The instructions indicate that relevant mathematical reasoning and/or calculations should be included in the responses to Questions 11 to 16. Candidates are reminded that where a question is worth several marks, full marks may not be awarded for an answer, even if the answer given is correct, if no working is shown.

This is because mathematical communication and reasoning are included in the objectives and outcomes assessed by the examination.

Candidates are advised to show all their working so that marks can be awarded for some correct steps towards their answer. A simple example is when candidates have to round their answer to a certain degree of accuracy. Candidates should always write their calculator display before rounding their answer. They should only round their answer in the last step of working, not in an earlier step. Markers can then see that candidates have rounded correctly, even if the answer is not correct.

### Question 11

(a) Most candidates obtained the correct solution by multiplying the numerator and denominator by the conjugate $\sqrt{5}+i$.

(b) In most responses, candidates realised that they were required to draw two circles. In weaker responses, common errors included incorrect centres of one or both circles, or incorrectly indicating the required region as the intersection of the two circles.

(c) Most candidates completed the square and obtained the correct solution.

(d) (i) In most responses, candidates found the modulus and argument of $z$. However, incorrect responses for the argument included $-\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{6}$ and $\frac{5\pi}{6}$, as candidates did not locate the complex number appropriately on the Argand diagram.

(ii) In most responses, candidates converted their answer from d (i) correctly into $x+iy$ form by the application of De Moivre's theorem.

(e) In better responses, candidates recognised that the primitive involved logarithms. In weaker responses, candidates made errors in the substitution of the limits of integration. Some candidates used the substitution method to answer this question.

(f) (i) In better responses, candidates sketched graphs, and marked all axes, intercepts or other significant features. In weaker responses, candidates shifted the curve $y=|x|$ in the wrong direction.

(ii) Candidates attempting this part using part (i) and multiplication of ordinates had various outcomes. In some better responses, candidates used the property of an odd function to assist their sketching.

**Question 12**

(a) This part was generally done very well with the method of *t*-substitution. In most responses, candidates quoted or deduced the relevant *t* expressions of $\cos\theta = \frac{1-t^2}{1+t^2}$ and $\frac{d\theta}{dt} = \frac{2}{1+t^2}$. The most common error was leaving the final answer in terms of $t$ rather than $\theta$. In some responses, candidates wrote $\tan^{-1}\frac{\theta}{2}$ instead of $\left(\tan\frac{\theta}{2}\right)^{-1}$ as their final answer, and these are not the same functions. In a small number of responses, candidates successfully found the primitive by using a substitution of $\cos\theta = 1-2\sin^2\frac{\theta}{2}$ or expressed as $\frac{1-\cos\theta}{\sin^2\theta}$.

(b) (i) Most candidates successfully derived the equation of the tangent using implicit differentiation.

(ii) In most responses, candidates correctly derived the equation of the normal. A large number of candidates then substituted $y=0$ correctly to obtain $-y_0 = \frac{a^2 y_0}{b^2 x_0}(x-x_0)$. However, they did not cancel $y_0$ and so had to deal with a more complicated process. In some weaker responses, candidates refrained from substituting $y=0$ until the end, making the process more arduous.

(iii) Candidates who attempted this part recognised the need to use part (i), in conjunction with $\frac{x_0^2}{a^2} + \frac{y_0^2}{b^2} = 1$ or equivalent expression. In better responses, candidates started with $\frac{x_0 x}{a^2}+\frac{y_0 y}{b^2}=1$ for the equation of the tangent at $P(x_0, y_0)$ to find point $T\left(\frac{a^2}{x_0}, 0\right)$.

(c) In the majority of responses, candidates recognised that integration by parts was required and many obtained $I_n = e^2 2^n - nI_{n-1}$. One common error involved $\frac{d}{dx}((\ln x)^n) = n(\ln x)^{n-1}$ rather than $n(\ln x)^{n-1} \times \frac{1}{x}$. In a small number of weaker responses, candidates unsuccessfully attempted mathematical induction.

(d) (i) In a significant number of responses, candidates did not use the correct terminology for the transformation required to move $A_1P$ to $A_1B_1$. In some weaker responses, candidates did what appeared to be a back-tracking process by stating $w_1 - u_1 = i(z-u_1) \Rightarrow w_1 = u_1 + i(z-u_1)$ without any other explanation. Even stating $A_1B_1 = A_1P \times i$ demonstrated use of the data given. In better responses, candidates made reference to the length of $A_1B_1 = A_1P$, $\angle B_1A_1P = 90^\circ$ and multiplying by $i$ constitutes an anticlockwise rotation of 90°.

(ii) The majority of candidates encountered difficulty in finding the correct value. Instead of $w_2 = u_2 - i(z-u_2)$ or equivalent expression, many stated $w_2 = u_2 + i(z-u_2)$ by not realising that the rotation is clockwise instead of anticlockwise. In a significant number of responses, candidates used $\frac{w_1 - w_2}{2}$ to find the midpoint, rather than $\frac{w_1 + w_2}{2}$. Few candidates realised that eliminating $z$ meant that the locus is a fixed point.

**Question 13**

(a) (i) In better responses, after finding the expression $\frac{dt}{dv} = \frac{40}{400 - v^2}$, candidates used partial fractions and correct integration to obtain $t = \ln\left[\frac{20+v}{20-v}\right]$. In better responses, candidates demonstrated their competency by changing the subject to obtain the required expression for $v$. In weaker responses, some candidates attempted to use the result for $\int \frac{dx}{a^2 - x^2}$ without resorting to partial fractions. In many of these responses, candidates did not obtain the correct result of this integration.

(ii) Having obtained $\frac{dx}{dv} = \frac{40v}{400 - v^2}$, the majority of candidates handled this part well and obtained the required result.

(iii) This part was again well done. The most successful approach was to substitute $t = 4$ into $v = \frac{20(e^t - 1)}{e^t + 1}$ and then substitute this expression into $x = 20\ln\left\{\frac{400}{400 - v^2}\right\}$.

Another approach was to integrate using $\frac{dx}{dt} = \frac{20\left(e^t - 1\right)}{e^t + 1}$. Only a small number of candidates used this approach, and of them very few successfully obtained the required result.

(b) (i) In most responses, candidates correctly proved the required result by using one of the following approaches:

- corresponding angles and alternate angles in parallel lines
- the external angle being equal to the sum of the two remote interior angles along with a pair of either corresponding or alternate angles in parallel lines
- the angle sum of a straight line and the angle sum of a triangle along with a pair of either corresponding or alternate angles in parallel lines.

(ii) In better responses, candidates used one of the following approaches:

- the intercept theorem on parallel lines
- the sine rule approach
- similar triangles $S'RS$ and $S'PQ$ and the fact that $\frac{S'R}{S'P} = \frac{S'S}{S'Q} \Rightarrow \frac{S'P + PR}{S'P} = \frac{S'Q + QS}{S'Q}$.

(c) (i) In better responses, candidates used the focus-directrix definition to obtain the required result. Candidates who used the distance formula had less success because this approach required them to use the trigonometric identity $1 + \tan^2\theta = \sec^2\theta$, as well as $b^2 = a^2\left(e^2 - 1\right)$ or variations of these results. In other weaker responses, candidates made errors in manipulating the expressions involved in the use of the distance formula.

(ii) In better responses, candidates solved the equation $\frac{PS}{QS} = \frac{PS'}{QS'} \Rightarrow \frac{a\left(e\sec\theta - 1\right)}{ae - x} = \frac{a\left(e\sec\theta + 1\right)}{ae + x}$ where $x = OQ$. In some other better responses, candidates substituted $ae - \frac{a}{\sec\theta}$ and $ae + \frac{a}{\sec\theta}$ for $QS$ and $QS'$ respectively. In a few responses, candidates used the ratio formula to divide $S'S$ internally at $Q$ in the ratio $e\ \sec\theta + 1 : e\ \sec\theta - 1$.

(iii) In better responses, candidates showed that $m_{PQ} = \frac{b\sec\theta}{a\tan\theta}$ was the gradient of the tangent at $P$. In other better responses, candidates found the equation of the tangent at $P$ and showed that $Q\left(\frac{a}{\sec\theta}, 0\right)$ satisfied the equation. In many responses, candidates found the equation of the tangent at $P$ without showing the required result.

**Question 14**

(a) In most responses, candidates who used partial fractions so that $3x^2 + 8 = a(x^2 + 4) + (bx + c)x$ tended to attain the answer most easily. The quickest solution was to rearrange the integrand $\int \frac{3x^2+8}{x(x^2+4)}dx = \int \frac{2x^2+8}{x(x^2+4)}dx + \int \frac{x^2}{x(x^2+4)}dx$.

A significant number of candidates found the correct values of $a$, $b$ and $c$. However, when substituting $b = \pm 1$ they obtained the primitive $\pm\frac{1}{2}\tan^{-1}(x^2+4)$. Candidates should include parentheses in writing expressions such as $\log_e(x^2+4)$.

(b) (i) In most responses, candidates showed some knowledge of either the shape of the curve or the vertical asymptotes. In weaker responses, many candidates did not draw an accurate diagram with the scale clearly marked or show the vertical asymptotes $x = 0$ and $x = \frac{3}{2}$.

(ii) Most candidates who made some progress found values for $a$, $b$ and $m$ using the division algorithm. In many responses, candidates tried to equate $\frac{x(2x-3)}{x-1}$ and $mx + b + \frac{a}{x-1}$ but took much longer or had less success. Having found values for $a$, $b$ and $m$, many candidates did not find equation $l$. Candidates should write the line as $y = 2x - 1$ not $l = 2x - 1$.

(c) Most candidates had difficulty in starting this part correctly. In most responses, candidates drew a diagram but did not recognise the need to use similar triangles to find the width of the rectangle. In many responses, candidates recognised the height of the triangle but did not go further. In most responses, candidates who found the correct width and height of the rectangle were able to correctly integrate to find the correct solution.

(d) (i) In many responses, candidates did not use the theorem that the angle between a chord and the tangent at the point of contact is equal to the angle in the alternate segment. Candidates need to be careful when abbreviating their reasoning and to copy the diagram from the question paper when planning their response. Candidates should label angles carefully and write legibly so that letters used in naming angles can be distinguished.

(ii) Many candidates recognised that $\frac{AP}{BP} = \frac{GP}{EP}$ from the similar triangles in the first part. In better responses, candidates also proved that triangles $EAP$ and $FBP$ are similar and that $\frac{AP}{BP} = \frac{EP}{FP}$.

### Question 15

(a) Candidates used a variety of successful methods to answer the question.

(i) This part was well done by most candidates. In weaker responses, candidates assumed the result before proceeding.

(ii) The easiest way to prove this result is to show that $LHS - RHS = (x-1)(y-x) \geq 0$, since both factors are positive. In many weaker responses, candidates assumed the result before proceeding and then often did not account for the direction of the inequality sign when multiplying or dividing by factors such as $(x-1)$ or $(x-y)$.

(iii) In quite a few responses, candidates noted that $j(n-j+1) \geq n$ was a consequence of the result in (ii), but fewer noted that $\sqrt{j(n-j+1)} \leq \frac{n+1}{2}$ was a consequence of the result in (i). In many weaker responses, candidates did not realise that the given inequality was a consequence of the two previous parts, or tried to prove the result without breaking up the inequality into two separate ones.

(iv) Only very few candidates observed that the given result was the product of $n$ inequalities from (iii), with $j = 1, 2, 3, \ldots n$. In a number of responses, candidates attempted to prove the result by induction, which wasted considerable time for small reward.

(b) (i) In a substantial number of responses, candidates did not note that the coefficients of the polynomial must be real in order that the conjugate of any zero is also a zero.

(ii) In the majority of responses, candidates chose to factorise rather than to employ the simpler option of expanding.

(iii) In the majority of responses, candidates did not note that since $z^2(z-k)^2 + (kz-1)^2 = 0$ and $z$ is real, then each term must be zero. In many responses, candidates assumed that since the monic polynomial had a constant term of 1, then the only possible real roots were $z = \pm 1$. This is not always true. For example, $(z^2 - z - 1)(z^2 - 3z - 1) = 0$ is monic with a constant term of 1 with four real roots, all of which are irrational.

(iv) In a number of responses, candidates used the fact that the product of the roots is 1 and that the four zeros had the same modulus to establish the given result. In many weaker responses, candidates wrote that $\alpha\overline{\alpha} = |\alpha|$ instead of $\alpha\overline{\alpha} = |\alpha|^2$ and some did not realise that $|\alpha| = |\overline{\alpha}|$.

(v) In the majority of responses, candidates considered the sum of the zeros. In some weaker responses, candidates thought that the conjugate of $ix - y$ was $ix + y$.

(vi) In better responses, candidates used a variety of creative approaches.

## Question 16

(a) (i) The correct response for this part was $\frac{(m+n)!}{m!n!}$ or ${}^{m+n}C_m$ or ${}^{m+n}C_n$. In weaker responses, candidates commonly included responses such as $(m+n)!$, $m!n!$ or $2m!n!$.

(ii) In better responses, candidates recognised the problem to be equivalent to arranging 10 identical coins and 3 identical separators, giving ${}^{13}C_3$ different allocations from part (i). The most common errors included ${}^{14}C_4$ or $4^{10}$.

(b) (i) In better responses, candidates first considered the tangent of either side of the identity before reaching a conclusion.

(ii) In the majority of responses, the check that the statement is true for the initial case $n = 1$ was done well. However, in most responses, candidates stated the inductive step but did not provide the working for a convincing proof.

(iii) In weaker responses, the most common error was $\tan^{-1}(0)$, obtained by mistakenly evaluating the limit of $\tan^{-1}\left(\frac{1}{2n^2}\right)$ as $n \to \infty$.

(c) (i) In better responses, candidates considered the number of ways of selecting $k$ distinct integers from the set $\{1,2,\ldots,n\}$, followed immediately by a selection of one of these $k$ integers. Dividing this number by $n^{k+1}$ gives $P(k)=\frac{{}^nP_k \times k}{n^{k+1}}$, which can then be simplified to $\frac{(n-1)!k}{n^k(n-k)!}$. In other better responses, candidates used an equivalent successive outcome approach by multiplying probabilities. In weaker responses, candidates did not explain their derivation of the result with clear, unambiguous reasoning.

(ii) In better responses, candidates gave the correct expression for $P(k-1)$ in terms of factorials before showing all working to derive the desired quadratic inequality $k^2-k-n\le 0$. In weaker responses, candidates made errors in determining $P(k-1)$ from the given expression for $P(k)$, by not correctly replacing $k$ with $k-1$.

(iii) In the best responses, candidates considered the role that the integer properties of $n$ and $k$ played in the proof of the result.

(iv) In better responses, candidates observed that the required integer $k$ is the largest integer for which $P(k)\ge P(k-1)$. In weaker responses, candidates did not refer to the stipulation of $4n+1$ or make use of the result in part (iii).