

EXECUTIVE SUMMARY OF

MATHEMATICS K-10 LITERATURE REVIEW

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1. Theoretical Frameworks

A comparison of the aims of the previous mathematics syllabuses with current syllabuses in New South Wales (NSW) suggests a change in emphasis. More recently, developing students' problem-solving abilities as well as an awareness of the mathematics in everyday situations has become a focus of the curriculum. The most recent syllabus for Stage 5, emphasises working mathematically and applying knowledge and skills to solve problems in a variety of contexts. Common aims in many earlier syllabuses included challenging students, encouraging the use of communication skills to justify mathematical solutions, and developing positive attitudes to mathematics. A review of curricula from other Australian States and Territories as well as overseas systems, identifies similar aims that emphasise the importance of investigations within meaningful contexts and preparing students for the work force.

The current courses in NSW define mathematics as a body of knowledge and skills as well as processes that include problem solving, observation, investigation, representation, generalisation and abstraction. The content is organised into the topics of number, algebra, geometry, measurement, chance and data, or a subset of these. A process strand, working mathematically, has been introduced into more recent syllabuses. Additional supporting documents have been produced that present outcomes and indicators for Years K-6 and outcomes for Years 7-8; an organisation that may lead to a fractured approach to programming in schools.

2. Learning Approaches that Cater for ALL Students

All current syllabus documents are prefaced with a description of the “Nature of Mathematics Learning” that includes the importance of the roles of motivation, interaction, investigation, language approaches, and individual development through intellectual, physical and social growth. Much more is now known about how students learn including the importance of the cultural and affective aspects of mathematics learning.

Learning theories have provided a framework for describing learning. Many learning theories are based on the idea that individual learners construct their own mathematics as the result of their experiences. Further, there is recognition that learning does not occur in a vacuum and that the socio-cultural context of the classroom and the students' lives are of great importance to the learning of mathematics.

In addition, teachers need to challenge each student's current knowledge and promote active engagement, encourage discussion with sharing of ideas, and support reflection. It is critical to determine the current knowledge of each student as well as consider the task demands such as cognitive and processing loads, particularly if concrete representations are to be used by students. In addition to these considerations, the complexity of mathematical tasks can be perceived in terms of linguistic complexity, contextual complexity, representational complexity, operational complexity, conceptual complexity, and intellectual complexity. All of these aspects are important considerations for the teacher if students are to have access to, or to engage with mathematics.

The importance of discussion and argumentation in classrooms during the development of concepts may take time. However, this reduces the need for exercises that may previously have been introduced by teachers' examples demonstrating procedures. Changes in teaching approach will move mathematics away from procedural knowledge to that of mathematical thinking and profound understanding.

3. Working Mathematically and the Role of Problem Solving

Working mathematically encompasses several different processes in mathematics including problem solving and problem posing. Students' ability to solve problems is affected by several factors. Task difficulty, problem representation, metacognitive thought, and the ability to apply knowledge to novel situations influence how students interpret, represent, and undertake problem solving. In addition, learning preference, personal beliefs, and attitudes towards mathematics affect problem-solving performance.

Problem-solving performance seems to be enhanced through regular experiences and systematic instruction that aim to assist the development of problem-solving schema and increase students' awareness of problem-solving strategies. Active engagement promotes higher problem-solving performance; this can be achieved through relevant or realistic contexts, and a variety of problem-solving experiences including open-ended tasks and investigations. Metacognitive strategies that include awareness, evaluation and regulation assist problem-solving performance and need to be developed through classroom discussions. In addition, encouraging students to pose their own problems promotes the process of thinking mathematically.

The use of visual and non-visual reasoning processes play an important role in aiding students' problem-solving abilities. Visual methods allow students to negotiate the difficulties associated with a problem when conceptual limitations do not allow a student to complete the problem quickly and analytically. Students need to be encouraged to evoke a range of different forms of imagery depending on the type of problem

encountered. In addition, visual representations, or diagrams, can be useful to assist learning in topics such as statistics, trigonometry, and spatial reasoning.

4. The ‘Big Ideas’ of Curriculum

The following section summarises the main ideas for consideration in the development of new syllabuses. Many recommended teaching and learning approaches are included in the main Literature Review.

Mathematisation

Mathematisation is a major aim of school mathematics education. It involves students in the process of reinventing mathematics with the guidance of the teacher and it can be defined in two ways. Horizontal mathematisation involves students discovering the mathematical tools that help to organise and solve real-life problems. Vertical mathematisation involves the process of reorganising knowledge within the mathematical system.

Connections

If students can make connections between their school mathematics and the real world, they are much more likely to be successful in their learning. Understanding is also aided if connections are made within the discipline of mathematics thus enhancing the notion that mathematics is an integrated field of study. In addition, it is helpful to make connections with the content of other school subjects.

Argumentation and Proof

The process of justifying one’s actions and solutions to problems is a critical component of the learning process. Students need to be encouraged to develop and present mathematical explanations as well as to listen and to attempt to make sense of the explanations of other students. In this way, they learn to justify not only their own mathematical thinking but also to distinguish between the strengths of other arguments. The type of argumentation and proof used in a mathematics classroom will depend on the mathematical sophistication of the learners and the opportunities which are afforded them to justify their conclusions and methods.

Models and Concrete Representations

While the use of manipulatives in mathematics education is well established, particularly in the early years of school, there is a deal of evidence to suggest that such manipulatives are not automatically helpful in the development of students’ mathematical ideas. For some students, and in some situations, it is the manipulatives and not the mathematics

which become most important. There is a need to reconsider the notion of concrete modelling of concepts since they can be used as models for activity and models for reasoning, rather than simply bridges between concrete and abstract.

Profound Understanding of Fundamental Mathematics

Teachers require a profound understanding of mathematics and should be striving to develop a profound understanding of mathematics in their students. A profound understanding has breadth, depth and thoroughness that enable making connections between topics in the curriculum. Teaching requires making connections for students, considers multiple solutions to problems, and revisits and reinforces basic ideas.

Algebraic Reasoning

Algebra is a language for generalisation, abstraction and proof; a tool for problem solving through equation solving or graphing; and an aid for modelling functions. If algebra begins from mathematical investigations, then teachers need to focus on establishing the recognition of pattern and the various ways of representing data. Verbalising relationships seems to be an important precursor to being able to use symbols satisfactorily. Algebraic thinking should begin with early number experiences and for a younger student may involve investigating a table of basic addition facts for patterns and relationships.

Students reason about their mathematical experiences. They need to be encouraged from an early age to use deductive reasoning that moves from the general to the specific case as well as inductive reasoning that moves from particular cases to generalisations. Several representational forms including tables, graphs, manipulatives, drawings, symbols or written forms need to be considered since students learn from each in different and idiosyncratic ways. It is important to consider the use of materials, as a gap seems to exist between the concrete and symbolic representations. Variables tend to become associated with labels for the concrete items themselves rather than having a numerical value.

Number Sense and Mental Computation

Children enter schooling with number knowledge and everyday number experiences. Teachers need to recognise these experiences and plan teaching activities that build on existing knowledge. Early counting strategies have been documented in several programs and support the assessment and further development of early number manipulation. Number sense development is aided by the use of mental computation strategies to solve problems followed by whole class discussion and sharing of problem-solving approaches. It is also critical that number sense enables students to evaluate the reasonableness of answers obtained on calculators.

Mental computation is viewed as an integral part of learning number operations from the beginning. The intention is to encourage students to develop flexible, idiosyncratic mental strategies, emphasising the mental processes involved. Mental computation represents a bridge between using school mathematics and that used outside the classroom. There needs to be a corresponding reduced emphasis in classrooms on written algorithms. Standard algorithms may develop in class discussions and teachers may, as part of that discussion, show standard algorithms if appropriate. Mental estimates and calculations can be supported by calculators or by paper-and-pencil strategies, often non-standard, that are developed by the students themselves.

It is noted that the development of understanding of the place value system has not necessarily been assisted by the algorithmic use of base-10 arithmetic blocks. Concrete materials do not of themselves provide transparent abstract mathematical concepts. While these blocks can be used as good representations to lead into the standard written algorithms, their use in a drilled, algorithmic way is not seen as necessarily promoting understanding of the algorithms. Nevertheless, making numbers and number relations using concrete materials, and matching these by orally counting by tens is important. Other groupings of materials show various relations, facilitate discussions, and help students develop mental imagery. On these images, numerical relationships are soundly developed. Movement away from concrete representations to mental images and strategies is emphasised. Non-counting-by-ones such as making number jumps are needed and will also assist proportional reasoning. The four constructs of counting, partitioning, grouping, and number relationships are central to the development of multidigit number sense.

Proportional Reasoning

Proportional reasoning is taken to include the understanding of fractions, decimal fractions, percentages, ratio, money and measurement. In early number development, the use of repeated addition for multiplication, the link between arrays and multiplication, and between sharing or grouping for division, need to be emphasised. Simple ideas of doubling and halving as well as sharing need to be considered in the early years to establish the development of proportional reasoning.

Advice regarding the development of proportional reasoning suggests that teachers need to:

- extend interpretations of rational numbers and develop connections between them;
- emphasise the interrelationships within the rational number domain (part-whole, decimal, ratio, measure and operator);

- delay procedures and operations until understanding of quantities and multiplicative relationships are established; and
- develop understandings via instructional models that reinforce links between concepts and procedures.

Overall advice suggests that establishing sound understanding of early ideas in the development of fractions and decimals helps to avoid the misconceptions that students seem to develop about these concepts. Many opportunities need to be provided that enable students to explore, investigate, establish a sound understanding, and develop for themselves procedures for dealing with these numbers.

There has been some challenge to the established sequence of teaching measurement concepts. Traditionally, measurement has been taught through a sequence of awareness of the attribute, comparison of objects, measurement using non-standard units, measurement using formal units, and applications to problems. Some research suggests that formal units and the use of measuring instruments can be successfully introduced earlier than this sequence may suggest.

Spatial and Geometric Thinking

Spatial thinking involves visual imagery processes such as recognition of shapes, transforming shapes, and seeing parts within shape configurations. Early spatial reasoning can be developed through play with a focus on position language, shape names, and making spatial patterns. Imagery and language are two key aspects of spatial thinking. Students need to reason about two dimensional shapes, three dimensional shapes, position in space, and different-sized spaces. Spatial thinking plays a role in making sense of problems and in representing mathematics in different forms such as diagrams and graphs.

There seems to be a focus on rote learning of space concepts, particularly in secondary classrooms. The predominance of naming shapes, often in standard form, is an insufficient activity for students to explore the properties of shapes and to use this knowledge to solve problems.

Data and Probability Sense

The understanding of data plays a critical role in students' ability to interpret and understand their world. Much information is transmitted through graphs, tables and using statistical ideas. Students need to be able to treat reports of data critically, and to establish the veracity of claims. Data handling and analysis can be explored by students as they collect, organise, describe, interpret, and represent their own data. Technology assists with the manipulation of this data.

The early introduction of probability language and experiences can assist in the avoidance of misconceptions in problems where intuition alone is insufficient to solve them. From an early age, children encounter the language of chance and begin to realise that there is an element of chance in many of the experiences they have in their lives. As they expand their experiences, students begin to quantify chance through comparison, to order the likelihood of more than two events, and to attach numerical values to these likelihoods. By developing their thinking about chance and its quantification, students will be able to make sensible decisions in situations of uncertainty.

The key concepts that should be taught in primary and middle school include:

- sample space;
- experimental probability of an event;
- theoretical probability of an event;
- probability comparisons;
- conditional probability; and
- independence.

While there is ample evidence of student misconceptions in the probability area, it would seem that these point to a need for further careful introduction of the probability ideas and reasoning rather than resulting in the deletion of probability from the curriculum, even in the early years.

5. Use of Calculators and Computer-Based Technology

The use of technology in classrooms has the power to change teaching and in turn can affect what students learn and how learning is accomplished. Several studies report that using technology to support mathematics learning has had significant effects on student engagement and understanding, however, availability of equipment and support for teachers seem to be critical components for successful outcomes. Calculator use in early number learning has met with overwhelming support from teachers since there were surprising outcomes as students explored bigger numbers and investigated number patterns. Geometry software packages such as Cabri Geometry and The Geometer's Sketchpad have the potential to significantly enhance students' spatial reasoning abilities. Spreadsheets can be used for high school mathematics by being a tool for data collection and manipulation and as a catalyst for the exploration and examination of patterns emerging from the data.

The use of graphing technologies has the potential to broaden the curriculum. Computer systems that can be used flexibly for teaching algebra show they improve facility in

algebraic manipulation and in use of different representational forms. The use of graphics calculators by students supports mathematical learning and explorations in unfamiliar contexts. However, time needs to be available for students to develop facility with these tools. The development of computer algebra systems (CAS), along with their potential for linking numerical, graphical, and symbolic representations, is significant and is expected to have an impact on curricula, assessment, and teaching, since it challenges the algorithmic nature of the current teaching of algebra and graphing.

The use of such technologies in classrooms has implications for assessment. Currently, Western Australia has distinguished tasks for the final examination that are especially for graphics calculator use and others not for calculator use. In NSW, graphics calculators can be used in the School Certificate Mathematics Test and in the Higher School Certificate examination for the new General Mathematics course.

6. Standards-Referenced Assessment in Mathematics K-10

In standards-based assessment, curricula provide syllabus and performance standards for different levels of achievement. For each task, teachers develop task-specific criteria, which are declared in advance of implementation of the task. High quality assessment integrates with learning, values contextualised teacher judgements, and produces fair and comparable results. In this sense, assessment must meet validity and reliability requirements; both of these can be obtained if assessment fits with educational goals, and if the contextualised judgements that involve the teachers are based on firm evidence across several tasks.

A variety of assessment tasks are recommended including the use of rich tasks, although it is acknowledged that these can be time consuming to design. Several general criteria for assessing responses to tasks are recommended. It is noted that a significant proportion of responses to multiple-choice and short-answer items in tests do not seem to match with responses to interview questions on the same topics, indicating that conceptual development is often not assessed well by tests. These findings highlight the need for a combination of tasks and approaches to assess students' understanding.

7. Developments in Other Australian States and Territories and Internationally including Continuity and Course Arrangements

To consider possible curriculum arrangements that promote continuity of learning across the continuum of K-10, curriculum structures from other Australian States and Territories were examined.

South Australia - a new curriculum is currently being trialed that covers the range from Birth-Year 12. The draft framework consists of four bands (Birth to Year 2, Years 3-5,

Years 6-9, and Years 10-12) that are not meant to parallel school structures. There is coherence across the bands from which curricula in each of the key areas can be developed. The draft framework statements do not differentiate courses of study for particular groups of students, although there are statements concerning the inclusion of all students.

Australian Capital Territory - the curriculum framework is based on the *National Statement* with four bands (Preschool-Year 1, Years 2-4, Years 4-7, and Years 7-10). The framework is written as one document with no differentiation for students of varying abilities although there is a suggestion that more able students should be provided access to some of the experiences from the Years 11-12 statements.

Queensland - a draft framework for Preschool to Year 10 suggests a coherent approach to the syllabus with no indication of separate courses across the years. However, there are certain outcomes, marked 'discretionary', that are considered desirable for those students who have demonstrated all previous outcomes.

Victoria - the *Standards Framework* provides a detailed mathematics curriculum for Years Prep-10 providing six levels of achievement. It is expected that most students would demonstrate achievement at Level 6 by the end of Year 10. In addition, there are Level 6 extension outcomes for students who have met Level 6 outcomes before they have completed Year 10.

Western Australia - the *Curriculum Framework* establishes learning outcomes for all students from Kindergarten to Year 12. This is not a curriculum or syllabus but is a document guiding the development of learning and teaching programs according to the circumstances of schools, teachers and the needs of students. It is presented as a coherent whole and does not prevent schools from offering programs that enable students to achieve outcomes beyond those specified in the document.

Tasmania - uses the *National Statement* as the basis of the mathematics syllabus.

Northern Territory - the *Curriculum Framework* is in the first Pilot stage and contains the notion of essential learnings that are "critical processes that all learners should develop as a result of their formal schooling". The essential learnings provide broad organisers for the syllabus.

In addition to the above, the curriculum structure from five countries selected from North America, Western Europe and Asia were considered. The first three had a higher level of

achievement than Australia in the recent Third International Mathematics and Science Study (TIMSS) whereas the last two had a lower level of achievement than Australia.

Singapore - the current curriculum organises mathematics into two separate courses (normal and extended) from Year 4 and then into four in secondary school (special, accelerated, normal academic, normal technical). In 1998, about 30% of the curriculum content was removed to allow more time for problem solving and investigation. It is noteworthy that almost one third of school students participate in extra tuition outside of normal school hours.

Japan - a highly centralised national curriculum with textbooks authorised by the Japanese Ministry of Education before they can be used in schools. All students are taught in same-age, mixed-ability classes with the whole class proceeding through the curriculum at the same pace. In addition, there is an extensive system of private education in the form of evening classes.

Netherlands - there is no centralised decision-making system, with schools developing their own curriculum. The curriculum is usually determined from the content of three sources of information including:- choice of a particular mathematics textbook series, key goals as determined by the government, and content recommendations from government documentation. More recently, trajectories of learning that are designed to provide an overview of how students' mathematical understanding might develop have been written.

United States of America - the *Principles and Standards for School Mathematics* (NCTM, 2000) is a key planning document that is consulted by the diverse education systems in the USA. In this document six principles and ten standards are used to provide coherence and to organise statements in each of the year bands.

Implementation of curricula is quite diversified but schools are supported by professional development and materials using approaches based on the *Standards*. These have been shown to be highly successful. There is no indication of diversity of courses although textbook series seem to be structured to cater for different ability groups.

England and Wales - there is one mathematics curriculum for government schools in England and Wales and this is embodied within the *National Curriculum*. Several revisions of this document have taken place since 1988, with the latest in conjunction with the National Numeracy Strategy. The curriculum is broken into four Key Stages with the fourth Stage split in mathematics into two courses (foundation and higher). The National Numeracy Strategy is highly prescriptive and has had a major impact on the teaching of mathematics in primary schools.

8. Teacher Reform

A key issue in change for effective mathematics learning is that of changing teachers' knowledge and practice. Several projects in Australia and overseas have sought to develop teachers' knowledge. These projects indicate that an increase in teachers' knowledge has the potential to change practice in classrooms. However, resources and time for individual student assessment need to be addressed.

A full understanding of the role of the teacher is necessary if new approaches are to be successfully implemented. An increased awareness of the teacher's role in implementing collaborative student groups and leading whole class discussion, as well as using problem solving and investigative approaches, is critical if these strategies are to be successful. In conclusion, research into teaching and learning has changed since researchers are neither equating learning with immediate recall, retention, and transfer, nor equating understanding with achievement. Learning is no longer only an end product but an ongoing activity, and the importance of social interaction on learning is recognised.