BOARD OF STUDIES
NEW SOUTH WALES

2012 HSC Mathematics Extension 2 ‘Sample Answers’

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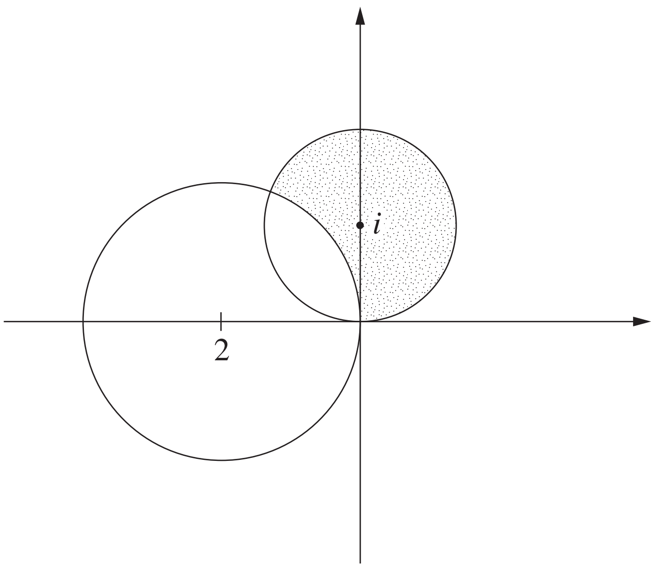
Section II

Question 11 (a)

$$\frac{2\sqrt{5}+i}{\sqrt{5}-i} = \frac{\left(2\sqrt{5}+i\right)\left(\sqrt{5}+i\right)}{5+1}$$

$$= \frac{10-1+3\sqrt{5}\,i}{6} = \frac{3}{2}+\frac{\sqrt{5}}{2}i$$

Question 11 (b)



Question 11 (c)

Completion of squares gives

$$x^2+4x+5 = x^2+4x+4+1$$

$$= \left(x+2\right)^2+1$$

Hence

$$\int \frac{1}{x^2+4x+5}\,dx = \int \frac{1}{\left(x+2\right)^2+1}\,dx$$

$$= \tan^{-1}\left(x+2\right)+C$$

Question 11 (d) (i)

$$z = 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right)$$
$$= 2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)$$

Question 11 (d) (ii)

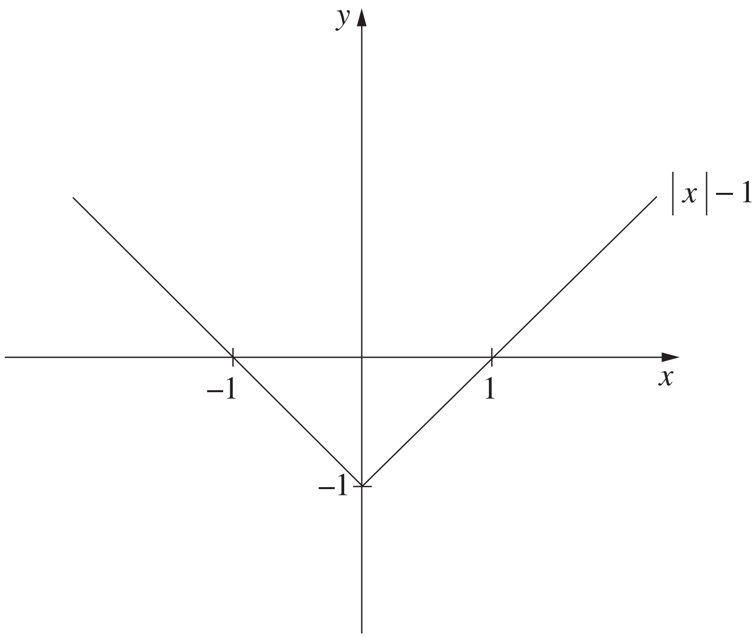
By de Moivre's theorem

$$z^9 = \left[2\left(\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right)\right]^9$$
$$= 2^9\left(\cos\left(-\frac{9\pi}{6}\right) + i\sin\left(-\frac{9\pi}{6}\right)\right)$$
$$= 2^9\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$$

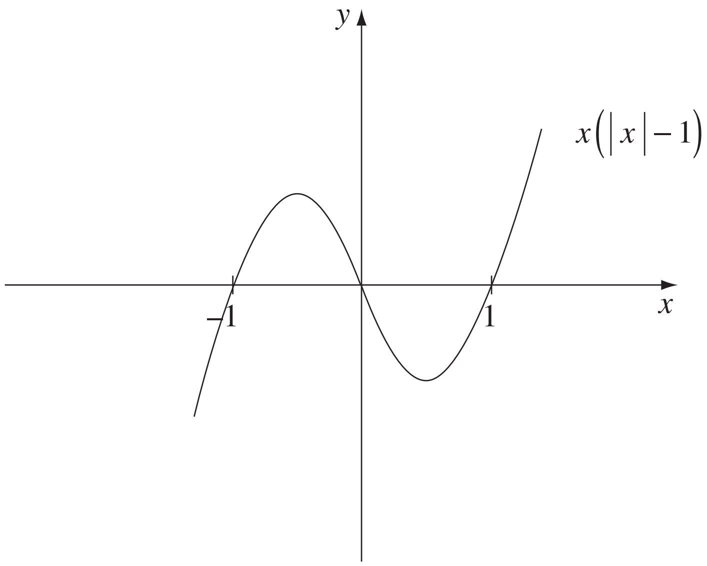
Question 11 (e)

$$\int_0^1 \frac{e^{2x}}{e^{2x}+1}\,dx = \frac{1}{2}\int_0^1 \frac{2e^{2x}}{e^{2x}+1}\,dx$$
$$= \frac{1}{2}\left[\log_e\left(e^{2x}+1\right)\right]_0^1$$
$$= \frac{1}{2}\log_e\left(e^2+1\right) - \frac{1}{2}\log_e 2$$
$$= \frac{1}{2}\log_e \frac{e^2+1}{2}$$

Question 11 (f) (i)



Question 11 (f) (ii)



Question 12 (a)

Use a *t*-substitution

$$\cos\theta = \frac{1-t^2}{1+t^2} \qquad d\theta = \frac{2}{1+t^2}\,dt$$

Hence

$$\int \frac{1}{1-\cos\theta}\,d\theta$$

$$= \int \frac{2}{\left(1+t^2\right)\left(1-\dfrac{1-t^2}{1+t^2}\right)}\,dt$$

$$= \int \frac{2}{\left(1+t^2\right)-\left(1-t^2\right)}\,dt$$

$$= \int \frac{1}{t^2}\,dt$$

$$= -\frac{1}{t} + C$$

$$= -\frac{1}{\tan\dfrac{\theta}{2}} + C$$

$$= -\cot\frac{\theta}{2} + C$$

Question 12 (b) (i)

To get the slope of the tangent at P, differentiate $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ implicitly with respect to x:

$$\frac{2x}{a^2}+\frac{2y}{b^2}y'=0$$

At $P\left(x_0, y_0\right)$ we get

$$\frac{2x_0}{a^2}+\frac{2y_0}{b^2}y'=0$$

Hence the slope is

$$y'=-\frac{-2x_0}{a^2}\times\frac{b^2}{2y_0}$$

$$=\frac{-b^2x_0}{a^2y_0}$$

and the equation of the tangent at P is

$$y-y_0=-\frac{b^2x_0}{a^2y_0}\left(x-x_0\right)$$

Question 12 (b) (ii)

The slope of the tangent at P is $-\dfrac{b^2 x_0}{a^2 y_0}$,

so the slope of the normal is $\dfrac{a^2 y_0}{b^2 x_0}$.

Hence the equation for the normal is

$$y - y_0 = \frac{a^2 y_0}{b^2 x_0}\left(x - x_0\right)$$

At N, $y = 0$.

$$\begin{aligned} -y_0 &= \frac{a^2 y_0}{b^2 x_0}\left(x - x_0\right) \\ &= \frac{a^2 y_0}{b^2 x_0}x - \frac{a^2 y_0}{b^2} \end{aligned}$$

$$\frac{a^2 y_0}{b^2 x_0}x = \left(\frac{a^2}{b^2} - 1\right)y_0$$

$$x = \left(\frac{a^2}{b^2} - 1\right)\frac{b^2 x_0}{a^2}$$

$$x = \left(1 - \frac{b^2}{a^2}\right)x_0$$

$$x = e^2 x_0 \qquad \left(\text{since } b^2 = a^2\left(1 - e^2\right) \text{ for an ellipse}\right)$$

Question 12 (b) (iii)

Find the x-coordinate of T by setting $y = 0$ in part (i)

$$-y_0 = -\frac{b^2 x_0}{a^2 y_0}\left(x - x_0\right)$$

$$a^2 y_0^2 = b^2 x x_0 - b^2 x_0^2$$

From the equation of the ellipse

$$a^2 y_0^2 + b^2 x_0^2 = a^2 b^2$$

$$\therefore\ a^2 b^2 = b^2 x x_0$$

Hence

$$x = \frac{a^2}{x_0}$$

From part (ii),

$$\begin{aligned} ON \times OT &= e^2 x_0 \cdot \frac{a^2}{x_0} \\ &= a^2 e^2 \\ &= OS^2 \end{aligned}$$

since S has coordinates $\left(ae, 0\right)$.

Question 12 (c)

Use integration by parts

$$u = \left(\log_e x\right)^n \qquad v' = 1$$

$$u' = \frac{n}{x}\left(\log_e x\right)^{n-1} \qquad v = x$$

Hence

$$I_n = \int_1^{e^2} \left(\log_e x\right)^n \, dx$$

$$= \Big[x\left(\log_e x\right)^n \Big]_1^{e^2} - n\int_1^{e^2} x \frac{1}{x}\left(\log_e x\right)^{n-1} dx$$

$$= e^2\left(\log_e e^2\right)^n - n\int_1^{e^2} \left(\log_e x\right)^{n-1} dx$$

$$= 2^n e^2 - n\, I_{n-1}$$

Question 12 (d) (i)

Rotate $z - u_1$ by 90° anti-clockwise:

$$w_1 - u_1 = i\left(z - u_1\right)$$

$$w_1 = u_1 + i\left(z - u\right)$$

Question 12 (d) (ii)

To get w_2 rotate $z - u_2$ by 90° clockwise:

$$w_2 - u_2 = -i\left(z - u_2\right)$$

$$w_2 = u_2 - i\left(z - u_2\right)$$

Midpoint of w_1 and w_2:

$$\frac{1}{2}\left(w_1 + w_2\right)$$

$$= \frac{1}{2}\left[u_1 + i\left(z - u_1\right) + u_2 - i\left(z - u_2\right)\right]$$

$$= \frac{1}{2}\left[u_1 + u_2 + iz - iz + i\left(u_2 - u_1\right)\right]$$

$$= \frac{1}{2}\left[u_1 + u_2 + i\left(u_2 - u_1\right)\right]$$

The midpoint is independent of z
therefore it is a fixed point.

Question 13 (a) (i)

Find t as a function of v:

$$\frac{dt}{dv} = \frac{1}{10 - \frac{v^2}{40}} = \frac{40}{400 - v^2}$$

Use partial fractions:

$$t = \int \frac{dt}{dv}\,dv$$

$$= \int \frac{40}{400 - v^2}\,dv$$

$$= \int \frac{dv}{20 + v} + \int \frac{dv}{20 - v}$$

$$= \log_e(20 + v) - \log_e(20 - v) + C$$

$$= \log_e \frac{20 + v}{20 - v} + C$$

If $t = 0$, $v = 0$

$$0 = \log_e \frac{20 + 0}{20 - 0} + C$$

$\therefore\ C = 0$

Hence

$$t = \log_e \frac{20 + v}{20 - v}$$

$$e^t = \frac{20 + v}{20 - v}$$

$$(20 - v)e^t = 20 + v$$

$$20e^t - 20 = v + ve^t$$

$$20\left(e^t - 1\right) = v\left(1 + e^t\right)$$

$$v = \frac{20\left(e^t - 1\right)}{e^t + 1}$$

Question 13 (a) (ii)

$$v\frac{dv}{dx} = \frac{dv}{dt}$$

$$= 10 - \frac{v^2}{40}$$

$$= \frac{400 - v^2}{40}$$

$$\therefore \frac{dv}{dx} = \frac{400 - v^2}{40v}$$

$$x = \int \frac{dx}{dv}\,dv$$

$$x = \int \frac{40v}{400 - v^2}\,dv$$

$$x = -20\log_e\left(400 - v^2\right) + C$$

$$x = 0 \text{ if } v = 0$$

$$0 = -20\log_e 400 + C$$

$$\therefore \quad C = 20\log_e 400$$

$$\text{So, } x = -20\log_e\left(400 - v^2\right) + 20\log_e 400$$

$$= 20\left(\log_e 400 - \log_e\left(400 - v^2\right)\right)$$

$$= 20\log_e\left(\frac{400}{400 - v^2}\right)$$

Question 13 (a) (iii)

From part (i), when $t = 4$

$$v = \frac{20\left(e^4 - 1\right)}{e^4 + 1}$$

So from part (ii)

$$x = 20\ln\frac{400}{400 - \left(\dfrac{20\left(e^4 - 1\right)}{e^4 + 1}\right)^2}$$

$$\approx 53 \text{ m}$$

Question 13 (b) (i)

$\angle S'RS = \alpha$ (corresponding angles $PQ \parallel RS$)

$\angle PSR = \alpha$ (alternate angles $PQ \parallel RS$)

Hence $\triangle PRS$ is isosceles with $PS = PR$

Question 13 (b) (ii)

$\dfrac{PR}{PS'} = \dfrac{QS}{QS'}$ (equal ratio of intercepts cut by parallel lines)

$\dfrac{PS}{PS'} = \dfrac{QS}{QS'}$ (as $PR = PS$ from part (i))

Hence $\dfrac{PS}{QS} = \dfrac{PS'}{QS'}$

Question 13 (c) (i)

The equation of the directrix is $x = \frac{a}{e}$.

By the focus-directrix definition of a hyperbola,

$$
\begin{aligned}
SP &= e \times \text{(distance } P \text{ to directrix)} \\
&= e\left(a\sec\theta - \frac{a}{e}\right) \\
&= ae\sec\theta - a \\
&= a\left(e\sec\theta - 1\right)
\end{aligned}
$$

Question 13 (c) (ii)

From part (c) (i) and part (b) $\frac{QS}{PS} = \frac{QS'}{PS'}$.

If x_0 is the x-coordinate of Q, then

$$
\begin{aligned}
\frac{ae - x_0}{a\left(e\sec\theta - 1\right)} &= \frac{QS}{PS} \\
&= \frac{QS'}{PS'} \\
&= \frac{ae + x_0}{a\left(e\sec\theta + 1\right)}
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left(ae - x_0\right)\left(e\sec\theta + 1\right) &= \left(ae + x_0\right)\left(e\sec\theta - 1\right) \\
2ae &= 2ex_0\sec\theta \\
x_0 &= \frac{a}{\sec\theta}
\end{aligned}
$$

Question 13 (c) (iii)

Slope of line *PQ* is

$$\frac{b\tan\theta}{a\sec\theta - \dfrac{a}{\sec\theta}}$$

$$= \frac{b\tan\theta\sec\theta}{a\left(\sec^2\theta - 1\right)}$$

$$= \frac{b\tan\theta\sec\theta}{a\tan^2\theta}$$

$$= \frac{b\sec\theta}{a\tan\theta}$$

This is the same as the slope of the tangent at *P*, so *QP* coincides with the tangent.

Question 14 (a)

Do a partial fraction decomposition

$$\frac{3x^2+8}{x\left(x^2+4\right)}$$

$$=\frac{a}{x}+\frac{bx+c}{x^2+4}$$

$$=\frac{a\left(x^2+4\right)+\left(bx+c\right)x}{x\left(x^2+4\right)}$$

Need $3x^2+8=a\left(x^2+4\right)+\left(bx+c\right)x$ for all x.

If $x=0$: $8=4a$, so $a=2$

If $x=1$: $11=10+b+c$

If $x=-1$: $11=10+b-c$

Solving simultaneous equations: $b=1$, $c=0$

Therefore

$$\int \frac{3x^2+8}{x\left(x^2+4\right)}\,dx$$

$$=\int\left(\frac{2}{x}+\frac{x}{x^2+4}\right)dx$$

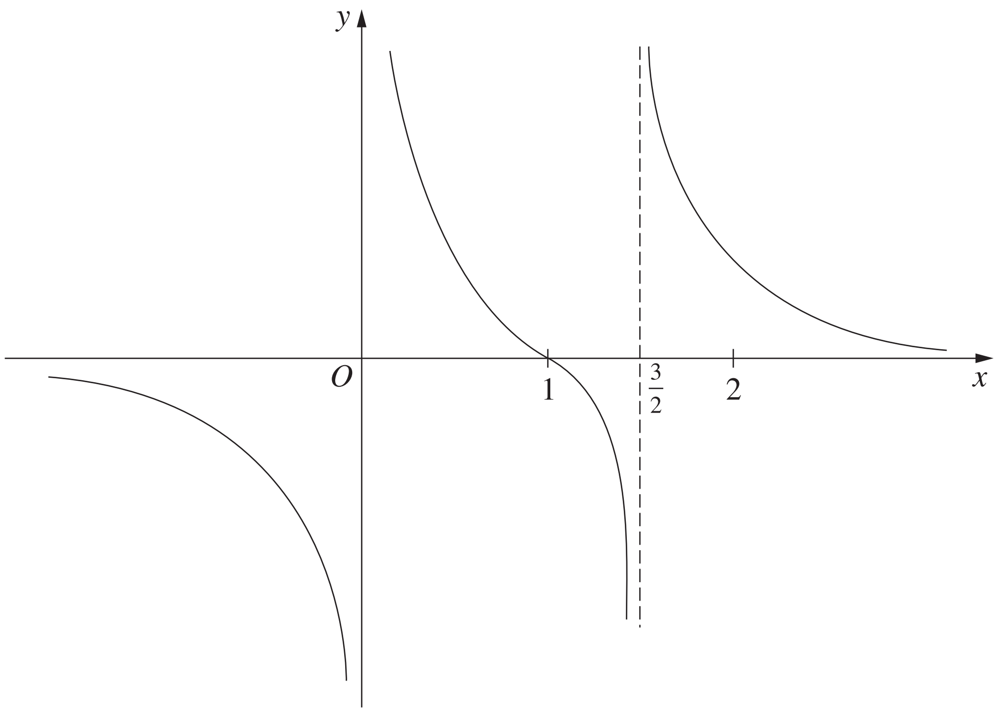
$$=2\int\frac{1}{x}\,dx+\frac{1}{2}\int\frac{2x}{x^2+4}\,dx$$

$$=2\ln x+\frac{1}{2}\ln\left(x^2+4\right)+C$$

Question 14 (b) (i)

Vertical asymptotes: $x = 0,\ x = \dfrac{3}{2}$

Horizontal asymptote: $y = 0$



Question 14 (b) (ii)

$$\frac{x(2x-3)}{x-1}$$

$$= \frac{(x-1)(2x-1)-1}{x-1}$$

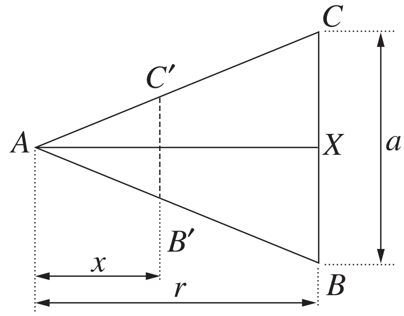
$$= 2x - 1 - \frac{1}{x-1}$$

As $x \to \infty$ we have $\dfrac{1}{x-1} \to 0$, so

the asymptote ℓ is $y = 2x - 1$

Question 14 (c)

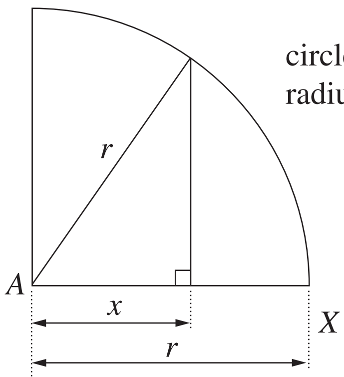
Let x be the distance from A along AX.



By similar triangles

$$\frac{x}{r} = \frac{B'C'}{BC}$$

Hence the length of the base $B'C'$ of a rectangular cross-section is $\dfrac{xa}{r}$.



Hence the length of the other side of the rectangular cross-section is $\sqrt{r^2 - x^2}$.

The area of the cross-section is $\dfrac{xa}{r}\sqrt{r^2 - x^2}$.

The volume is

$$\frac{a}{r}\int_0^r x\sqrt{r^2 - x^2}\,dx$$

$$= \frac{a}{r} \times \frac{-1}{2}\int_0^r -2x\left(r^2 - x^2\right)^{\frac{1}{2}}\,dx$$

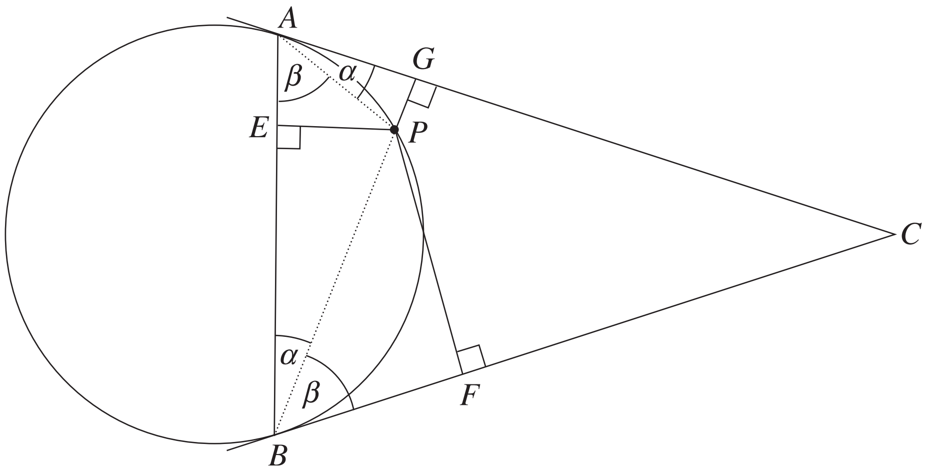
$$= -\left[\frac{a}{3r}\left(r^2 - x^2\right)^{\frac{3}{2}}\right]_0^r$$

$$= -\frac{a}{3r}\left(0 - \left(r^2\right)^{\frac{3}{2}}\right)$$

$$= \frac{a}{3r}r^3$$

$$= \frac{ar^2}{3}$$

Question 14 (d) (i)



By the angle in the alternate segment theorem

$\angle GAP = \angle EBP$

$\angle AGP$ and $\angle BEP$ are 90° (given)

Hence $\triangle APG \,|||\, \triangle BPE$ (equiangular)

Question 14 (d) (ii)

As AC and BC are tangents, we get

$\angle CAB = \angle CBA$ and so by part (i) $\angle EAP = \angle FBP$

(or same argument using alternative segment theorem, as in part (i))

Hence $\triangle EAP \,|||\, \triangle FBP$.

By the similar triangles from part (i)

$\dfrac{EP}{AP} = \dfrac{FP}{BP}$ and $\dfrac{EP}{PB} = \dfrac{PG}{AP}$

Hence $\dfrac{EP}{FP} = \dfrac{AP}{BP} = \dfrac{PG}{EP}$ and so $EP^2 = FP \times PG$

Question 15 (a) (i)

$$0 \le \left(\sqrt{a} - \sqrt{b}\right)^2$$

ie $\quad 0 \le a - 2\sqrt{ab} + b$

$$\therefore \sqrt{ab} \le \frac{a+b}{2}$$

Question 15 (a) (ii)

$x\left(y - x + 1\right) - y$

$= xy - x^2 + x - y$

$= \left(x - 1\right)\left(y - x\right)$

≥ 0 since each factor is non-negative $\left(x \ge 1,\ y \ge x\right)$

$\therefore \quad x\left(y - x + 1\right) \ge y$

Question 15 (a) (iii)

From part (i) with $a = j$ and $b = n - j + 1$

$$\sqrt{j\left(n - j + 1\right)} \le \frac{j + n - j + 1}{2}$$

ie $\sqrt{j\left(n - j + 1\right)} \le \dfrac{n+1}{2}$

From part (ii) with $x = j$ and $y = n$

$$j\left(n - j + 1\right) \ge n$$

$\therefore \sqrt{j\left(n - j + 1\right)} \ge \sqrt{n}$

$\therefore \sqrt{n} \le \sqrt{j\left(n - j + 1\right)} \le \dfrac{n+1}{2}$

Question 15 (a) (iv)

Let j take the values from 1 to n. We have

$$\sqrt{n} \leq \sqrt{1 \times n} \leq \left(\frac{n+1}{2}\right)$$

$$\sqrt{n} \leq \sqrt{2 \times (n-1)} \leq \left(\frac{n+1}{2}\right)$$

$$\vdots$$

$$\sqrt{n} \leq \sqrt{n \times 1} \leq \left(\frac{n+1}{2}\right)$$

Multiplying these:

$$\left(\sqrt{n}\right)^n \leq \sqrt{n! \times n!} \leq \left(\frac{n+1}{2}\right)^n$$

ie $$\left(\sqrt{n}\right)^n \leq n! \leq \left(\frac{n+1}{2}\right)^n$$

Question 15 (b) (i)

Since the coefficients of the polynomial are real, complex zeros occur in pairs of complex conjugate numbers. Hence $\overline{\alpha}$ is a zero since α is a zero, and $\overline{i\alpha} = -i\overline{\alpha}$ is a zero since $i\alpha$ is a zero.

Question 15 (b) (ii)

$$\begin{aligned} P(z) &= z^4 - 2kz^3 + 2k^2z^2 - 2kz + 1 \\ &= \left(z^4 - 2kz^3 + k^2z^2\right) + \left(k^2z^2 - 2kz + 1\right) \\ &= \left(z^2 - kz\right)^2 + \left(kz - 1\right)^2 \\ &= z^2\left(z - k\right)^2 + \left(kz - 1\right)^2 \end{aligned}$$

Question 15 (b) (iii)

If z is real, then $z^2(z-k)^2$ and $(kz-1)^2$ are real and either positive or zero.

Hence if

$P(z) = z^2(z-k)^2 + (kz-1)^2 = 0$,

then $z(z-k) = 0$ and $(kz-1) = 0$

We deduce $z - k = 0$ and $kz - 1 = 0$.

Therefore $z = k$ and $k^2 = 1$, so $k = 1$ or $k = -1$.

If $k = 1$, then

$P(z) = z^2(z-1)^2 + (z-1)^2 = (z^2+1)(z-1)^2$.

If $k = -1$, then

$P(z) = z^2(z+1)^2 + (-z-1)^2 = (z^2+1)(z+1)^2$.

Question 15 (b) (iv)

If $P(z)$ has real zeros, then from part (iii) the zeros are $1, 1, i, -i$ or $-1, -1, i, -i$ which have modulus one.

If $P(z)$ does not have real zeros, then the zeros are $\alpha, \overline{\alpha}, i\alpha, -i\overline{\alpha}$ which are all different (the assumption is that $\overline{\alpha} \neq i\alpha$).

The product of the zeros is

$\alpha\overline{\alpha}(i\alpha)(\overline{i\alpha}) = |\alpha|^4 = 1$,

so $|\alpha| = 1$.

It follows that the modulus of all zeros is one.

Question 15 (b) (v)

The sum of the zeros is $\dfrac{2k}{1} = 2k$,

so

$\alpha + \overline{\alpha} + i\alpha - i\overline{\alpha}$

$= (x+iy) + (x-iy) + i(x+iy) - i(x-iy)$

$= 2x - 2y$

$= 2k$

Hence $k = x - y$.

(This also applies if $P(z)$ has real zeros.
If $k = 1$, then $\alpha = 1$. If $k = -1$, then $\alpha = i$.)

Question 15 (b) (vi)

From part (iv) $x^2 + y^2 = |\alpha|^2 = 1$, and part (v) $y = x - k$.

Hence

$$x^2 + (x-k)^2 = 1$$

$$2x^2 - 2kx + k^2 = 1$$

$$2x^2 - 2kx + k^2 - 1 = 0$$

To have a solution, the discriminant can't be negative, so

$$4k^2 - 8(k^2 - 1) \geq 0$$

$$-4k^2 + 8 \geq 0$$

$$2 \geq k^2$$

$$-\sqrt{2} \leq k \leq \sqrt{2}$$

Alternative: To have a solution, the line $x - y = k$ must meet the circle $x^2 + y^2 = 1$. This is only possible if the distance from $(0, 0)$ to the line is less than or equal to 1.

$$\left| \frac{0 - 0 - k}{\sqrt{2}} \right| = \frac{|k|}{\sqrt{2}} \leq 1,$$

that is, $-\sqrt{2} \leq k \leq \sqrt{2}$

Question 16 (a) (i)

We consider $m + n$ objects with repetition, so the number is

$$\frac{(m+n)!}{m!\,n!} = \binom{m+n}{m} = \binom{m+n}{n}$$

Question 16 (a) (ii)

The question is equivalent to arranging 10 coins and 3 separators in a row.
From part (i) the number of possibilities is

$$\frac{13!}{10!\,3!} = \binom{13}{3} = 286$$

Question 16 (b) (i)

$$\tan\left(\tan^{-1} x + \tan^{-1} y\right)$$

$$= \frac{\tan\left(\tan^{-1} x\right) + \tan\left(\tan^{-1} y\right)}{1 - \tan\left(\tan^{-1} x\right)\tan\left(\tan^{-1} y\right)}$$

$$= \frac{x + y}{1 - xy}$$

As $|x| < 1$ and $|y| < 1$ we have

$$-\frac{\pi}{4} < \tan^{-1} x < \frac{\pi}{4},$$

$$-\frac{\pi}{4} < \tan^{-1} y < \frac{\pi}{4}.$$

Hence $-\frac{\pi}{2} < \tan^{-1} x + \tan^{-1} y < \frac{\pi}{2}$ and so

$$\tan^{-1}\left(\tan\left(\tan^{-1} x + \tan^{-1} y\right)\right)$$

$$= \tan^{-1} x + \tan^{-1} y$$

$$= \tan^{-1}\left(\frac{x + y}{1 - xy}\right)$$

Question 16 (b) (ii)

For $n = 1$

$$\sum_{j=1}^{1} \tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{1}{2}\right)$$

$$= \tan^{-1}\left(\frac{1}{1+1}\right)$$

as required.

Now assume that for $n = k$

$$\sum_{j=1}^{k} \tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{k}{k+1}\right)$$

Then

$$\sum_{j=1}^{k+1} \tan^{-1}\left(\frac{1}{2j^2}\right) = \sum_{j=1}^{k} \tan^{-1}\left(\frac{1}{2j^2}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$$

$$= \tan^{-1}\left(\frac{k}{k+1}\right) + \tan^{-1}\left(\frac{1}{2(k+1)^2}\right)$$

As $\dfrac{k}{k+1} < 1$ and $\dfrac{1}{2(k+1)^2} < 1$ we can apply the identity from part (i):

$$\sum_{j=1}^{k+1} \tan^{-1}\left(\frac{1}{2j^2}\right) = \tan^{-1}\left(\frac{\dfrac{k}{k+1} + \dfrac{1}{2(k+1)^2}}{1 - \dfrac{k}{(k+1)}\dfrac{1}{2(k+1)^2}}\right)$$

$$= \tan^{-1}\left(\frac{2k(k+1)^2 + k + 1}{2(k+1)^3 - k}\right)$$

$$= \tan^{-1}\left(\frac{(k+1)(2k(k+1)+1)}{2k^3 + 6k^2 + 5k + 2}\right)$$

$$= \tan^{-1}\left(\frac{(k+1)(2k^2 + 2k + 1)}{(k+2)(2k^2 + 2k + 1)}\right)$$

$$= \tan^{-1}\left(\frac{k+1}{(k+1)+1}\right)$$

Hence the formula holds for $n = k + 1$.

Question 16 (b) (iii)

$$\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}}$$

As $n \to \infty, \dfrac{1}{1+\frac{1}{n}} \to 1$

$$\therefore \lim_{n\to\infty} \sum_{j=1}^{n} \tan^{-1}\left(\frac{1}{2j^2}\right)$$

$$= \lim_{n\to\infty} \tan^{-1}\left(\frac{n}{n+1}\right)$$

$$= \tan^{-1} 1$$

$$= \frac{\pi}{4}$$

Question 16 (c) (i)

$$P(k) = \underbrace{\frac{n}{n} \times \frac{(n-1)}{n} \times \frac{(n-2)}{n} \times \ldots \times \frac{n-k+1}{n}}_{k \text{ different numbers from } n} \times \frac{k}{n}$$

$$= 1 \times \frac{1}{n^{k-1}} \times \frac{(n-1)!}{(n-k)!} \times \frac{k}{n}$$

$$= \frac{(n-1)!\,k}{n^k\,(n-k)!}$$

Question 16 (c) (ii)

For $P(k) \geq P(k-1)$,

$$\frac{(n-1)!\,k}{n^k\,(n-k)!} \geq \frac{(n-1)!\,(k-1)}{n^{k-1}\,(n-k+1)!}$$

$$k(n-k+1)! \geq n(k-1)(n-k)!$$

$$k(n-k+1) \geq n(k-1)$$

$$kn-k^2+k \geq nk-n$$

ie $\quad k^2-k-n \leq 0$

Question 16 (c) (iii)

Suppose $\sqrt{n+\frac{1}{4}} > k-\frac{1}{2}$

then $n+\frac{1}{4} > \left(k-\frac{1}{2}\right)^2$

ie $\quad n+\frac{1}{4} > k^2-k+\frac{1}{4}$

ie $\quad n > k(k-1)$

Since n, k integers

$$n \geq k(k-1)+1$$

ie $\quad n \geq k^2-k+1$

$\therefore \quad n > k^2-k+\frac{1}{4} = \left(k-\frac{1}{2}\right)^2 \qquad$ (k is positive)

$\therefore \quad \sqrt{n} > k-\frac{1}{2}$

Question 16 (c) (iv)

$P(k)$ is greatest when k is the greatest integer such that

$P(k) \geq P(k-1)$

ie such that $k^2 - k - n \leq 0$ (from part (ii))

The positive root of the quadratic equation is

$\dfrac{1+\sqrt{4n+1}}{2}$, which is not an integer since $4n + 1$ is not a perfect square.

$\therefore$ we want the largest integer k such that

$$k < \frac{1+\sqrt{4n+1}}{2} = \frac{1}{2} + \sqrt{n + \frac{1}{4}}$$

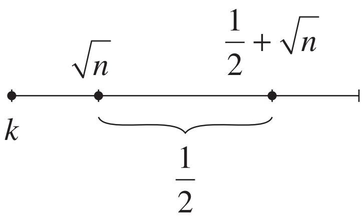
ie such that $k - \dfrac{1}{2} < \sqrt{n + \dfrac{1}{4}}$

which is equivalent to $k - \dfrac{1}{2} < \sqrt{n}$ (from part (iii) and converse)

$\therefore$ k is the largest integer such that

$$k < \frac{1}{2} + \sqrt{n}$$

ie k is the closest integer to $\sqrt{n}$.



or