

# Literature Review: Report on Investigational Tasks in Mathematics in Years 9–10 for Advanced and Intermediate Students

## Preface

The current *Mathematics Years 9–10 Syllabus* was released by the Board of Studies in 1996 for implementation in Year 9, 1997. Schools were advised at that time that the *Mathematical Investigations* strand of the Advanced (pp 173–179) and Intermediate (pp 151–155) courses was to be considered draft until further notice.

The Board advised in Official Notice 16/99 (*Board Bulletin* vol. 8, no. 3, April 1999) of a process for the review of the status of *Mathematical Investigations*. The first stage of this process, undertaken in 1999, consisted of a review of research literature on the use of longer investigations in the teaching and learning of Mathematics.

The Board decided that remaining work in the review of the status of *Mathematical Investigations* should form part of the review of the Mathematics syllabuses for Years 7–10 announced by the NSW Government. The Years 7–10 review is being undertaken to ensure that the syllabuses for Years 7–10 meet the needs of all students and appropriately prepare students for courses in the New Higher School Certificate. In the meantime, the *Mathematical Investigations* strand will remain optional.

## Background

This literature review,<sup>1</sup> *Report on Investigational Tasks in Mathematics in Years 9–10 for Advanced and Intermediate Students*, was undertaken by academics from The University of New South Wales, The University of Sydney and The University of Western Sydney, all of whom specialise in mathematics education. Mr Lindsay Grimison and Associate Professor Lloyd Dawe of The University of Sydney were responsible for 'Part A: Report Supporting Investigations for the Advanced and Intermediate Courses of the NSW Mathematics Years 9–10 Syllabus'. Dr Paul Ayres, then of The University of Western Sydney (Nepean),<sup>2</sup> and Professor John Sweller, University of New South Wales, were responsible for 'Part B: Why Investigations Should Not Be Compulsory in the Years 9–10 Mathematics Courses'. Part C represents the joint conclusions of the authors of the report.

**Note:** The views expressed in this literature review are those of the authors and not necessarily those of the Board of Studies NSW.

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<sup>1</sup> For this review, each of the two research teams was required to nominate a maximum of 15 key references in support of their arguments.

<sup>2</sup> Dr Ayres joined the School of Education at The University of New South Wales in 2000.

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## Part A

# Report Supporting Investigations for the Advanced and Intermediate Courses of the NSW Mathematics Years 9–10 Syllabus

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### Introduction

This paper is being written in the context of vast changes to the School Certificate and Higher School Certificate in NSW. The NSW Government, in an effort to increase the equity, rigour and quality of the school curriculum, has initiated a set of reforms that touch every area, including teaching, learning and assessment. In recent years students, conscious of the changes made to scaling procedures, have preferred easier options to maximize their UAI ranking, an issue which is only now being addressed. In the case of HSC mathematics, the enrolment of students in the more demanding 3 and 4 unit courses has fallen dramatically. The 2 unit courses in Mathematics in Society and Mathematics in Practice, which together account for over half the total mathematics candidature, have been replaced by a single course. Further, the attempt to introduce technology and widen assessment practices has met with resistance from teachers. The calculus-based 2, 3 and 4 unit mathematics courses have remained untouched. In the senior school, teaching and learning experiences have tended to remain generally dull and uninspiring, with teachers preferring direct instructional techniques to maximize marks. At the same time, the number of students electing to train for mathematics teaching as a career has also fallen dramatically. There is an urgent need to change this situation, and inject a new vitality into mathematics courses.

In 1996, the NSW Board of Studies released the syllabus documents for the three courses of Stage 5 (Years 9–10) Mathematics — Advanced, Intermediate and Standard. They were implemented in Year 9 in 1997, and the 1998 School Certificate Test based on the new syllabus. The Advanced and Intermediate courses included a strand called *Mathematical Investigations*, which was designed as an integral part of the core in each course. The syllabus writers have clearly linked this approach to expected outcomes addressing equity issues; problem-solving; values and attitudes towards mathematics and mathematics learning; language development; collecting; analysing and organising information; using technology; and working with others and in teams. Thus the use of mathematical investigations is seen as a far-reaching teaching strategy which has implications for future mathematical study and the world of work. It is not just a tacking-on to traditional content. It is one among many teaching strategies which the syllabus writers acknowledge as having a rightful place in the mathematics classroom (see syllabus, *Mathematics Years 9–10 Advanced Course Syllabus*, p 19).

The outcomes for the Advanced and Intermediate courses are organised into seven categories: *Values and Attitudes*, *Working Mathematically*, *Geometry*, *Number*, *Measurement (and Trigonometry, Advanced Course)*, *Chance and Data* and *Algebra (and Coordinate Geometry, Advanced Course)*. The outcomes for *Working Mathematically* relate to the overarching skills that are expected to be achieved by students in the strands. There are nine stated outcomes for this category (*Advanced Course*, p 30). One of them involves planning, carrying out and reporting on an extended mathematical investigation with persistence, autonomy and flexibility.

## **The Issue**

It is the inclusion of an *extended* mathematical investigation, particularly for Advanced level students, which has caused concern for a few university mathematicians and, to a lesser extent, some mathematics educators. There is the fear that this will take teaching time away from content, and could lead to reduced standards. But lying behind this is the belief that direct instruction is the most efficient method of teaching secondary mathematics. According to research carried out by Sweller (1999) and his colleagues, there is good evidence to support this claim. However this should not rule out the appropriate use of other methods, as learning is much more than a matter of efficiency. The issue is how to accommodate student-centred learning with teacher-centred instruction.

Each course mandates that students at Advanced and Intermediate level undertake at least one extended mathematical investigation task that would take about five hours, and ‘... could be done when the core has been completed or could be integrated into the teaching of the core’ (*Advanced Course*, p 22).

In 1996 it was determined that *Mathematical Investigations* would be moved from the core to the options in each course, as a result of concerns raised by some who considered that *Mathematical Investigations* were taking away from core hours in mathematics. Since then, some schools have still incorporated investigations into their Advanced and Intermediate programs in Years 9–10; others have chosen to omit them completely. It is now time to settle the matter.

As an integral part of the *Chance and Data* strand of the Advanced, Intermediate and Standard courses, students are expected to carry out an investigation (individually or in groups) in which they experience the main aspects and methods of planning, organising, analysing and evaluating data. The support document further expands on the possible details of such strategies. It is interesting that *all* three levels incorporate a statistical investigation as a part of this strand. It is assumed that these sections of the courses are not under threat, and have not been moved from the core to the options. It seems to us that the enunciated approach would greatly enrich the teaching and learning of *Chance and Data* for both teachers and students.

In summary, the issue is one of compulsory or voluntary use of an extended piece of investigational work in the core of the syllabus at Advanced and Intermediate levels. Those who are arguing that it should be voluntary are not concerned with the lower ability level. Further, they claim that if there are advantages to investigations, the onus of proof lies with those who support them, proof being confined to experimentally controlled studies as the only valid research methodology. This paper is an attempt to answer that criticism.

## **What is a Mathematical Investigation?**

Before we start it is important to understand what we mean by a 'mathematical investigation'. Is it any different from problem-solving generally? In fact, yes. However the difference is not often made explicit in the literature, and people still often use the terms synonymously.

In 1982, the *Committee of Inquiry into the Teaching of Mathematics in Schools* (Cockcroft, 1982) clearly distinguished between the two (see paragraphs 249–252). The Committee recommended that mathematics teaching at all levels should include opportunities for:

- exposition by the teacher
- discussion between teacher and pupils and between pupils themselves
- appropriate practical work
- consolidation and practice of fundamental skills and routines
- problem-solving, including the application of mathematics to everyday situations
- investigational work.

However, teachers at the time were far more comfortable with problem-solving, which they described as:

'making maths relevant; interpreting the question; looking for a solution; using the teacher as a resource; original thought; applying maths to different, perhaps everyday situations; closed; answering exam questions.'

On the other hand investigational work was characterised by these teachers as:

'open ended; finding patterns; self-discovery; reducing the teacher's role; not helpful for examinations; not worthwhile; not doing real maths; using one's own methods; being exposed; limited to the teacher's experience; not being in control; divergent.' (Edmonds & Knights 1983, p 76)

Leaving aside teachers' misunderstandings and fears, the most helpful classification is to think of problem-solving and investigational work in terms of being more 'convergent' and more 'divergent' in purpose respectively. In problem-solving, students converge towards a solution and the focus is on obtaining that solution. Investigations are characterised as open-ended activities, where the focus is on the processes of thinking rather than the solution. Students are not expected to produce the 'right answer' but are required to explore

possibilities, make conjectures, and convince themselves and others of what they find. The emphasis is on exploring a piece of mathematics in all directions. The journey, not the destination, is the goal. The idea is to enable the student, more particularly advanced students, to make more connections within and between different branches of mathematics by using contrasting representations, reflecting on false trails and mistakes, as well as making exciting breakthroughs in thinking mathematically. Metacognitive skills are called into play to monitor students' thinking, and articulating them adds strength to the process.

Cockcroft argues that:

'It is necessary to realise that much of the value of an investigation can be lost unless the outcome of the investigation is discussed. Such discussion should include consideration not only of the method which has been used and the results which have been obtained, but also the false trails which have been followed and mistakes which may have been made in the course of the investigation.'

(Cockcroft 1982, p 74)

Secondly, looking at the finished product, students' mathematical investigations share two other important characteristics. On the whole, it is the students' work and not the teacher's, and the students have written about the mathematics that they have been doing. The Years 9–10 syllabus writers (p 173) have suggested a written and/or verbal report of the findings of an extended investigation. One can only applaud the committee for the opportunity to strengthen mathematical literacy skills. Far from being an imposition on teachers, this is a positive step towards encouraging mathematics teachers to take literacy seriously, even if it is confined to their own subject area. It is also an encouragement to use technology as a tool to aid the investigation, a strategy which will have enormous potential benefits for both students and teachers.

## **The Brief**

The brief commissioned by the Board of Studies was to 'review the literature relating to research on the use of longer investigations in the teaching and learning of mathematics'. To maintain a balance in reporting, two academics supporting the case for inclusion of an extended investigation in the Core, and two opposing this view, were asked to respond.

## **The New Philosophy of the Course**

The 1996 syllabus outlines of the three Years 9–10 courses in Mathematics contain recommendations for teaching which depart markedly from previous syllabus documents (*NSW Secondary Schools Board Mathematics Syllabus and Notes* 1962, 1975, 1983). In general these documents were concerned with content for each of the Years 9–10 levels

and made few recommendations about other matters. The 1996 syllabus takes up a variety of issues about which the mathematics teacher should be made aware — including equity principles, problem-solving and investigating, communication (especially the role of language in the mathematics classroom), appropriate use of technology, group work, and alternatives in assessment. Many teachers in the past have not given much concern to these issues, and it was opportune that they should be highlighted. Australian and overseas research in mathematics education has addressed these issues in the past 20 years, and it was important for our teachers to be made aware of these factors as they impinge on the teaching and learning process in the secondary mathematics classroom.

Good teaching in mathematics is enhanced by a consideration of a variety of issues that influence what occurs in the classroom. Certainly the traditional approach to mathematics learning, where teacher-directed information is transmitted to the passive learner, works for many students — in fact it probably worked well for most mathematics teachers when they were students of mathematics at secondary school. Yet this was perhaps the only approach to learning mathematics that these teachers, as students, had ever known. It is our experience that this traditional transmission of mathematics knowledge is still far and away the most popular model occurring in most secondary mathematics classrooms of this State. However, it is certainly not the only model. A problem-solving approach is also useful for the effective teaching of some topics to some learners. We are not saying that a problem-solving approach to teaching mathematics is better than the more traditional method. Problem-solving also has its limitations. We are saying that a raft of teaching/learning approaches should be used, including open-ended problem-solving tasks incorporating the planning, execution and reporting of an extended mathematical investigative task.

Research in the last 20 years has shown that many students regard secondary mathematics as a set of isolated topics which must be handled by using the correct formulae, recalled from previous similar exercises. It is widely believed that there is only one correct way to solve a mathematics problem, and that the goal of doing mathematics is to obtain the correct answer (Schoenfeld, 1988, 1992; Lambert, 1990; Greer, 1993). However, students have little understanding or ownership of these algorithms and certainly don't have any idea of how the various mathematical topics can be integrated, or have any application in the real world. Schoenfeld writes:

‘Mathematics curricula have been chopped into small pieces, which focus on the mastery of algorithmic skills. Most textbooks present problems that can be solved without thinking about the underlying mathematics, but by blindly applying the procedures that have just been studied. Indeed, typical classroom instruction subverts understanding even further by providing methods for solving problems

that allow students to answer problems correctly, without making any attempt to understand them.’ (Schoenfeld 1988, p 163)

What is most significant about a mandatory investigational task is that students are required to apply their mathematical knowledge and skills to solving an open-ended problem for which the answer is not immediately obvious, and cannot be determined by use of the correct formula or learned procedure. For most students this will be a new challenge. If accompanied by careful planning, done individually or in groups and with appropriate teacher involvement, it should result in some real ownership of the process and product of the assigned task. An oral presentation often accompanies the final written report. Again, this could be an individual or group activity. Allocating five hours of teaching time out of a minimum of 200 hours over Years 9 and 10 to this task seems appropriate, and would not really take away from other topics. Such an activity would consolidate knowledge and skills already learned and integrate them in a way that other approaches would find more difficult. Students completing investigational tasks would not see themselves as passive consumers of others’ mathematics (usually the teacher’s), but instead would have a real hand in choosing strategies, not immediately obvious, to solve these problems. In so many classrooms, mathematics is usually presented as a body of facts and procedures. What is needed is a sense of exploration, or, as Schoenfeld says:

‘the possibility that the students could make sense of the mathematics for themselves.’ (Schoenfeld 1988, p 161)

By contrast, what usually happens is that:

‘students gain the clear impression that someone else’s mathematics was theirs to memorize and spit back.’ (loc. cit.)

The *Handbook of Educational Psychology* (Berlinger & Calfee, 1996) contains a comprehensive and international review of research in mathematics teaching and learning undertaken in the last 20 years from an educational psychology point of view (Chapter 16). The chapter authors (De Corte, Greer and Verschaffel) make it quite clear that mathematics teaching and learning at the school level must contain experiences wider than those exemplified in the traditional transmission-of-knowledge model. They should include the solution of open-ended problems in which the solution is not immediately obvious to the learner and has not been directly assigned by the teacher. They claim that recent research has shown:

‘that traditional teaching tends to divorce mathematical problem-solving from children’s real-world knowledge by using an impoverished and stereotyped diet of standard word problems that can be modelled and solved by the straightforward

application of one or more arithmetic operations with given numbers.’ (De Corte et al. 1996, p 505)

They advocate:

‘a radical departure from traditional, weak classroom environments based on the view that mathematics learning is a highly individual activity consisting mainly in absorbing and memorising a fixed body of decontextualised and fragmented knowledge and procedural skills transmitted by the teacher.’ (ibid., p 521)

The philosophy inherent in the 1996 NSW Years 9–10 syllabus reflects this new thinking and represents international trends in the teaching of mathematics. The review, furthermore, recommends the broadening of assessment practices in the classroom beyond traditional pencil and paper tests. The inclusion of broader assessment tasks which count towards the internal school assessment of mathematics ensures that teachers and students use these methods on occasion. These alternative assessment devices include the assessment of students’ work through:

‘extended tasks lasting many hours, often spread over several weeks. The range of such tasks include investigations in pure mathematics, exercises in mathematical modelling, statistical investigations, and design/construction tasks.’ (ibid., p 531)

### **Investigational Work in Other States and Overseas**

Over the past ten years, investigational work in Years 9–10 mathematics has appeared in the syllabuses of the USA, the United Kingdom and most of the States of Australia. The 1989 *NCTM Curriculum and Evaluation Standards Document for School Mathematics* made it mandatory in the USA and despite the ‘math wars’ there in recent years, the 1998 newly drafted *Principles and Standards for School Mathematics* continues to advocate Problem-solving as a Standard in Mathematics for K–12. Within the Standards for Grades 9–12, open-ended problem-solving tasks continue to be recommended and advocated (*NCTM*, 1998).

In the United Kingdom, investigational work has assumed an integral part of the coursework for the GCSE (equivalent to our School Certificate) throughout the 90s and was cemented into the 1996 National Mathematics Curriculum for England and Wales for Stages 3 and 4 (equivalent to our Years 7–10). This was seen to be part of Using and Applying Mathematics, equivalent to our *Working Mathematically* strand, in which students should be given opportunities to:

- ‘1. use and apply mathematics in practical tasks, in real-world problems and within mathematics itself;

2. work on problems that pose a challenge;
3. encounter and consider different lines of mathematical argument.'

(Department of Education, UK, 1996)

Typically, investigations form an important part of the coursework in UK Stage 3 and 4 classes, and this coursework contributes 20% of the marks in the external assessment at age 16. The method of assessment for the coursework is usually by open-ended investigational tasks (eg SMP 11–16 examination guidelines). The UK Government is currently reviewing the National School Curriculum, and in May 1999 released its draft for discussion (United Kingdom Qualifications and Curriculum Authority, 1999). In the Stage 4 Foundation and Stage 4 Higher Mathematics Syllabus, 'Using and Applying Mathematics' remains an overarching strand, but has been more tightly linked to each of the other content areas (number and algebra, shape, space and measures and handling data). However, there is no suggestion that investigational tasks should not continue. The Handling Data section contains a specific investigational task, which is similar to that found in the 1996 NSW Years 9–10 *Chance and Data* strand for Advanced and Intermediate levels.

The other States of Australia have all incorporated problem-solving and investigational approaches into their secondary mathematics curriculum, alongside more traditional methods. It is well known that in Victoria, Queensland and Western Australia this approach has been adopted throughout the secondary mathematics curriculum and has been reported quite widely in the literature and at relevant conferences. South Australia and Tasmania have also incorporated such developments, and these are reflected in their assessment of a broader range of performance in mathematics. In Victoria, especially, the changes have most affected the Years 11 and 12 VCE Mathematics courses and these changes have flowed down the years of secondary schooling into Years 9 and 10 and below (Stephens & Money, 1993). Investigational tasks have been an integral part of the secondary mathematics curriculum in Victoria for nearly 10 years and we can learn much from their experience. But the most important thing that can be learned is that these tasks endeavour to offer experiences in learning mathematics at the school level which complement the more traditional approaches, and certainly influence the students' attitude positively to the application of the subject to real-life issues.

## **Literature Review**

We have chosen fifteen articles from the recent literature, all of which support and encourage the use of an investigational approach in the mathematics classroom at the mid-

secondary level. Some of these are chapters from books; others are articles appearing in international refereed journals. One is a book on the topic.

Boaler, J, 1997a, 'Equity, empowerment and different ways of knowing', *Mathematics Education Research Journal*, vol. 9, pp 325–42.

Boaler, J, 1997b, *Experiencing School Mathematics: Teaching Styles, Sex and Setting*, Open University Press, Buckingham, UK.

Boaler, J, (1998), 'Open and closed mathematics: student experiences and understandings', *Journal for Research in Mathematics Education*, vol. 29, pp 41–62.

Chapin, S, 1998, 'Mathematical investigations — powerful learning situations', *Mathematics Teaching in the Middle School*, vol. 3, pp 333–38.

Doyle, W, 1988, 'Work in mathematics classes: the context of students' thinking during instruction', *Educational Psychologist*, vol. 23, pp 167–80.

Frobisher, L, 1994, 'Problems, investigations and an investigative approach,' in *Issues in Teaching Mathematics*, eds A Orton & J Peel, Cassell, London.

Garfield, J, 1993, 'An authentic assessment of students' statistical knowledge', in *Assessment in the Mathematics Classroom*, eds N Webb & A Coxford, NCTM, Reston, USA.

Haines, C & Izard, J, 1994, 'Assessing mathematical communications about projects and investigations', *Educational Studies in Mathematics*, vol. 27, pp 373–86.

Maher, C, 1991, 'Is dealing with mathematics as a thoughtful subject compatible with maintaining satisfactory test scores?: A nine-year study', *Journal of Mathematical Behavior*, vol. 10, pp 225–48.

Oliveira, H, Segurado, I, Da Ponte, J & Cunha, M, 1997, 'Mathematical investigations in the classroom: a collaborative project', in *Developing Practice: Teachers' Inquiry and Educational Change*, eds V Zack, J Mousley & C Breen, Deakin University Press, Geelong.

Orton, A & Frobisher, L, 1996, *Insights into Teaching Mathematics*, chapter 4, Cassell, London.

Sigurdson, S & Olson, A, 1992, 'Teaching mathematics with meaning', *Journal of Mathematical Behavior*, vol. 11, pp 37–57.

Stephens, M & Money, R, 1993, 'New developments in senior secondary assessment in Australia', in *Cases of Assessment in Mathematics Education: An ICMI Study*, ed M Niss, Kluwer, Dordrecht.

Sullivan, P, 1992, 'Using open questions for teaching: a classroom experiment', in *Space — the First and Final Frontier: Proceedings of the Mathematics Education Research Group of Australasia*, pp 510–20.

Zehavi, N, Bruckheimer, M & Ben-Zvi, R, 1988, 'Effect of assignment projects on students' mathematical activity', *Journal for Research in Mathematics Education*, vol. 19, pp 421–38.

Frobisher traces the inclusion of an investigative approach in the teaching of secondary mathematics in Britain back to the work of the Association of Teachers of Mathematics in the 1960s (Frobisher in Orton & Wain, 1994; Orton & Frobisher, 1996). The recommendations of the Cockcroft Report (Cockcroft, 1982), that problem-solving and investigations should be incorporated into the mathematics curriculum, really reinforced this thinking for teachers of mathematics at all levels. The aim was to change the role of schoolchildren from that of passive recipients of transmitted knowledge to that of active participants in their own learning. These recommendations were widely read and, in time, stimulated changes in the philosophy of mathematics education around the world (Ernest, 1991). Frobisher distinguishes between problem-solving and investigations by defining the former as associated with known goals, while the latter present open situations which for children have no known outcomes. Frobisher takes the argument further by suggesting that it is not the teaching of investigations as another topic in the syllabus which is important; it is instead the creation of an investigative environment in the classroom which encourages the students to regard their mathematics as a relevant pursuit.

Writing in 1996, Orton and Frobisher distinguished investigational tasks from problems by defining the former as problems which are open-ended, and provide the students with the freedom to determine the goals they wish to attain. They state:

'This independence and autonomy [are] not possible in problems having a precise and unambiguous goal with a known and well-established method of solution ... In practice, an investigation is characterised by a spirit of dynamic engagement on

the part of the investigator. There is a strong element of curiosity as the investigator moves into the unknown with many possible paths to choose and follow. Some of these paths lead to exciting and novel revelations as well as mathematics which for the child is new, others quickly lead nowhere with few conclusions reached.' (Orton & Frobisher 1996, p 32)

Orton and Frobisher discuss in detail the way that such an investigative approach should be gradually introduced into the mathematics classroom. They include statements that suggest the initial reaction of many mathematics teachers to this change in methodology is actively resisted and it certainly poses a challenge to their past experience in the classroom. The 'tacking on' of investigations to the traditional content is not appropriate and mirrors the 'tacking on' of problem-solving as an additional topic by many teachers to the 1989 Years 7–8 Mathematics Syllabus for NSW. What is needed, according to Orton and Frobisher, is the 'introduction of an investigative approach to the entire curriculum,' rather than a mere 'remodelling of the curriculum to involve investigations' (p 34).

Orton and Frobisher go on to describe the implementation of a number of longer investigations into a classroom (pp 39–52). They all contain these characteristics:

- they relate directly to the learning of a part of the program of study or the set curriculum;
- they either set up mathematical learning activities or situations for which no goal is specified, or set up problems for which the goals are clearly stated, but which encourage the students to internally and/or externally investigate the problem;
- they allow children to determine for themselves which knowledge and skills are needed for the learning activity;
- they provide children with the opportunity to use processes and strategies in their learning.

Walter Doyle conducted a seminal study of work practices in a number of US mathematics classrooms in the 1980s which showed quite clearly that most students in most mathematics lessons follow repetitive teacher-directed tasks that never call on the students to make decisions on how to apply their learned mathematics to a novel situation.

He describes mathematics lessons in which teachers almost always:

'neglect interpretive analysis and strategic decisions in presenting lessons or structuring tasks for their students. Instead they focus instruction on

computational procedures and accuracy of calculations. In addition, for many maths assignments, students know in advance which computational procedures they are to use in solving problems. Word problems involving the addition of fractions are assigned within the context of instruction on adding fractions. As a result, students have little need to formulate semantic models of problems. Under such circumstances, students' decisions are quite limited, and it is unlikely that they will learn when to use computational skills or how to apply them to unfamiliar situations.' (Doyle 1988, p 171)

Doyle selected two junior high school mathematics classes (Year 7 and Year 8) and used participant observation, analysis of students' work and interviews with teachers and selected students over a period of time. Familiar and novel tasks were compared in the mathematics classrooms. Doyle characterises familiar work as having:

'little ambiguity about what to do and how to do it and little risk that things will go wrong along the way. Such work creates only minimal demands for students to interpret situations or make decisions within the content domain.'

The results indicated that familiar work accounted for most of the academic tasks that students accomplish in most classrooms.

Novel work is characterised by tasks which require the student to assemble information and operations from a number of sources in ways that have not been explicitly laid out beforehand by the teacher, where emphasis is put on students making decisions about what to produce and how to produce it. Doyle found that some teachers avoid struggles over work demands by simply eliminating any novel activity at all (p 174). Few real challenges are given to the students. Instead, stereotyped step-by-step procedures are presented for most of the time, in most of the mathematics lessons, with little real understanding occurring (see also Schoenfeld, 1988). There should be real concern if this is the case. An open-ended investigational task will at least give students one opportunity to use their acquired mathematical knowledge and skills in a 'novel' fashion to solve an extended problem which does not draw immediately upon rehearsed methods.

Although he wrote in 1988 in the American context, Doyle could have been describing a typical mathematics lesson in a junior or senior secondary classroom in NSW in 1999. Clearly these types of lessons will continue to be taught often, but other types of mathematical experiences should also be introduced occasionally to the class where students are able to

feel a greater deal of ownership of the mathematical processes they can use. The mandatory inclusion of an extended investigational task is a start in this direction.

Boaler conducted a three-year study of some 300 children and the mathematics staff of two secondary schools in the United Kingdom. This study was written up in the book *Experiencing School Mathematics: Teaching Styles, Sex and Setting* (Boaler, 1997b) and in a number of journal articles included in our literature (Boaler, 1997a, 1998). A wide range of qualitative and quantitative data is marshalled in this study to produce case studies of two contrasting mathematics departments — one traditional and the other fairly progressive. The different school approaches were compared and analysed using student interviews, lesson observations, questionnaires given to students and teachers and a range of different assessments, including the GCSE examinations. The study indicates that students did just as well on external tests at the GCSE level using either method. In fact, the students in the less traditional classrooms did slightly better. They were certainly not disadvantaged by using less traditional teaching and learning styles. Boaler gives evidence that students under the less traditional approaches were more motivated, enjoyed their mathematics learning much more, and were able to more easily apply their acquired mathematical knowledge and skills to solving real-world problems requiring mathematical thinking.

Students in the less-traditional mathematics department (the fictitious name Phoenix Park School was used in the study) frequently applied a problem-centred approach to their mathematics learning, and made use of a number of longer mathematical investigations. In the more traditional classrooms at Amber Hill School (the fictitious name for the other school), students followed a fairly rigid textbook approach which involved them in following a topic through from beginning to end, and then moving on to another one. Boaler (1998) comments that these studies:

‘show the ways that these two approaches encourage different forms of knowledge. Students who followed a traditional approach developed a procedural knowledge that was of limited use to them in unfamiliar situations. Students who learned mathematics in an open-ended, project-based environment developed a conceptual understanding that provided them with advantages in a range of assessments and situations. The project students had been “apprenticed” into a system of thinking and using mathematics that helped them in both school and non-school settings.’ (p 41)

Boaler comments on how many of the students at Amber Hill had developed an *inert* knowledge of mathematics that they found difficult to apply or use in all but textbook

questions. When an examination question demanded anything other than the rehearsal of a rule or procedure, such students found great difficulties in applying their mathematical knowledge to this slightly unfamiliar situation because they had no real understanding of the underlying mathematical concepts. To them, mathematics was merely a set of unconnected rules and procedures, and these were the teachers' rules — there was no internalisation of the mathematical concepts involved. By contrast, the students at Phoenix Park were able to interpret and perceive what to do in different mathematical situations and could apply their mathematical knowledge and skills accordingly. These students could use their mathematics in novel situations.

In the NSW context we are not advocating a complete transformation of Years 9–10 Advanced and Intermediate Mathematics classrooms to those studied by Boaler at Phoenix Park School. Quite the contrary. We are advocating only that these students be afforded an opportunity to complete a five-hour, teacher-directed investigation in mathematics that will give them a unique chance to apply their acquired mathematical knowledge and skills to the planning, execution and reporting of a worthwhile task, for which the solution is not immediately obvious through the use of a rule, formula or procedure.

Boaler's findings are supported by two related studies which were reported in the United States and Canada by Maher (1991) and Sigurdson & Olson (1992). The Maher study reports on an analysis of standardised mathematics scores covering a nine-year period in K–8 schools in New Jersey, USA. Over a period of time, traditional teaching and assessment methods were changed by an interventionist policy, and an attempt was made to make the learning and teaching of mathematics in selected classrooms more meaningful. In these classes, students were engaged actively in the building of mathematical models to establish a strong experiential foundation. The teachers were endeavouring to

‘build classroom environments in which the content of the mathematics curriculum would be based on the learner's active construction of meaning’. (ibid., p 226)

Students achieved better results in those classrooms in which mathematics was taught using less traditional methods compared to students in control groups. These students had experienced their mathematics in an environment which encouraged them to solve their problems by exploring patterns, making conjectures about their character, posing hypotheses for their resolution, and reflecting on their ideas in other problem-presenting situations. They had also worked together in groups to solve their problems in a cooperative fashion, and had not always worked individually and competitively. Most important is the finding that these students were doing no worse than those who did not experience the

interventionist strategy of more meaningful types of mathematics experiences in the classrooms. She adds:

‘In moving from a curriculum that emphasised computation to one that emphasised thinking and problem-solving and de-emphasised computation, students improved markedly in concepts and problem-solving, while generally maintaining their scores in computation.’ (ibid., p 245)

Maher adds another dimension to her conclusions by recommending that the thoughtful assessment of mathematics needs to include investigative tasks and projects to ensure that students can avail themselves of this change in instructional practice. This issue will be taken up later in this report.

Sigurdson and Olson at the University of Alberta, Canada, reported in 1992 a study of forty Year 8 mathematics classrooms. It investigated the effect on student achievement of classroom teaching that emphasises meaning. The results were similar to those reported above by Maher. Students in the innovative classes were compared to a control group in conventional classes and, in general, teaching with meaning was found to increase student achievement — especially for those in the upper two-thirds of the groups. In these innovative classrooms, rote learning (characterised typically by much practice at solving mathematical algorithms leading to performance skills that have a high degree of automaticity) was de-emphasised. Instead, meaning was *front and centre* in the teaching process. The authors refer to these contrasting teaching methodologies as ‘low roads’ and ‘high roads’ to learning.

The researchers found that teaching with meaning resulted in much improved performances, although investigational tasks at this Year 8 level were not specifically included in the array of meaningful teaching techniques displayed. However, the research did show that students who had teachers who departed from the strict drill and practice paradigm enjoyed their mathematics more and achieved better results in tests.

In 1988, the Israeli researchers, Zehavi et al. reported the result that the mathematical activity of high-achieving students in Year 9 was enhanced by giving them open problem-solving assignments. They developed assignment projects for these high-achieving Year 9 students to encourage activities leading to higher levels of cognitive thought. These activities were chosen to result in the non-avoidance of unfamiliar problems, an improvement in the quality of the solution to a particular problem, the extension of that solution and the extension of that problem. These investigations were always designed to be open-ended mathematical

problems whose scope was wide but related to the regular mathematics curriculum for Year 9 and its core mathematics.

The study was conducted with 351 students whose ability level would be the equivalent of our Advanced students in Year 9. A control group of high-achieving Year 9 students who were not assigned the investigational tasks was included in the study. The students in the experimental classes completed about four investigational tasks — some sections were done in class with some teacher direction; other sections of the investigations were completed as homework. The results indicated that those students who had grappled with these mathematical investigations were able to achieve higher results in the achievement tests, particularly when those tests contained items unfamiliar to either group. Clearly, this study has some ramifications for our NSW Advanced Mathematics students — they too can profit from becoming involved in completing an investigative task.

Haines and Izard (1994) offered a number of informed recommendations for the assessment of mathematical investigations. They were concerned mainly with issues of modelling tasks assigned chiefly to university undergraduates, but drew much on their knowledge of investigational work assigned in the secondary mathematics classrooms of the United Kingdom and Victoria (Australia). Haines and Izard emphasised the advantages of the oral communication of the results of such investigations to fellow students. They addressed such issues for the ranking of these tasks as:

- ‘1. Does similar performance on similar tasks result in similar assessment?
2. Is there an agreed meaning of ‘superior performance’?
3. Is superior performance recognized in a fair way?
4. What are the practical resource implications for the assessment procedure?’ (Haines & Izard 1994, p 374)

They believed that the use of projects and investigations as assessment tasks has the potential for ensuring that the tasks mirror the desired skills, consistent with the aims and objectives of the activity, hence contributing to valid assessment. Thus, like Maher (1991), Haines and Izard recommended that assessment in mathematics should include projects and investigations to ensure that a wider range of mental mathematical abilities contributes to the final result.

Four articles are included which describe in detail investigational tasks that were given to secondary mathematics classes in the USA, Portugal and Australia (Chapin, 1998; Garfield, 1993; Oliveira et al., 1997; Stephens & Money, 1993). Chapin described a mathematical investigation in her middle-grade Boston (USA) classroom in which her students learned and

applied the concepts of area, symmetry, pattern and function to problems involving squares in larger squares. Some students chose to work individually, others in groups and the teacher's role was active throughout — without prescribing a particular method for all to follow. Educational technology was available for students to use when appropriate. Chapin concluded:

'Mathematical investigations offer opportunities for all students to explore a topic in depth and make connections among various representations. The investigation is rich with mathematics but open enough for teachers and students to pursue a variety of paths ... Investigations offer many opportunities for students to enter and to become at home with mathematics, while actively engaged in exploring interesting questions. When students study a topic in detail, they not only learn a great deal of mathematics, they also learn the power of careful reasoning, thoughtful discourse, and perseverance.' (Chapin 1998, p 338)

Joan Garfield, of the University of Minnesota, described a series of investigations that have been used in the Years 9–10 age range to complement the teaching of statistical concepts (Garfield, 1993). The statistics outcomes in the NCTM Curriculum and Evaluation Standards (similar to those in the NSW Years 9–10 Advanced and Intermediate *Chance and Data* strand) advocate that students should:

'learn to apply probability and statistical concepts to solve problems and evaluate information in the world around them. The statistics standards suggest hands-on activities involving collecting and organising data, representing and modelling data including the use of technology, and communicating ideas verbally and in written reports.' (Garfield 1993, p 187)

These tasks encourage teachers to have students work on statistical projects individually or in groups; teachers 'engage students in learning about statistics and help them to integrate the knowledge they have learned' (p 187). Typical projects were related to the detailed gathering of information about, and analysis of, a phenomenon that could be observed by the student and that was of interest to them, and the reporting of this in written and oral form. This formed part of the internal assessment of coursework. Details were included for scoring the statistical projects. Garfield commented on how these investigational tasks affected the students' attitudes to mathematics in a positive way. The statistical investigational task in the NSW syllabus is similar to that described above and should remain an integral part of the core *Chance and Data* strand.

Oliveira, Segurado, Ponte and Cunha described in a similar fashion a number of mathematical investigations in somewhat lower age groups (Years 6–7) in a number of schools in Portugal (Oliveira et al., 1997). These tasks had been developed through a collaborative project involving mathematics teachers and mathematics educators. They were aimed at the development of classroom tasks involving students in mathematical investigations and the study of related teaching styles. Oliveira and her co-authors believed that mathematical investigations, based on open-ended problem-solving tasks:

- ‘1. are indispensable to provide a complete view of mathematics, since they are an essential part of mathematical activity;
2. stimulate the sort of student involvement required for significant learning;
3. provide multiple entry points for students at different ability levels; and
4. stimulate a holistic mode of thought, relating many topics, an essential condition for significant mathematical reasoning.’ (Oliveira et al. 1997, p 136)

The authors returned to the theme of the ideal teacher involvement in the overall structuring of investigational tasks, and suggested that a correct balance needs to be struck between ‘guiding the students and leaving not enough thinking for them to do’ and ‘implying that the students move into different paths and perhaps miss the ideas most related to the mathematical concepts in the curriculum’ (p 141). They also reported on the high level of student engagement which occurred with the assigned investigational tasks, even in ‘difficult classes’.

As a general reflection, these classroom researchers commented thus:

- ‘1. these tasks are suitable for all students and not just the better students;
2. students can really get involved in significant mathematical activity and hence have a sense of doing something significant;
3. it is possible that students develop autonomy;
4. there is an advantage in stimulating interaction between students, working in groups or in pairs, since a lot of power emerges from their talking and arguing with each other.’ (ibid., p 141)

Stephens and Money reported internationally on the experience in Victoria of investigational tasks in mathematics as part of the assessment in the Year 12 final examination. They also returned to the theme enunciated by others, that:

‘Expanding the range of performance assessed in mathematics in order to reflect more fully the objectives of the mathematics curriculum is not likely to be achieved without major reforms in the design of the mathematics curriculum, at system and school levels, and by ensuring that assessment procedures are driven by the curriculum, and not the other way around.’ (Stephens & Money, 1993, p 155)

Since 1990, the Victorian Certificate of Education (now called the Higher School Certificate) for mathematics has incorporated extended independent investigations as an integral part of coursework and examination tasks. The Year 12 investigational task on a centrally set theme is assessed by panels of mathematics teachers across the whole State. This is similar to the Year 10 GCSE level in England and Wales. This final Year 12 task is extensive, and may take up to 15 to 20 hours of independent work. These changed classroom directions and expanded assessment practices have percolated down into earlier years of secondary mathematics education, with teachers ‘extending students’ ability to work through non-routine problems and justifying their solutions’ (Stephens & Money, p 168). In 1999, at the Years 9–10 level, such investigational work is universal in Victoria, but of course is programmed and assessed at the local school level. Extended tasks taking up to about five hours are common.

Stephens and Money made a range of suggestions for the assessment criteria of these tasks under the headings of ‘Conducting the Investigations’, ‘Mathematical Content’ and ‘Communication’. These are as follows:

*‘Conducting the Investigations*

- Identifying important information
- Collecting appropriate information
- Analysing information
- Interpreting and critically evaluating results
- Working logically
- Breadth or depth of investigation.

*Mathematical Content*

- Mathematical formulation or interpretation of problem, situation or issues
- Relevance of mathematics used
- Level of mathematics used
- Use of mathematical language, symbols and conventions
- Understanding, interpretation and evaluation of mathematics used
- Accurate use of mathematics.

### *Communication*

- Clarity of aims of project
- Relating topic to theme
- Defining mathematical symbols used
- Account of investigation and conclusions
- Evaluation of conclusion
- Organisation of material.' (ibid., p 165)

The final article (Sullivan, 1992) addresses the important issue of the kinds of mathematical investigations, both short and extended, which are crucial in providing opportunities for higher mathematical thought and advanced mathematical discourse. Under the present arrangements it is the advanced students who stand to lose most from the decision to make such tasks optional. We have defined these tasks as 'open ended'. However, Sullivan draws attention to the need to make them also 'content specific'. In other words, that the activity is not just mathematics generally, but the specific content that the classroom program seeks to address. With this in mind, Sullivan carried out a controlled experiment with two classes of Grade 6 children, randomly assigned to a control group and an experimental group, to ascertain 'whether students can learn substantive mathematics content from a program based only on content-specific open questions' on length, area and perimeter. The control group followed a traditional program of direct instruction. The experimental group's program consisted of content-specific open questions, with a minimum of teacher explanation and very few closed questions. Students spent most of the time exploring mathematical concepts in their own way. No opportunity for practice was given. Both groups had seven one-hour lessons.

The pupils in the experimental group performed as well as those in the control group on the skill items of the content, even though they had had no practice. Surprisingly, however, these pupils did not give multiple or general answers to the open items. The researcher concluded that it may help to supplement constructive activity with some skill practice, and that it may be necessary for teachers to intervene in the reviews of the investigations to make the learning and significance of the investigation more explicit.

### **Conclusion**

The literature on the importance of investigational work for mathematics learning is fairly extensive, but actual research in the area is meagre. In particular, the brief to examine extended investigations has shown a big gap in the literature. To research the mathematical

thinking process, as opposed to the products of mathematical teaching of content, is quite difficult because it necessitates the use of qualitative methods. What has been attempted is to get quantitative evidence of the relative effectiveness of direct instruction over problem-solving approaches to learning (Sweller, 1999). What has emerged from these comparisons is a picture of higher gains from direct instruction using worked examples, albeit in the short term. However, they do not address the central features of investigational work.

Nevertheless, there are compelling reasons for arguing that teachers provide opportunities for investigational work, together with exposition, practical work, discussion, practice of skills and routines, and problem-solving in their classrooms. As open-ended learning experiences, investigations are deliberately designed to encourage divergent thinking. The emphasis is on the journey, not the destination. This distinguishes investigational work from general problem-solving, where the focus is on getting a solution. Investigational work has a deliberate metacognitive intent. This refers to the development of the student's knowledge concerning their own cognitive processes, and the active monitoring and consequent regulation and orchestration of these processes, in the service of a particular goal such as the proof of a theorem, or a problem to be solved (Schoenfeld 1992, p 347). However, to realise the potential of a mathematical investigation to contribute to the development of their ability to think mathematically, it is important, as Cockcroft (1982, p 74) argues and the syllabus recommends, that students discuss and write about their thinking.

To summarise, the Board of Studies can be confident of the potential value of investigational work in the realisation of the desired outcomes of the Stage 5 syllabus. This includes learning to think mathematically; equity issues; command of language; using technology; and working with others and in teams. However, the current situation in schools is that teachers are unlikely to include investigational work in their teaching while it is not compulsory. Direct instruction is seen to be of more immediate effect, easier to control, and easier to assess. In an examination-driven curriculum the need for students to score high marks is paramount. Student-centred activities, even if they amount to only five hours in a two-year course, may be regarded as a waste of good teaching time.

In our view, this situation is a great shame: it excludes students from a powerful learning experience, particularly advanced students who have (arguably) the most to gain. It is our future mathematicians, scientists and engineers who most need to appreciate the human face of mathematics as a discipline. It is they who need to appreciate the extraordinary hard work and perseverance required to make a breakthrough in mathematics. A beautiful example is Amir Aczel's recent book *Fermat's Last Theorem*, which we recommend highly as an enjoyable, readable account of the longest successful investigation in the history of

mathematics. In 1637 the French mathematician Pierre de Fermat made an intriguing mathematical proposition. In 1993, after a seven-year struggle, Andrew Wiles presented to an astonished conference in Cambridge a 200-page proof using number theory, algebra, analysis, geometry and topology. At a time when student enrolment in higher-level mathematics courses for the HSC is falling, and attitudes of senior students hardening against teaching mathematics as a career, mandatory investigational work at School Certificate level may have great long-term potential for change.

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## **Part B**

# **Why Investigations Should Not Be Compulsory in the Years 9–10 Mathematics Courses**

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## Part B

### Why Investigations Should Not Be Compulsory in the Years 9–10 Mathematics Courses

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#### Background

A strand called *Mathematical Investigations* was originally written into the Core of the Mathematics Years 9–10 syllabus for the Advanced and Intermediate courses. However, this strand was made optional in August 1996 because of concerns over the reduction of Core hours in mathematics. Each of these courses specifies that at least one longer investigation be completed which might take up to five hours. In addition, an investigation should be undertaken in the *Chance and Data* strand.

‘Within this strand, students should carry out an investigation (individually or in a group) in which they experience the main aspects and methods of planning, organising, analysing and evaluating data.’ (*Advanced Course*, p 120)

Furthermore, for each described section of content, there is a section on suggested activities and sample questions. The activities:

‘reflect current research on the teaching and learning of mathematics. They give a range of problem types and investigations to aid the teaching and learning process. The activities included highlight the relevance of mathematics. Their use within the teaching program facilitates a problem-solving approach to student learning experiences.’ (*ibid.*, p 121)

The mandatory investigation in *Chance and Data*, coupled with the longer investigation, could mean that students spend as much as 10 hours investigating. This would equate to over 6% of Core time. This figure could be substantially higher if teachers adopted a wider investigative approach as promoted by the syllabus.

Closely associated with investigations is the methodology known as *problem-solving*. Many mathematics educators believe that a problem-solving approach is more effective than the ‘traditional’ approach. Unfortunately the term ‘problem-solving’ means different things to

different people. Similarly, the notion of a traditional way of teaching mathematics is open to interpretation. The notions of problem-solving and traditional teaching are therefore examined here.

## **Problem-solving**

Some mathematics educators tend to define problem-solving differently from psychologists who regard problem-solving as the process involved in solving any problem where the answer or solution is not known.

Classic examples of problem-solving used in psychology experiments include the *orcs and hobbits* problem, the *water jar* problem and the *Tower of Hanoi* problem. These problems have featured in countless experiments and have proved to be a rich resource for understanding how humans think and solve problems. Obviously, their uses in everyday life are somewhat limited.

In contrast, mathematics problems occur everywhere, both in school and in everyday life. Consequently, mathematics problems have also been the focus of considerable research in experimental psychology.

For decades students have been asked to solve mathematics problems under various conditions within the experimenters' laboratories. These problems come from many domains and vary from the simple to the complex. Solving the linear equation ( $2x + 5 = 8 - x$ ) or completing the long division ( $12674/58$ ) are both examples of a mathematical problem. However, many mathematics educators would not consider these examples as needing 'true' problem-solving because they are considered to be too routine, or they are insufficiently open-ended, or they do not involve real-world contexts, or they do not meet some other criteria defined by Polya and others. In other words, mathematical problem-solving only involves particular types of mathematics problems. In some respects this is a very shallow description and ignores the huge differences in abilities and knowledge of school children.

The 'routine' tasks given above would be found by many students to be very difficult or impossible, generating high cognitive loads. No problem can be defined as being routine or non-routine without simultaneously referring to the knowledge level of the solver. All successful mathematics teaching results in formerly non-routine problems becoming routine for those students who have learned. Furthermore, most psychologists accept that learning has no other function.

## **Traditional Mathematics Teaching**

It is worth considering what goes on in the so-called *traditional mathematics classroom*. A period of teacher transmission on a topic is usually followed by textbook work, a teaching model which has attracted much criticism from the reform movements. The textbook period is often referred to as consolidation and practice. But is it? For the students who do not understand the concept being taught, this period is hardly consolidation. They may have a very poorly developed knowledge base for the area and consequently fail to solve problems successfully. In most classrooms, students are asked to solve problems that they cannot complete. Indeed, some textbook problems are not standard drill questions at all, but are genuine transfer problems. Within this framework, the experienced teacher overcomes student difficulties by moving around the room and helping small groups and individuals or by providing more worked examples on the blackboard. Students will also discuss problems amongst themselves and have access to worked examples. There is no doubt that many teachers provide these additional support systems and are extremely effective teachers, although on the surface 'traditional'. Presumably, ineffective teachers do not supply the necessary support mechanisms and rely too much on textbooks without building understanding.

Many educators believe that the traditional approach is not the best way to teach or learn mathematics. Reformists argue that the best way is to substitute a problem-solving/investigations/discovery-learning approach for *direct* instruction. Ironically, other researchers also believe that the traditional method is inefficient and seek to change classroom practices — not, however, by including more problem-solving, but less. Researchers from the cognitive-load school, for example, believe that asking students to solve problems without adequate attempts to build understanding and develop schema acquisition is inefficient. Unfortunately, reformists do not accept that students working from textbooks are problem-solving.

Regardless of the philosophical differences and syntactic arguments, both schools of thought want to change mathematics teaching. However, the cures are very different, as one group wants to reduce problem-solving, while the other wants to increase it.

## **Philosophy of the Syllabus**

As a consequence of these differences, the emphasis placed on problem-solving as a methodology in the Years 9–10 Syllabus must be considered controversial. The syllabus not

only documents the content but also how it should be taught. This approach contrasts with the *1995 National Curriculum in Mathematics* for England and Wales where the content is listed with a number of expected mathematical outcomes — no attempt is made to recommend a particular teaching style. More recent developments in the United Kingdom have seen the Teacher Training Agency in 1998 specify a whole range of effective teaching methods that trainees should be taught. These methods range from whole-class teaching, exposition and other interactive methods through to thinking mathematically when solving problems. Sensibly, no technique is favoured, and certainly none is made compulsory.<sup>3</sup>

Although the NSW syllabus does not explicitly state that an investigative approach is superior to any other methodology, there is an implicit underlying acceptance of its value written within the general philosophy of the courses. The Stage 5 Syllabus and the Support Document, as well as various other Board of Studies publications, make a strong link between problem-solving and investigations.

‘Investigation and problem-solving are integral to this course.’ (*Advanced Course*, p 173)

‘It [the syllabus] emphasises the ability to investigate and reason logically, to solve non-routine problems ...’ (*ibid.*, p 7)

Furthermore, investigations directly involve solving problems as stated above. In the case of problem-solving, more explicit statements are made:

‘There is a general recognition that the process of mathematical problem-solving will prepare students more appropriately to function competently in society and that a problem-solving approach actually aids mathematical learning.’ (*loc. cit.*)

Whereas most people would agree that problem-solving forms an important focus of mathematics, it is not generally accepted that the best way to learn mathematics is through a methodology of problem-solving. In fact, there is strong evidence to suggest that the reverse is true. For example, the findings of a whole array of research associated with *cognitive load theory* has shown, time and again, that more direct instructional techniques appear to

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<sup>3</sup> It is worth noting that since this report was originally written in 1999, the new American *Standards* (NCTM, 2000) has been released. This version appears more flexible than before by acknowledging the effectiveness of different strategies: ‘Teachers have different styles and strategies for helping students learn particular mathematical ideas, and there is no one “right way” to teach.’ (p 18)

be more efficient than strategies involving problem-solving. These results have been obtained in unfamiliar domains, including non-routine problems. Although a problem may be labelled non-routine, it will become routine as students become familiar with the underlying mathematical principle. Cognitive load theory research suggests that a direct approach using worked examples (for example) is more likely to be successful than an undirected problem-solving approach. Although the syllabus acknowledges cognitive load theory (p 14) and indicates that goal-free and worked examples should be encouraged, it is implied that they should be used within a general problem-solving framework — which would negate their effectiveness.

There are inherent dangers in advocating within a syllabus a philosophy which has not been rigorously researched. The syllabus writers acknowledge (p 8) the influence of the *National Statement on Mathematics for Australian Schools* (Curriculum Corporation, 1991) on their document. This publication is now ten years old, and was highly influenced by publications produced in the late eighties by the National Council of Teachers of Mathematics (NCTM) in the USA. Both the Australian *National Statement* and the American *Standards* make statements about the learning and teaching of mathematics. Although many mathematics educators would agree with many of the principles contained in these documents, there seems to have been little research completed to back up many of the recommendations. Ten years down the track, there is still a fundamental shortage of empirically-based research to support such recommendations associated with investigations and problem-solving, and furthermore the US document has led directly to the outbreak of the 'math wars' in that country (see below).

It should be noted that the *Mathematics Years K–6 Syllabus* (DSE, 1989) and *Mathematics Years 7–8 Syllabus* (1988) were introduced with a problem-solving focus. The *Years 7–8 Syllabus* states that students learn mathematics *for, about and through* problem-solving (p 3) and recommends that 'problem-solving should encompass a wide variety of problem types including open investigations ...' Although the *Years 7–8 Syllabus* in particular has a problem-solving focus, its effect on teaching and learning has never been directly evaluated. Anecdotal evidence emerges from time to time that few teachers implemented a true problem-solving program. If anything, it is treated as a separate chapter where students solve problems like the ubiquitous handshake problem. Questions might be asked about the wisdom of a wider introduction without an evaluation of the effects on learning and teaching in the younger years. If it hasn't worked, why broaden it now?

Far from seeing a universal adoption of new methodologies such as problem-solving, there has been in the last couple of years a drift away from such positions by many governments and educational boards. A backlash against so-called progressive ideologies seems to be taking place as public perceptions grow that the new ways have not worked. In the USA, there have been the so-called 'math wars'. In a reaction to a perceived fall in standards and parental concern, the California State Board of Education approved a new mathematics framework which restored an emphasis on basic skills, computation and rigour. A group called *Mathematically Correct* emerged as an organisation opposed to many of the reforms of NCTM, which they called *fuzzy mathematics*. *Mathematically Correct* argues that many weak programs have resulted from NCTM guidelines. They believe that no theory of learning or method of teaching should be promoted above others and generally advocate a return to many of the practices of the past, such as an emphasis on algorithms.

In England, under the Blair Government, there have been dramatic changes to the teaching of mathematics. The Numeracy Task Force has implemented many initiatives, especially in primary schools, which have included a focus on whole-class teaching and mental arithmetic. There has also been concern in England about students not studying enough mathematics. A strong influence on much of this 'back to basics' push has been the poor results gained by many Western countries in the Third International Mathematics and Science Study (TIMSS; see Lokan et al., 1996). Even NSW itself has seen a back-to-basics approach with the introduction in 1998 of a non-calculator section in the Year 10 School Certificate Mathematics Test. Similarly, in May 1999 the Federal Government budget provided significant funds for literacy and numeracy programs.

If NSW were to continue to advocate a philosophy based on problem-solving and investigation, it would do so against a background of change internationally away from such 'reforms'. This decision might be unwise unless extensive proof can be found that the methodology works.

### **Example of an Investigation**

The following example is given in the Syllabus:

'A soft drink can will have a volume of 375 mL. Find the dimensions that would require the least amount of sheet metal to construct it. Comment on its suitability for use.' (*Advanced Course*, p 175)

This is a typical 'max-min' question in calculus. Students studying 2 Unit or higher learn to solve these types of problems routinely in their study of calculus. However, most Years 9–10 students would not be able to attempt this problem without some directions and pointers. For the most able students it would be a suitable enrichment/extension type of question where they can apply and practise previous knowledge (volumes, surface areas, substitution, graph sketching) to an unusual problem. Without calculus the problem has to be solved by trial and error or by graphical means. It is an area of mathematics where there are more unknowns than equations. It is called linear programming, which was a lobe in the previous syllabus, and is now part of the mathematical modelling lobe. An investigation approach would require students to try to solve this problem themselves and discover various aspects about it. A more direct approach would give an example of a similar problem (eg relations between the perimeter and area of a rectangle) and then ask students to complete similar examples leading up to the given problem. Which method is best? If an investigative approach is recommended, then it needs to be proven that it is the best. Otherwise, teachers should be free to choose.

Does this problem have value? Perhaps. As a preparation for calculus it will get some students thinking about this type of problem. However, in some respects this is asking students to solve a problem in an inefficient manner, as calculus does it very quickly. Some of the other investigations listed are of the same type: asking students to solve problems which would be quite routine with Year 11 and 12 knowledge, or which involve mathematical modelling.

### **How is an Investigation to be Recorded?**

The *Mathematics Years 9–10 Support Document* (p 15–17) details in its assessment section how an investigation might be assessed. One example (p 17) for a Data Investigation provides a sample proforma broken down into the following headings: Aim and introduction, Method, Results, Discussion and conclusion, and Communication skills. This is the format for writing an empirical research report — a valuable skill, but usually employed by research graduates or in the writing up of science experiments. Such a focus is wasted in a mathematics syllabus.

### **What is Claimed from Using Investigations?**

The Syllabus states:

'In doing such an investigation, students would use all of the processes of *Working Mathematically*, ie investigating, conjecturing, solving problems, applying and verifying, communicating and working in context.' (*Advanced Course*, p 173)

It is worth looking at each of these claims. The first point, investigating, is certainly true as it is a circular statement. The third point, solving problems, is also true as a problem lies at the heart of any investigation. To solve a problem, knowledge needs to be applied, and 'verifying' is usually used in conjunction with application and checking an answer. Communication depends on having somebody to communicate with, and working in context depends on the problem itself. In the example of the soft-drink can (above) the problem certainly has a real-life context, although not the sort of problem that many would meet in real life. The amount of conjecturing that goes on depends on the problem. For the soft-drink can problem, what might the conjecture be? Given a fixed volume, a cylinder has many dimensions but only one combination that gives a minimum area. Students might arrive at that conclusion. They could learn this fact from transmission too. They certainly learn it when they study calculus.

Apart from the first point, it is possible that the others could be achieved through many of the other techniques that teachers employ routinely in any class. Nevertheless, it must be added that these processes seem quite vague. It might be expected that clearer outcomes would be defined in a syllabus. To do that, we suspect that there has to be a clearer link with specific content areas.

## Summary

From an analysis of the syllabus documents it is clear that a methodology of problem-solving and investigations is strongly advocated. It is claimed explicitly that problem-solving is a highly effective way to learn mathematics. As the Syllabus generally emphasises investigations, and specifies that, in the Advanced and Intermediate courses, both a longer investigation and a *Chance and Data* investigation be completed, the expected time spent on investigations may have been understated previously. As for the expected outcomes of investigations, these seem quite vague and consist of five processes which contribute to *Working Mathematically*. One aspect which is clearly defined, however, is that of reporting, where a research-type format is proposed. Furthermore, many of the investigations given as examples seem to consist of classes of problems which can be solved through mathematical modelling or with the aid of further mathematical knowledge. Finally, the introduction of such a problem-solving emphasis at this time goes against an international trend.

## Resolution of the Problem

To determine whether investigations should be included in the Syllabus, the following questions need to be answered:

1. Does the literature support the claim that investigations are an effective method of teaching mathematics?
2. Do investigations produce the required outcomes of *Working Mathematically* as claimed?
3. Are the outcomes of *Working Mathematically* as specified too vague?
4. Will too much time be spent on reporting investigations?
5. Are the investigations as suggested just other content areas of mathematics?

It is also worth considering the following:

- 6 As problem-solving and investigations have been integral recommendations of the Years K–6 and Years 7–8 syllabuses, a review of their impact should be completed before an expansion policy is implemented.
- 7 In view of the 'back-to-basics' trend, NSW should not rush to adopt 'reformist' methodologies which are being rejected elsewhere.
- 8 In view of the strong connection between problem-solving and investigations, the focus on problem-solving within the syllabus should also be reviewed.

## The Effectiveness of Problem-solving and Discovery Learning: Studies from Experimental Psychology

### Theory in a historical light

'Investigations' is an enquiry/discovery learning/constructivist construct. Historically, its genesis can probably be traced back to Rousseau, but in modern times first Piaget and then Bruner can be considered its progenitors. Piaget (1928) and Bartlett (1932) originated modern schema theory according to which humans do not receive information or knowledge in the way in which a tape recorder receives information, but rather restructure or reconstruct information so that it accords with previously acquired knowledge. That knowledge is in the form of a schema. Thus, when we see an equation such as  $(a + b)/(c - d) = xy/z$ , the extent to which we can deal with it depends on the extent to which we can relate it to a general structure, a schema, for this type of equation. In relating new information to a schema, we restructure it rather than process it in the exact form it is

presented. In Bruner's (1961) terms, we 'discover' this restructured knowledge, we do not receive it.

This cognitive structure is not controversial. However, the educational consequences claimed by some to flow from it are. Many educationists have interpreted the structure to mean that instruction should not be direct, but rather that students should 'discover', 'inquire', 'investigate', or 'construct' knowledge (the terms keep changing as opposition to a particular term develops) because they cannot properly process direct instruction. In fact, students must construct a knowledge base of schemas irrespective of whether instruction is direct or not. That construction is made easier with direct instruction.

Because there is no necessary development from constructing a knowledge base to enquiry-based learning, resulting in no sensible theoretical base, the meaning and purpose of enquiry learning is quite impenetrable. Is it intended to help students acquire and understand particular mathematical procedures? In other words, will someone who has learned geometry by investigations be better at geometry than someone who has it presented by direct instruction? Or is the intention of enquiry methods just to teach people how to conduct research? Or perhaps the intention is to improve thinking skills? Any one or even all of these benefits have at various times been suggested as benefits of enquiry learning. In fact, there are no theoretical reasons to suppose that any of these benefits eventuate.

One alternative theoretical framework suggests that all skills develop from automatic schemas held in long-term memory. Those schemas must be constructed in a very limited working memory; the more that unnecessary working memory load is reduced, the more is available for schema acquisition. Investigations are very expensive of working memory resources while direct instruction reduces working memory load. This theoretical perspective encourages direct instruction and suggests that enquiry-based methods are ineffective because they impose a heavy cognitive load (Sweller, 1999).

## **Research designs**

How can we determine the effectiveness of these opposing viewpoints? Almost all the evidence in favour of discovery-learning methods is based on designs which consist of observing students learning by discovery and finding some favourable measure which increases after the exercise compared to before. These exercises are almost useless. It would be hard to imagine a teaching/learning procedure that did not result in improvement in something. The real issue is whether the improvement could be larger using an alternative

technique. That means using a controlled experimental design, for example one that compares enquiry approaches with direct instruction.

### **Data from controlled experimental designs**

In making radical changes to an education system, there should be an expectation and confidence that the changes will work. Consequently, there should be strong research evidence to support the changes. There is an onus to provide evidence for any new theory. The studies described below were completed within an experimental psychology context and have sought to find this proof. In particular, they focus on comparing worked examples with problem-solving techniques and analysing the effectiveness of discovery learning. Transfer is also examined, as one of the main criticisms of traditional teaching is that little transfer occurs.

### **Worked examples**

The Sweller and Cooper (1985) and Cooper and Sweller (1987) experiments were mainly conducted in the algebra domain of *changing the subject* and included students from Grades 8 and 9. The procedures were the same for both grades. A 'conventional' group was usually compared with a 'worked-example' group. Typically, both groups received a common introduction of two worked examples, which was followed by an acquisition phase and then a test phase. For the conventional group, the acquisition phase would require students to complete about eight problems. Students were not allowed to advance from one problem to the next unless they completed it correctly. If after five minutes they had not completed the problem successfully they were given a worked example. Consequently, the conventional groups were following a format often seen in traditional mathematics classrooms, where a period of teacher transmission is followed by textbook work. It should be noted that these students received at least two worked problems, and some of the poorer students would have received more. In contrast, the worked-example groups during the acquisition phase were given the same problems in pairs. Firstly, they observed a worked example, then they were asked to solve a similar problem themselves. This pairing of problems is central to the worked-example format as Trafton and Reiser (1993) found that giving a whole block of worked examples was not effective.

The results from these experiments indicated that the worked-example groups needed less acquisition time and made fewer errors on the test problems. In addition, it was discovered that the effects of worked examples are somewhat sensitive and dependent upon several variables, including: ability of students; acquisition times; and transfer problems. If the

acquisition period is too long, the between-groups effects disappear. More able students with better-developed schemata need less acquisition time. However, less able students also benefit from worked examples. For transfer problems, it was discovered that rule automation is very important for facilitating transfer. Students with poor rule automation do not easily transfer their 'incomplete knowledge'. Nevertheless, worked examples were effective in obtaining transfer.

Zhu and Simon (1987) found that 13-year-old Chinese students performed better using worked examples in factorising, laws of indices and geometry than students who received the normal lecture mode. Learning times were also quicker. Verbal protocols suggested that the students did not simply learn rules by rote. Zhu and Simon argued that students do not simply memorise rules but understand the process with considerable depth. A Beijing class completed three years of middle-school mathematics in two years using a worked-example approach and performed above average for the whole of China. Furthermore, they performed better than three similar classes who did the normal three years of study.

Paas (1992) studied 16- to 18-year-old students learning basic statistics. Three groups were compared: a conventional group (solving a problem); a worked-example group; and a 'complete a partly-worked problem' group. The last two groups involved solving a problem following two worked examples. Prior to treatment all groups were given a general introduction to the topic. Results indicated that students performed worst in the conventional group, in training time and on the tasks. On transfer tasks the worked-example group performed best. Similarly, Paas and van Merriënboer (1994) found that worked examples were more effective (involving less time and greater accuracy) than a conventional format in geometrical problem-solving with groups of 19- to 23-year-olds. In particular, worked examples were very effective for transfer when the problems included in the instructional phase had a high degree of variability. A significant aspect of the Paas and van Merriënboer study was a self-rating scale of *mental load*. The conventional group, who had to do more problem-solving, had much higher measures of cognitive load.

Further evidence for the effectiveness of worked examples came from Carroll (1994). Using 15- to 17-year-olds and algebra work problems in a classroom setting, Carroll found that a worked-example group was more effective than a conventional group, with the latter requiring more assistance in class. Significantly, worked examples were particularly helpful for low-achieving students.

## Summary

A number of controlled experiments are described which show that worked examples are more effective in teaching than solving the equivalent problems. These studies have been conducted with a range of secondary-age students in a number of mathematical disciplines. Worked examples have also been shown to promote the transfer of mathematical knowledge. Furthermore, the effectiveness of worked examples has been demonstrated in a classroom setting (Carroll, 1994) and over an extended period of time (Zhu & Simon, 1987). Significantly, conventional problem-solving promotes a high cognitive load (Paas & van Merriënboer, 1994). It should be noted that there are many other studies demonstrating the effectiveness of worked examples and indicating the conditions under which they should be used.

## Empirical experiments involving discovery learning

This section describes four controlled studies which featured discovery learning. It should be noted that the studies were conducted with university students using classic problem-solving tasks or computing tasks.

To investigate the effects of discovery learning and transfer, McDaniel and Schlager (1990) used some classic transformation problems. In the first experiment, three groups of psychology students were given the *hobbits and orcs* problem to practise on and then asked to solve the *jealous husbands* problem as transfer tasks. In the acquisition phase, one group was given explicit instructions on the strategy and moves needed to solve variations of the *hobbits and orcs* problem. A second group was given instructions on the strategy but had to discover the moves (partial discovery), while the third group had to discover both the strategy and the moves (full discovery). It was found that the direct instruction group was superior to the other two groups during the acquisition phase. However, on the transfer problems there were no between-group differences. McDaniel and Schlager reasoned that discovery learning would be most effective if transfer tasks were less like the acquisition tasks.

A further experiment compared partial discovery and full discovery groups with a control group (having no acquisition period). Using variations of the *water jug* problem it was found that the full discovery group performed best on the transfer tasks in terms of time to solution and the least number of moves. Also, both discovery groups tended to perform better than the control. Surprisingly, on a two-jar remote transfer problem the control group performed best, if not quite significantly. This result was not really explained very satisfactorily, as the

authors used it to argue that the full discovery group showed less *Einstellung* (mental set) than the partial discovery group who performed worst of all on this task. *Einstellung* occurs when a learned rule is applied to the wrong situation. It is often used as a sign that learning has occurred, which would mean that the partial discovery group had learnt the most, although they applied that knowledge incorrectly. Regardless of what the *Einstellung* effect means in this situation, a group who had no training performed best. Unfortunately, no comparison was made with a more direct instructional technique.

Carlson et al. (1992) studied information networks with psychology undergraduates. In one experiment, four instruction modes were compared: fixed template; variable template; discovery learning; and procedural instructions. In all groups, students were given instructions on how to work the network simulation on a computer and the overall goal of the problems to be solved. In the *discovery learning* group, students were required to discover solutions. The *fixed template* group was given guidance based on an algorithm, whereas the *variable template* group was given the same guidance but had to make choices at each step of the solution. In contrast, the *procedural instruction* group received direct instructions on the algorithm. It was found that procedural instructions and the variable template mode were more effective than discovery learning during the acquisition and test phases, although the variable template mode required memory aiding to be effective. In contrast, the fixed template mode was least effective.

Charney et al. (1990) used twelve spreadsheet commands to compare undergraduate students using an explorative process with those receiving interactive instruction. Both groups were given an instruction manual on the twelve commands. In addition, the latter group was exposed to half of the commands through problems which were worked in tutorial form. In contrast, the other six commands were presented in problem-solving form only. Solutions were given after the students had attempted the problem. The exploration group was told that:

'there was no constraint on the order in which they could study or practice the commands. These subjects experimented with the commands at their own initiative, looked back at the manual at any time, and freely created their own spreadsheets or modified the set of practice-problem spreadsheets that were stored on line.' (Charney et al. 1990, p 332)

The results of this study showed that the average time spent problem-solving on a command during acquisition was significantly greater than the time for the tutorial or discovery mode. However, on the test phase that followed instruction, the best performance (time and

accuracy) was found on the commands which experienced a problem-solving approach. No difference was found between tutorials and exploration.

On the surface, this result appears to reverse the previous finding and show that problem-solving is more effective than worked examples. This analysis is false as the tutorials were not in a worked-example format of tutorial followed by a similar problem to be solved. This design is crucial to the effectiveness of worked examples (see Trafton & Reiser, 1993). Also of concern is the amount of time spent problem-solving, which was substantially greater than the other two modes. Tuovinen and Sweller (1999) argue that differences in training times have significant effects which cannot be smoothed over by regression analysis as Charney et al. claimed.

In summary, the results of this study are inconclusive. The problem-solving approach has not been shown to be more effective than worked examples, as the design format did not include the necessary pairing of worked examples. It should also be noted that this form of problem-solving is quite similar to the textbook style. Putting aside the experimental design faults, the unguided discovery group did not show any signs of effectiveness.

Partially in response to the study by Charney et al., Tuovinen and Sweller (1999) made a direct comparison between worked examples and discovery learning. In a computer-based environment, Diploma of Education students learned about certain aspects of FileMaker Pro databases. Following a common introductory instructional period, the worked example group received a period of instruction which paired examples with problems. In contrast the discovery group was told to:

'... try out the functions in each of the lessons in situations you create yourself, saving your files on the floppy disk provided. You may use any of the databases on the floppy disk if you wish. You will be asked to solve problems similar to the one shown in the lessons, in the test on this work. So direct your exploration towards gaining adequate mastery of the program to deal with such questions.' (Tuovinen & Sweller 1999, p 12)

Both groups received the same time for acquisition. In addition, throughout all aspects of the study, students were asked to record the mental effort required for each task completed. The overall results of the study showed that worked examples were more effective than exploration. However, this result was only true for database novices. For students with database experience there was no difference between the groups. Similarly, the mental effort ratings for novices showed high cognitive load readings for the discovery group.

Through efficiency measures it was shown that high cognitive load interfered with learning rather than facilitated it. A further significant finding from this study was that the effectiveness of worked examples was reduced by previous domain knowledge. Consequently, it was concluded that:

‘... in this experiment, we found that combining worked examples and problem-solving produced better learning for students totally unfamiliar with a new domain, but exploration practice was just as good as this combined approach for students with some domain experience. Evidence that these results were due to cognitive load factors rather than other factors were obtained by recording mental effort ratings.’ (ibid., p 22)

## **Summary**

There is little evidence to support discovery learning. McDaniel and Schlager found that under some circumstances discovery learning was more effective than partial discovery or no instruction on more extreme transfer tasks. This result was not very convincing as no comparison was made with more meaningful forms of instruction. In fact, in a second experiment they found that direct instruction produced a better performance than either of the discovery modes. Carlson et al. (1992) also found that direct instruction was superior to a discovery mode. In the Charney et al. (1990) study it was found that a problem-solving approach was superior to discovery learning. This result may argue badly for discovery learning as the problem-solving approach has been shown above to be inefficient when compared with worked examples. Finally, Tuovinen and Sweller (1999) showed that worked examples were more effective than discovery learning. In addition, the potential for discovery modes to facilitate learning may be further diminished when it is considered that all the students in these studies were at university. It might be expected that this population would have more experience of student-initiated processes.

## **The Effectiveness of Investigations: Evidence from Mathematics Education**

In this section we review the articles nominated by Lindsay Grimison and Lloyd Dawe. The articles fall into several categories: studies on investigations and other reform methodologies; studies on issues related to investigations; studies on assessment and teacher beliefs. Several of the studies are not experimental in nature.

## **Studies on investigations and other reform methodologies**

### ***Boaler (1997a, 1997b, 1998)***

In this English study, Boaler examined the teaching and learning of mathematics of a year cohort over a three-year period in two schools. Commencing in Year 9, the students were tested and both cohorts were found to be equal mathematically, and to have similar socioeconomic backgrounds.

Boaler reported that Amber Hill School had an authoritarian head teacher who ran the school by enforcing traditionalism in an attempt to lift the academic record. The students followed the SMP 11–16 scheme and textbooks in Years 9–11. Typical lessons were of the ‘talk and chalk’ variety for 15 minutes, followed by textbook work. Mathematics classes were streamed and generally the students were on task and orderly. Boaler found that the students disliked the lack of variety and lack of practical work and believed that mathematics was all about memorising rules and rote learning. Students tried to remember similar problems rather than attending to the detail of a problem. Students tended to give up easily on novel problems and found mathematics boring. Many students disliked the textbook and teaching approach. However, the students were hard-working and motivated, but more boys enjoyed mathematics than girls.

In contrast, Phoenix Park School was committed to progressive education, encouraging students to be independent thinkers and to take responsibility for their own actions. There were few school rules and classes consisted mainly of mixed-ability groupings. Students learned their mathematics through open-ended projects lasting two to three weeks. There was a philosophy of developing a need to use mathematics in situations that were meaningful and realistic.

‘If a student or group needed to learn some mathematics they didn’t know, the teacher would teach it to them.’ (1998, p 49)

Boaler reported that many students liked the variety of activities and relaxed atmosphere which stimulated discussion between students. However, many students did little work and were off-task more than the other school. Consequently, the method seemed to polarise students. More girls enjoyed mathematics.

Two measured comparisons were made between the two schools. On a problem-solving task based around a model and a plan of a house, two questions were asked involving council design rules. To solve the problem, Year 9 students needed to find information from

different sources, choose methods, combine content areas, etc. Prior to this, students were given pencil-and-paper type tests on relevant content. These tests indicated that both sets of students knew the individual content areas associated with the main tasks and there was no difference between the schools. On the problem-solving task the progressive school did better, even though the traditional school had more higher-ability students in the sample. One interesting observation made by Boaler was that the top class of the traditional school used trigonometry inappropriately. Where an estimate was needed of an angle, this class tried to calculate it using trigonometry. The same experiment was repeated for Year 10 students with similar results, although the results were not as strong as in Year 9.

On the surface, these results seem to indicate that the progressive school has outperformed the traditional school. This is certainly true; however the cause of the difference cannot be easily deduced. For a start, the teachers are different in each school and this could have a major impact on the results. Boaler also acknowledges that the progressive school may have had more practice on the type of problem asked. This might be a significant influence. In addition, it is highly likely that the top class of the traditional school who tried trigonometry were demonstrating the Einstellung (mental set) effect. For this to happen, it is possible that they had done similar problems which involved trigonometry, and were naturally trying the method they had previously found to work.

In external examinations the progressive school performed significantly better than the traditional school (88% compared with 71% success rate). At the traditional school, boys performed better than girls; however, there was no gender difference at the progressive school. Overall, the progressive school did better than the national average in GCSE mathematics. Boaler also made the observation that at the traditional school students complained after their trials that they could not interpret unfamiliar questions. In contrast, the progressive school's students seemed more comfortable with unfamiliar situations and in choosing correct procedures. Again, these results look positive for the progressive school, but some questions could be asked about the competency of the teachers at the traditional school in preparing their students for examinations. On the difference in the national examination marks, it would be interesting to know how the students performed in their other subjects. Perhaps the progressive school is a better school. Similarly, Boaler records that the progressive school students start in Year 9, whereas in the traditional school the students start in Year 7. What effect does two extra years in school with a dictatorial head have?

## General comments

This may be an example of a mathematics department using investigations in an effective way. Nevertheless, there are many non-controlled variables in this study which make it difficult to compare any of the between-school results. Furthermore, in the traditional school, the mathematics teachers seem not to prepare their students very well, the books seem to be out of date and do not match GCSE-type problems. They adhere to a method which nobody would promote. (We have shown previously how worked examples are superior to this traditional approach.) These teachers might be experienced and hard-working, but are they any good? In the progressive school, the GCSE results are above average, indicating that their methods seem to be working for them. However, these results cannot be generalised any further, as the stringent conditions of a comparative study have not been met.

### ***Zehavi, Bruckheimer and Ben-Zvi (1988)***

In this Israeli study a group of Year 9 (top 40%) students were given a set of three or four algebra assignment projects to complete. The projects seemed quite structured and consisted of many sub-questions. Some of the questions were open and students were also asked to invent similar problems. The projects were completed at home as well as at school. During the completion of these projects, the researchers (not teachers) gave verbal and written feedback to the students for each project. A control group was used to make comparisons, although no statement was made on how the control group was taught or what assignments they did. Presumably they had 'normal' mathematics lessons.

The two groups were given pre- and post-tests on algebra assignments and a multiple-choice algebra achievement test. The pre- and post-assignments were marked according to the quality of the result and the procedure. An attitude questionnaire was given to test if work on projects affected students' attitudes towards mathematical activity.

In the first phase of this study it was found that the experimental group did significantly better on their post-test compared with their pre-test for both the assignment and the achievement tests, whereas for the control group there were no significant changes between pre- and post-tests. On the most unfamiliar problem, fewer students in the experimental group avoided it. On the attitude measure there was a significant decline for the control group, but no change for the experimental group.

Zehavi et al. make the following observation:

‘But the treatment was too “heavy” and demanding for teacher and students.’  
(p 430)

It might be assumed from this remark that the researchers considered this approach was impractical. It is also difficult to generalise or conclude too much about these results as we do not know what the control group did. A second factor involves time. How much more time did the students spend on these projects? The improvements shown in algebra could be entirely due to the excessive time spent on the projects. The comment stated above suggests that a significant amount was taken. Obviously more time aids learning.

### ***Sigurdson and Olson (1992)***

In this study three groups of Grade 8 students were taught mathematics differently over a five-month period. This was a big study as thirty classes were involved. Pre-, post- and retention tests were given. The teachers were grouped according to the following:

- A conventional group. Students taught as they always are.
- A direct-teaching group. Emphasis on algorithmic procedures and practice.
- A ‘meaning’ group emphasising teaching with meaning. Four categories were identified as promoting ‘meaning’ activities: representations with physical objects; pictorial representations; familiar applications of mathematics; and mathematical interpretations. Some examples of activities were given which could loosely be described as investigations. Some were clearly not. The researchers state, ‘None of the meaning activities were really open-ended’ (p 56) and ‘can vary from manipulating objects to drawing diagrams ... to proving mathematical theorems.’ (p 56)

Teachers were not randomly assigned to groups but were grouped according to location. No measures in difference of effectiveness of teachers were made. All teachers underwent a training program. The *meaning* and *direct* group were asked to teach in an Active Teaching format consisting of review, development, seatwork and homework to standardise lesson formats. The training took the following form:

- *Conventional* teachers were given 10 hours of training over the five-month period where conventional ideas only were reinforced.
- *Direct* teachers were asked not to emphasise meaning and concentrate on procedures and skills. They were given 10 hours of training and also learnt the Active Teaching format.

- *Meaning* teachers had 25 hours to learn the Active Teaching format and develop meaning activities.

The results of the Active Teaching format (both meaning and direct) was superior to the conventional group, especially for high-achieving students. For low-achieving students the class format made no difference. For high-achieving classes, the meaning group performed best. In contrast, for below-average achieving classes, the direct group was best. On six measures of attitude, only one was significant. The meaning group were more dependent upon their teacher. Researchers interpreted these results as indicating that teachers are important to students' learning of mathematics.

### **General comments**

This result is quite uncontroversial. It demonstrates that meaning is important when learning mathematics. The meaning group did not take an investigative approach and investigations are not required for students to derive meaning.

### ***Maher (1991)***

This study used standardised tests over a period of time for one school in New Jersey after it had undergone a school-wide intervention in Grades K–8. Components of the test included mathematical concept scores, mathematical problem-solving and mathematical computation. In a collaboration between Rutgers University and a K–8 New Jersey School, the intervention set out to create classrooms in which children would be engaged actively in the building of mathematical models to establish a strong experiential foundation. A goal was to build classroom environments in which the content of the mathematics curriculum would be based on the learner's active construction of meaning. It is mentioned that the intended spirit would be similar to the 'Standards' approach of the NCTM. An attempt was made to produce a 'thoughtful' approach to mathematics. The results showed that students in the intervention classes, compared with pre-intervention classes at the school, generally improved in concepts and problem-solving, while retaining their scores in computation.

The difficulty with assessing this study is that the intervention is not detailed and certainly is not exclusively (if at all) concerned with investigations. Improvement does seem to have occurred but it is unclear why. It is quite feasible that the results could be explained by the Hawthorne Effect (see Tuckman, 1994). The effect of being involved in a study which the teachers believed in could have generated extra interest and reflection. No evidence is provided about how the teachers changed their practices, or any other factors like new staff, extra hours, and any other confounding variable. Again the question needs to be

asked: did the results in other subjects change within the school? It may be that the changes in teaching practices were the main reason for an improvement in the teachers' enthusiasm. It could also be the effects of being involved in the study, and the influence of the administrators. The main problem with this study is that, like the previous one, it appears not to have tested investigations.

### ***Doyle (1988)***

Doyle based his research around the hypothesis that classroom tasks affect understanding of the curriculum and act as a context for students' thinking during and after instruction. In this study, teachers were observed teaching Years 7 and 8 over a period of three to seven weeks. Both students and teachers were interviewed. The teachers selected had good classroom management and used a wide variety of tasks, especially involving higher-order thinking. Teachers were also selected according to measured effectiveness. Academic work within the classroom was divided into familiar and novel work. The latter's main features were that students had to make decisions based on what to produce and how to produce it.

Doyle found that familiar work tends to produce classes that are smooth, whereas novel work tends to be more bumpy, takes longer, involves more work, has lower production, involves more errors, and students often negotiate with teachers for more explicit specifications. It was more demanding on classroom management. Teachers often simplify tasks to cope with these problems. In contrast, tasks in high production classrooms were often very familiar, with content given in small chunks with stepwise instruction.

Doyle also found a link between assessment and tasks. Familiar work tended to be subjected to strict accountability, whereas for novel tasks the accountability softened. Doyle concluded that there is a real problem in sustaining novel work within the classroom and makes the following point about the need to understand what goes on in the classroom:

'The point here is not that current practices in mathematics classes should be preserved. Rather, the argument is being advanced that any attempt to reform teaching in mathematics must come to grips with the situational forces that shape the curriculum and hold it in place as a classroom event. Only if such classroom knowledge is incorporated into reform proposals will the quality of students' work change.' (p 179)

There is no empirical evidence from this study that investigations are beneficial.

## **Teacher beliefs**

Perry et al. (1999) examined the beliefs of mathematics teachers from the perspective of a two-factor model of Transmission and Child-centred. They found that head mathematics teachers believed more in the child-centred approach than less experienced teachers. They also concluded that the transmission approach is so ingrained that the 'impact of any reform agenda will be minimal' (p 50). Again, an interesting article which adds to the literature on teacher beliefs. However, it hardly supports the introduction of investigations.

## **General papers on investigations**

A number of opinion papers have been written on investigations. Because of the near absence of empirical research supporting investigations, they are not research-based.

Chapin (1998) defined investigations and explained how they can be used in the classroom. Examples were given and a rationale for their inclusion expressed.

Oliveira et al. (1997) gave a rationale as to why they think investigations should be included. They then described a project between them and teachers to develop tasks. From this process some questions were developed. The teachers then reported back on how their 'experiments' in the classroom went. Anecdotal evidence was given that the students generally react positively to the tasks and enjoy them. No assessments were made nor any student work provided.

Frobisher (1994) defines investigations and distinguished them from problem-solving, although making the point that they are connected. The Cockcroft Report and NCTM Standards and the need for reform were discussed. A detailed description was given of the processes used (developed) during problem-solving and investigating. Polya's work was quoted. The role of the teacher was examined, especially for the novice attempting investigations for the first time. The importance of the teacher's beliefs and attitude was emphasised. Some examples were given. An interesting article but no research was conducted.

## **General papers on assessment**

Garfield (1993) discussed the need to broaden assessment techniques. The practical project as an assessment method was described and data analysis was used as an example. An analytic scoring method was then described as a method of assessing a project. Different

scenarios were presented which were then assessed according to the given method. A rating sheet to score projects was given. This paper is an interesting description of how a project might be assessed but no research was conducted.

Haines and Izard (1994) explored the issues related to changes in Higher Education curricula, and the link with assessment. Eisner's eight criteria for creating and appraising extended tasks were stated. The focus was on how to assess projects and oral communication. It did not comment on the actual learning through such tasks.

Stephens and Money (1993) examined the changes to curriculum and assessment in Victoria. They explained how the new VCE was introduced in 1990 and how this has influenced changes to junior secondary years. They detailed the three work requirements in the study of mathematics: skills practice and standard applications; problem-solving and modelling; projects (extended independent investigations). The four common assessment tasks used were examined, including an investigative project. This may be an interesting article but have the effects of these changes been assessed? Victoria performed very poorly in the TIMSS at the lower secondary level, which must throw some doubt on its effectiveness. A second observation concerns the amount of verification that teachers need to undertake to ensure that the students have completed the tasks themselves.

## **Conclusions**

From an analysis of the above articles we conclude that no empirical evidence has been provided that a teaching methodology of longer investigations is more effective than any other teaching method. To be fair to the authors of the articles, they do not necessarily claim that their studies were designed with this in mind. In contrast, the laboratory-style methods of experimental psychology are quite conclusive. Direct instruction techniques such as worked examples are more effective than problem-solving or discovery methods.

Many articles in the mathematics education literature (including the paper at Part A by Grimison & Dawe) refer to the reform changes that have been made in the USA and England. Victoria is also cited as an example of progressive change (see Stephens & Money, 1993). It is rather ironic that the TIMSS results indicate that the USA and England were placed in the last band for secondary mathematical achievement. Victoria was also placed in this band. In contrast, Australia overall was placed in the middle band, a considerable distance behind Singapore, Korea, Japan and Hong Kong in the top band. These international results hardly instil confidence in the progressive systems.

If there are advantages to investigations, the onus of proof lies with the technique's protagonists. The evidence is missing and until it becomes available the sensible course for the Board is to leave the technique as a voluntary exercise for those who believe in its efficacy. The 'math wars' of the US provide a salutary warning of the risks associated with compelling teachers to use extremely controversial techniques based on little or no evidence. The current policy of providing choice eliminates that risk.

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## Part C

### Joint Conclusions of the Authors

1. The 'mathematics industry' is funded by society primarily to ensure that citizens have technical competence in those areas of mathematics deemed essential to the functioning of an advanced industrial society. The four authors are unanimous that this function must have priority in any mathematics syllabus and that the current syllabus serves this function well.
2. All four authors agree that flexible mathematical problem-solving is an important outcome of mathematics teaching. The authors of Part A of this review feel that the experience of open-ended, content-specific investigational work (which emphasises process) is an important way of realising this outcome. The authors of Part B feel that a deep knowledge base is essential for flexible problem-solving and that therefore investigational work should not be compulsory.
3. The four authors agree that direct instruction with worked examples has been shown to be an efficient and useful method for mathematics learning. However, teachers should be aware that a wide range of other strategies such as group work which incorporates discussion, practical work, investigational work and problem-solving can also be effective under some circumstances. Consequently, teachers should be free to choose their strategies in order to achieve their goals.
4. There is a need for the syllabus committee to clarify the mathematical purposes of investigational work, and the written report, as opposed to general problem-solving. The four authors suggest that additional material be prepared for teachers with better examples and well-defined outcomes, regardless of whether investigational work is optional or mandatory.