BOARD OF STUDIES
NEW SOUTH WALES

2012
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2–8

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 9–19

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Let $z = 5 - i$ and $w = 2 + 3i$.

What is the value of $2z + \overline{w}$?

(A) $12 + i$

(B) $12 + 2i$

(C) $12 - 4i$

(D) $12 - 5i$

2 The equation $x^3 - y^3 + 3xy + 1 = 0$ defines y implicitly as a function of x.

What is the value of $\frac{dy}{dx}$ at the point $(1, 2)$?

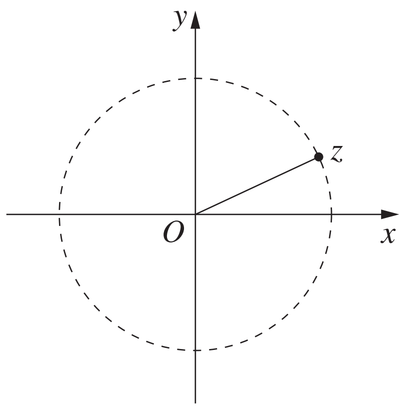
(A) $\frac{1}{3}$

(B) $\frac{1}{2}$

(C) $\frac{3}{4}$

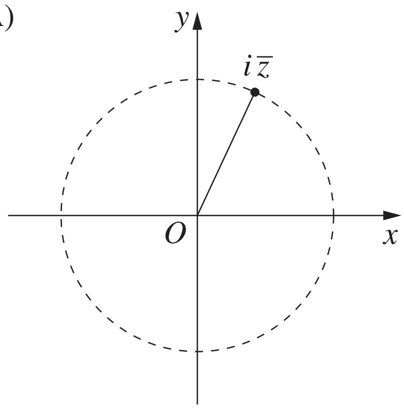
(D) 1

3 The complex number z is shown on the Argand diagram below.

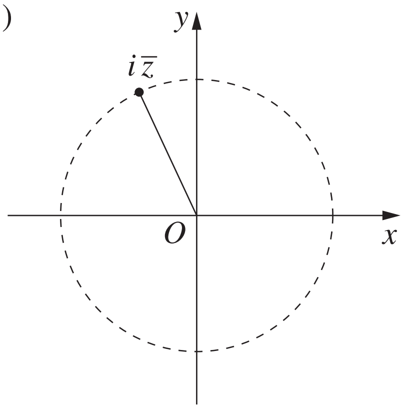


Which of the following best represents $i\overline{z}$?

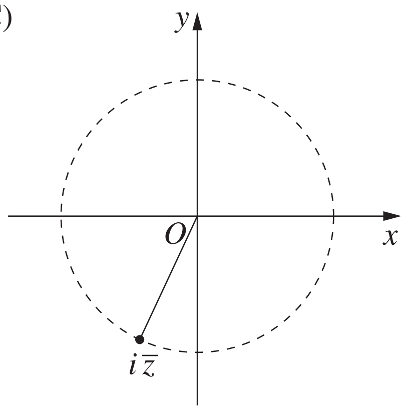
(A)



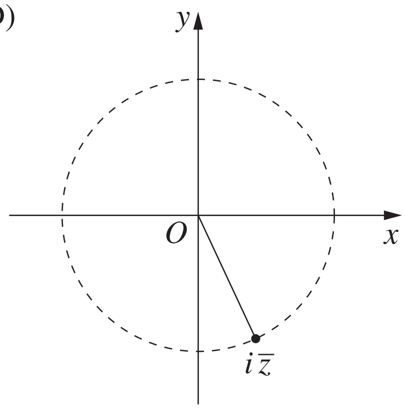
(B)



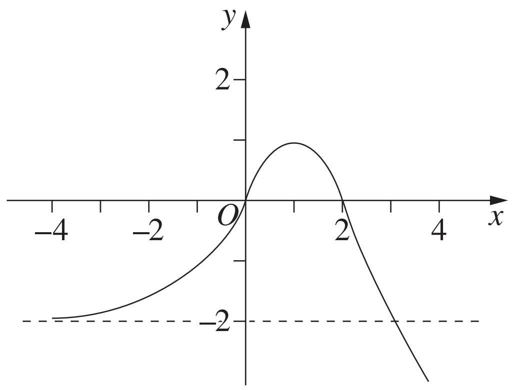
(C)



(D)

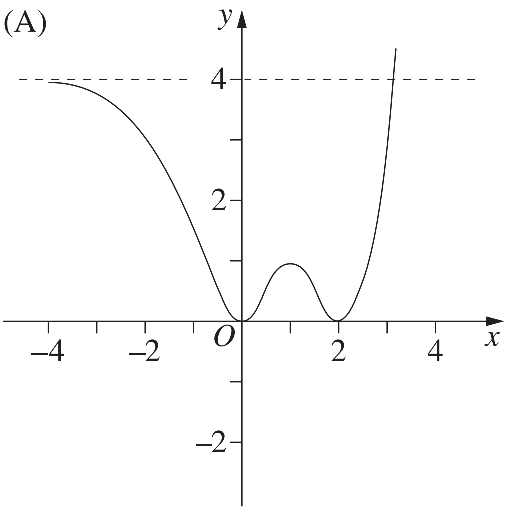


4 The graph $y = f(x)$ is shown below.

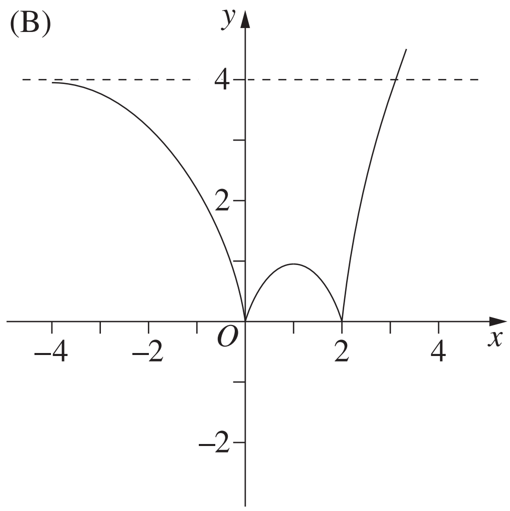


Which of the following graphs best represents $y = \left[f(x)\right]^2$?

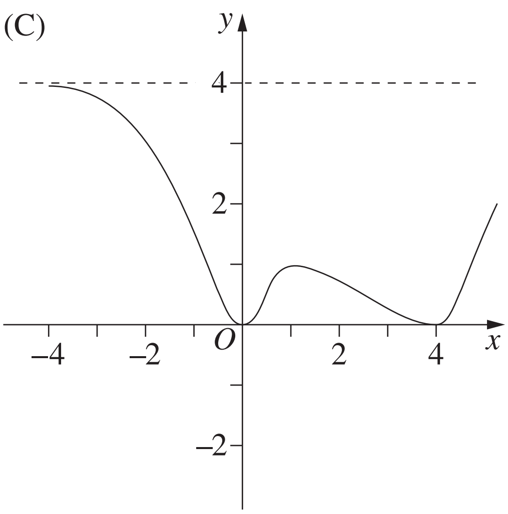
(A)



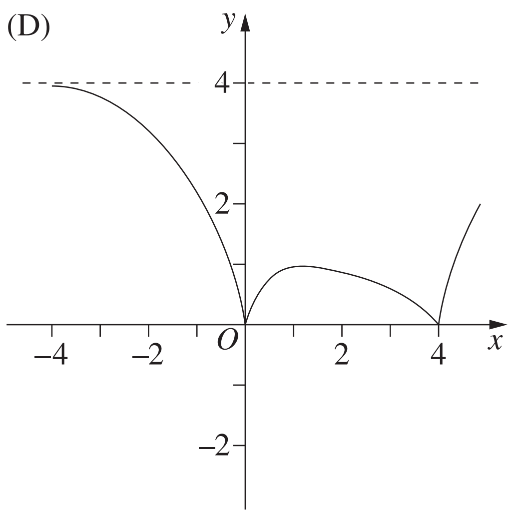
(B)



(C)



(D)



5 The equation $2x^3 - 3x^2 - 5x - 1 = 0$ has roots α, β and γ.

What is the value of $\dfrac{1}{\alpha^3\beta^3\gamma^3}$?

(A) $\dfrac{1}{8}$

(B) $-\dfrac{1}{8}$

(C) 8

(D) -8

6 What is the eccentricity of the hyperbola $\dfrac{x^2}{6} - \dfrac{y^2}{4} = 1$?

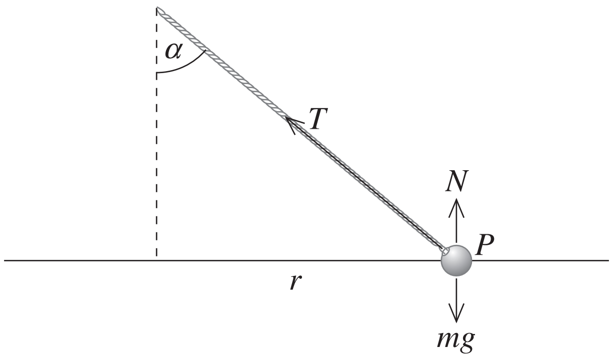
(A) $\dfrac{\sqrt{10}}{2}$

(B) $\dfrac{\sqrt{15}}{3}$

(C) $\dfrac{\sqrt{3}}{3}$

(D) $\dfrac{\sqrt{13}}{3}$

7 A particle P of mass m attached to a string is rotating in a circle of radius r on a smooth horizontal surface. The particle is moving with constant angular velocity ω. The string makes an angle α with the vertical. The forces acting on P are the tension T in the string, a reaction force N normal to the surface and the gravitational force mg.



Which of the following is the correct resolution of the forces on P in the vertical and horizontal directions?

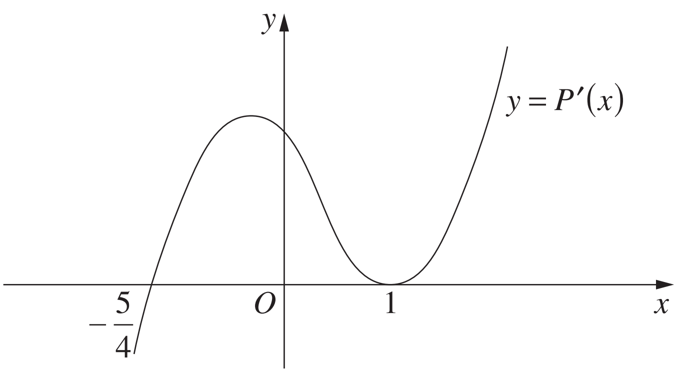
(A) $T\cos\alpha + N = mg$ and $T\sin\alpha = mr\omega^2$

(B) $T\cos\alpha - N = mg$ and $T\sin\alpha = mr\omega^2$

(C) $T\sin\alpha + N = mg$ and $T\cos\alpha = mr\omega^2$

(D) $T\sin\alpha - N = mg$ and $T\cos\alpha = mr\omega^2$

8 The following diagram shows the graph $y = P'(x)$, the derivative of a polynomial $P(x)$.



Which of the following expressions could be $P(x)$?

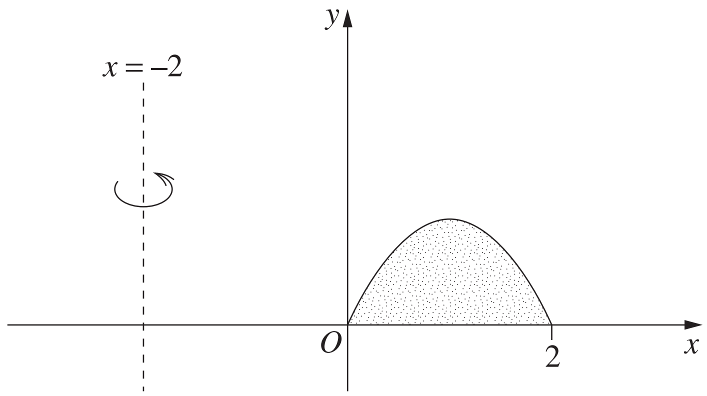
(A) $(x-2)(x-1)^3$

(B) $(x+2)(x-1)^3$

(C) $(x-2)(x+1)^3$

(D) $(x+2)(x+1)^3$

9 The diagram shows the graph $y = x(2 - x)$ for $0 \le x \le 2$. The region bounded by the graph and the x-axis is rotated about the line $x = -2$ to form a solid.



Which integral represents the volume of the solid?

(A) $2\pi \int_0^2 x(2 - x)^2 \, dx$

(B) $2\pi \int_0^2 x^2(2 - x) \, dx$

(C) $2\pi \int_0^2 x(2 - x)(2 + x) \, dx$

(D) $2\pi \int_0^2 x(2 - x)(x - 2) \, dx$

10 Without evaluating the integrals, which one of the following integrals is greater than zero?

(A) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x}{2+\cos x}\,dx$

(B) $\displaystyle\int_{-\pi}^{\pi} x^3 \sin x\,dx$

(C) $\displaystyle\int_{-1}^{1} \left(e^{-x^2}-1\right)dx$

(D) $\displaystyle\int_{-2}^{2} \tan^{-1}\left(x^3\right)dx$

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Express $\dfrac{2\sqrt{5}+i}{\sqrt{5}-i}$ in the form $x+iy$, where x and y are real. **2**

(b) Shade the region on the Argand diagram where the two inequalities **2**

$$|z+2| \geq 2 \quad \text{and} \quad |z-i| \leq 1$$

both hold.

(c) By completing the square, find $\displaystyle\int \frac{dx}{x^2+4x+5}$. **2**

(d) (i) Write $z=\sqrt{3}-i$ in modulus–argument form. **2**

(ii) Hence express z^9 in the form $x+iy$, where x and y are real. **1**

(e) Evaluate $\displaystyle\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx$. **3**

(f) Sketch the following graphs, showing the x- and y-intercepts.

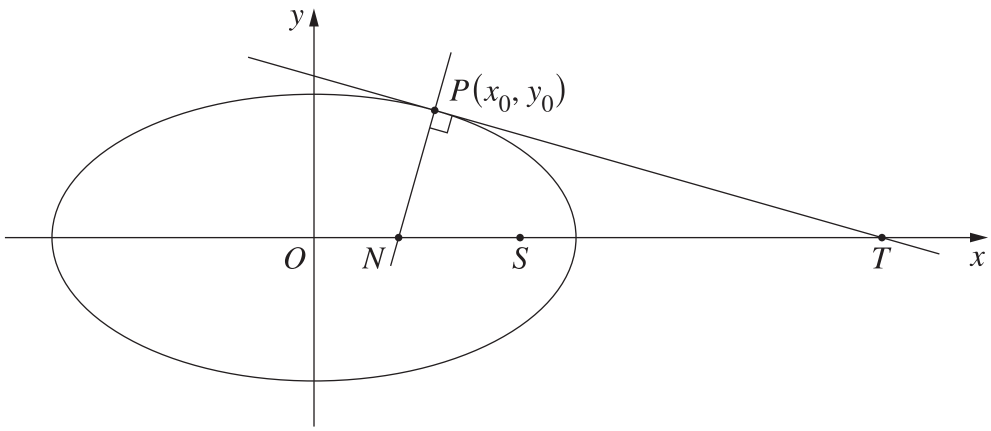
(i) $y=|x|-1$ **1**

(ii) $y=x\left(|x|-1\right)$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Using the substitution $t = \tan\dfrac{\theta}{2}$, or otherwise, find $\displaystyle\int \frac{d\theta}{1-\cos\theta}$. **3**

(b) The diagram shows the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with $a>b$. The ellipse has focus S and eccentricity e. The tangent to the ellipse at $P(x_0, y_0)$ meets the x-axis at T. The normal at P meets the x-axis at N.



(i) Show that the tangent to the ellipse at P is given by the equation **2**

$$y - y_0 = -\frac{b^2 x_0}{a^2 y_0}\left(x - x_0\right).$$

(ii) Show that the x-coordinate of N is x_0e^2. **2**

(iii) Show that $ON \times OT = OS^2$. **2**

Question 12 continues on page 11

(c) For every integer $n \geq 0$ let **3**

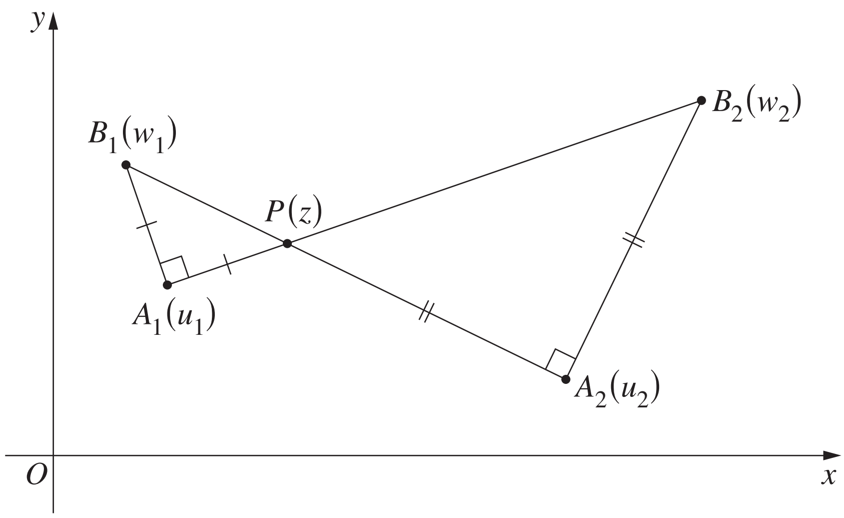
$$I_n = \int_1^{e^2} \left(\log_e x\right)^n \, dx.$$

Show that for $n \geq 1$

$$I_n = e^2 2^n - nI_{n-1}.$$

(d) On the Argand diagram the points A_1 and A_2 correspond to the distinct complex numbers u_1 and u_2 respectively. Let P be a point corresponding to a third complex number z.

Points B_1 and B_2 are positioned so that $\triangle A_1PB_1$ and $\triangle A_2B_2P$, labelled in an anti-clockwise direction, are right-angled and isosceles with right angles at A_1 and A_2, respectively. The complex numbers w_1 and w_2 correspond to B_1 and B_2, respectively.



(i) Explain why $w_1 = u_1 + i(z - u_1)$. **1**

(ii) Find the locus of the midpoint of B_1B_2 as P varies. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) An object on the surface of a liquid is released at time $t = 0$ and immediately sinks. Let x be its displacement in metres in a downward direction from the surface at time t seconds.

The equation of motion is given by

$$\frac{dv}{dt} = 10 - \frac{v^2}{40},$$

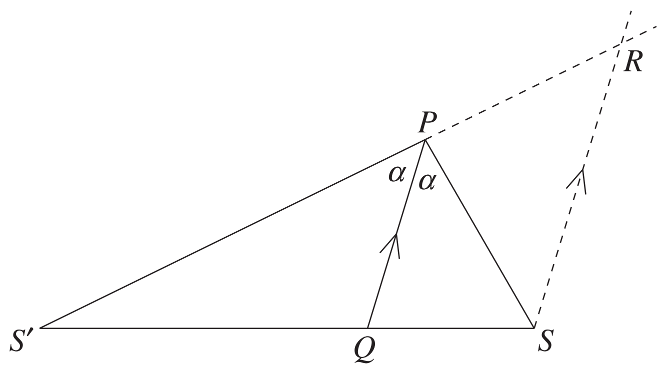
where v is the velocity of the object.

(i) Show that $v = \dfrac{20\left(e^t - 1\right)}{e^t + 1}$. **4**

(ii) Use $\dfrac{dv}{dt} = v\dfrac{dv}{dx}$ to show that $x = 20\log_e\left(\dfrac{400}{400 - v^2}\right)$. **2**

(iii) How far does the object sink in the first 4 seconds? **2**

(b) The diagram shows $\triangle S'SP$. The point Q is on $S'S$ so that PQ bisects $\angle S'PS$. The point R is on $S'P$ produced so that $PQ \parallel RS$.

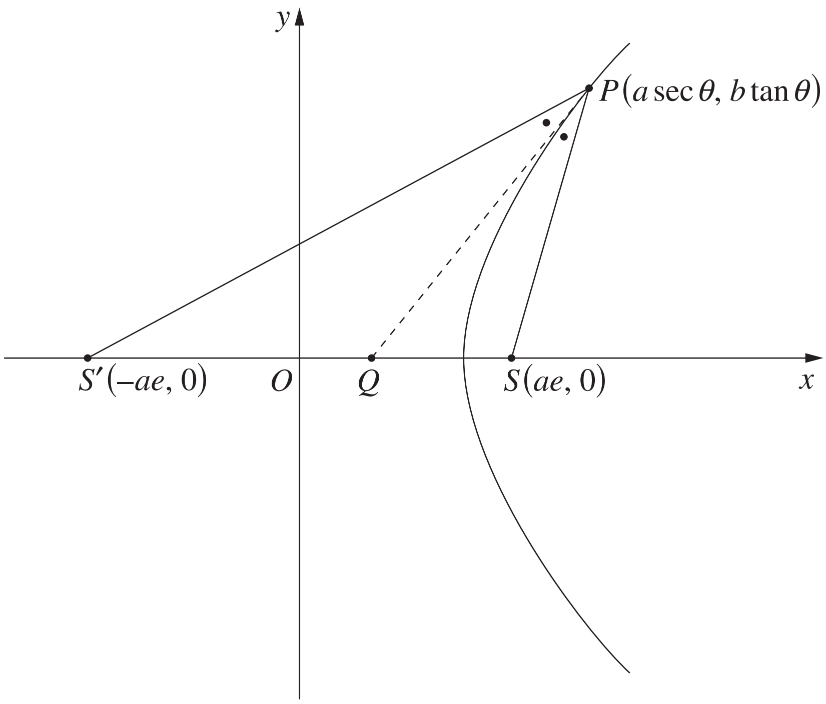


(i) Show that $PS = PR$. **1**

(ii) Show that $\dfrac{PS}{QS} = \dfrac{PS'}{QS'}$. **2**

Question 13 continues on page 13

(c) Let P be a point on the hyperbola given parametrically by $x = a\sec\theta$ and $y = b\tan\theta$, where a and b are positive. The foci of the hyperbola are $S(ae, 0)$ and $S'(-ae, 0)$ where e is the eccentricity. The point Q is on the x-axis so that PQ bisects $\angle SPS'$.



(i) Show that $SP = a(e\sec\theta - 1)$. **1**

(ii) It is given that $S'P = a(e\sec\theta + 1)$. Using part (b), or otherwise, show that the x-coordinate of Q is $\dfrac{a}{\sec\theta}$. **2**

(iii) The slope of the tangent to the hyperbola at P is $\dfrac{b\sec\theta}{a\tan\theta}$. (Do NOT prove this.) **1**

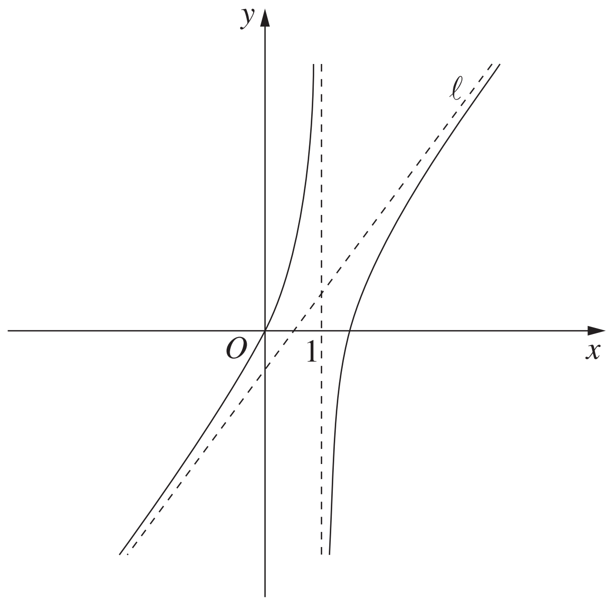
Show that the tangent at P is the line PQ.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\displaystyle\int \frac{3x^2+8}{x\left(x^2+4\right)}\,dx$. **3**

(b) The diagram shows the graph $y = \dfrac{x(2x-3)}{x-1}$. The line ℓ is an asymptote.



(i) Use the above graph to draw a one-third page sketch of the graph **2**

$$y = \frac{x-1}{x(2x-3)}$$

indicating all asymptotes and all x- and y-intercepts.

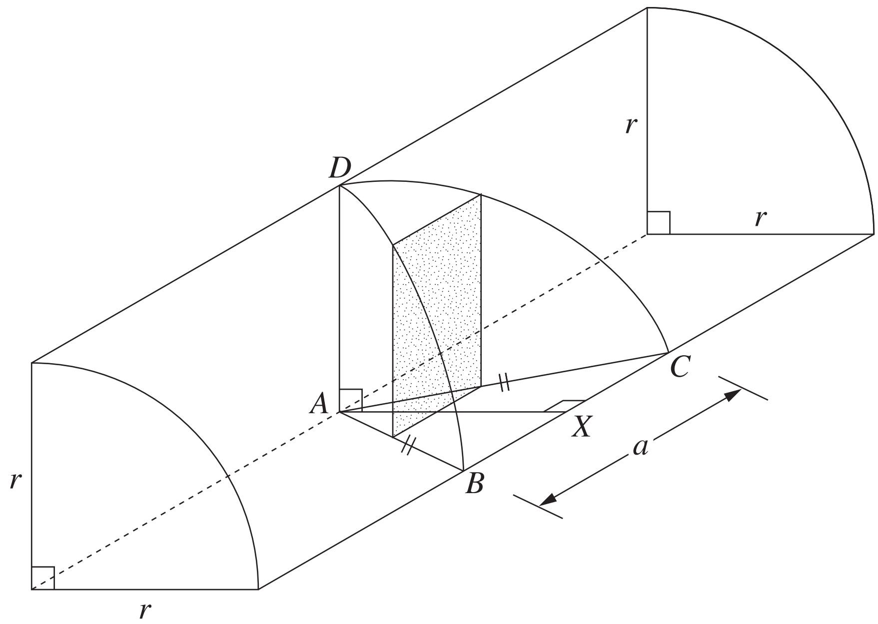
(ii) By writing $\dfrac{x(2x-3)}{x-1}$ in the form $mx+b+\dfrac{a}{x-1}$, find the equation of the line ℓ. **2**

Question 14 continues on page 15

Question 14 (continued)

(c) The solid $ABCD$ is cut from a quarter cylinder of radius r as shown. Its base is an isosceles triangle ABC with $AB = AC$. The length of BC is a and the midpoint of BC is X. **4**

The cross-sections perpendicular to AX are rectangles. A typical cross-section is shown shaded in the diagram.

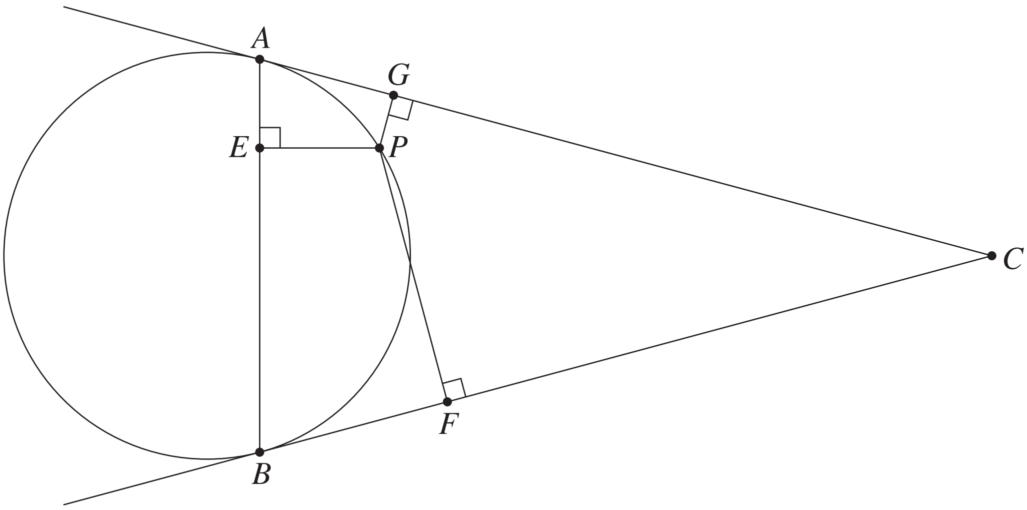


Find the volume of the solid $ABCD$.

Question 14 continues on page 16

Question 14 (continued)

(d) The diagram shows points A and B on a circle. The tangents to the circle at A and B meet at the point C. The point P is on the circle inside $\triangle ABC$. The point E lies on AB so that $AB \perp EP$. The points F and G lie on BC and AC respectively so that $FP \perp BC$ and $GP \perp AC$.



Copy or trace the diagram into your writing booklet.

(i) Show that $\triangle APG$ and $\triangle BPE$ are similar. **2**

(ii) Show that $EP^2 = FP \times GP$. **2**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove that $\sqrt{ab} \le \dfrac{a+b}{2}$, where $a \ge 0$ and $b \ge 0$. **1**

(ii) If $1 \le x \le y$, show that $x(y - x + 1) \ge y$. **2**

(iii) Let n and j be positive integers with $1 \le j \le n$. Prove that **2**

$$\sqrt{n} \le \sqrt{j(n-j+1)} \le \frac{n+1}{2}.$$

(iv) For integers $n \ge 1$, prove that **1**

$$\left(\sqrt{n}\right)^n \le n! \le \left(\frac{n+1}{2}\right)^n.$$

(b) Let $P(z) = z^4 - 2kz^3 + 2k^2z^2 - 2kz + 1$, where k is real.

Let $\alpha = x + iy$, where x and y are real.

Suppose that α and $i\alpha$ are zeros of $P(z)$, where $\overline{\alpha} \ne i\alpha$.

(i) Explain why $\overline{\alpha}$ and $-i\overline{\alpha}$ are zeros of $P(z)$. **1**

(ii) Show that $P(z) = z^2(z-k)^2 + (kz-1)^2$. **1**

(iii) Hence show that if $P(z)$ has a real zero then **2**

$$P(z) = \left(z^2+1\right)(z+1)^2 \quad \text{or} \quad P(z) = \left(z^2+1\right)(z-1)^2.$$

(iv) Show that all zeros of $P(z)$ have modulus 1. **2**

(v) Show that $k = x - y$. **1**

(vi) Hence show that $-\sqrt{2} \le k \le \sqrt{2}$. **2**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) In how many ways can m identical yellow discs and n identical black discs be arranged in a row? **1**

(ii) In how many ways can 10 identical coins be allocated to 4 different boxes? **1**

(b) (i) Show that $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\dfrac{x+y}{1-xy}\right)$ for $|x| < 1$ and $|y| < 1$. **1**

(ii) Use mathematical induction to prove **3**

$$\sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{n}{n+1}\right)$$

for all positive integers n.

(iii) Find $\displaystyle\lim_{n\to\infty} \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{2j^2}\right)$. **1**

Question 16 continues on page 19

(c) Let n be an integer where $n > 1$. Integers from 1 to n inclusive are selected randomly one by one with repetition being possible. Let $P(k)$ be the probability that exactly k different integers are selected before one of them is selected for the second time, where $1 \le k \le n$.

(i) Explain why $P(k) = \dfrac{(n-1)!\,k}{n^k (n-k)!}$. **2**

(ii) Suppose $P(k) \ge P(k-1)$. Show that $k^2 - k - n \le 0$. **2**

(iii) Show that if $\sqrt{n + \frac{1}{4}} > k - \frac{1}{2}$ then the integers n and k satisfy $\sqrt{n} > k - \frac{1}{2}$. **2**

(iv) Hence show that if $4n + 1$ is not a perfect square, then $P(k)$ is greatest when k is the closest integer to $\sqrt{n}$. **2**

You may use part (iii) and also that $k^2 - k - n > 0$ if $P(k) < P(k-1)$.

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$