BOARD OF STUDIES
NEW SOUTH WALES

2001 HSC Specimen Paper

Mathematics Extension 2

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2001 Higher School Certificate Specimen Examination Paper

Mathematics Extension 2

Introduction

This booklet contains the specimen examination paper for the 2001 Higher School Certificate examination in Mathematics Extension 2. A mapping grid is also included, showing how each question in the examination relates to the syllabus outcomes and content, and to the performance bands.

The specimen paper shows the format of the New HSC examination. It has been printed on A4 paper and side-stapled to make it convenient for use in schools. Actual examination papers will be produced as A4 booklets. All New HSC papers will be printed on white paper.

The 2001 HSC specimen papers have been produced in accordance with the Board's *Principles for Setting HSC Examinations in a Standards-Referenced Framework*, published in Board Bulletin Volume 8 Number 9 (Nov/Dec 99). Questions are closely related to the outcomes of the course, and the paper as a whole is structured to allow for appropriate differentiation of student performance at all levels on the performance scale.

The papers have been designed so that students have a clear understanding of what they are required to do in each question and in working through the paper. Instructions have been standardised, and the demands of the questions have been made explicit. Key words in questions, such as 'discuss', 'analyse', and 'explain', have been used consistently in accordance with the glossary published in the Board's *Assessment Support Document.*

This specimen paper is an example of the type of examination that could be prepared within the current examination specifications. Examinations will be based on the syllabus, and will test a representative sample of syllabus outcomes. Therefore, the range and balance of outcomes tested in HSC examinations in 2001 and subsequent years may differ from those addressed in the specimen paper.

The mapping grid is an important feature of the development of the examination. It aids in ensuring that the examination as a whole samples a range of content and outcomes, and allows all students the opportunity to demonstrate their level of achievement. Where courses have components in the examination other than written papers, the grid indicates the wider range of outcomes that are assessed by including these other components.

In considering the Mathematics Extension 2 specimen paper, it should be noted that the paper is based on the 1999 Mathematics 4 Unit (Additional) HSC Examination, with minor changes to ensure consistency with the 2001 HSC Examination format and the Board's *Principles for Setting HSC Examinations in a Standards-Referenced Framework.*

Mathematics Extension 2

HSC Specimen Examination Mapping Grid

For each item in the examination, the grid shows the marks allocated, the syllabus content and syllabus outcomes it relates to, and the bands it is targeting on the Draft Mathematics Extension 2 performance scale. The range of bands shown indicates the performance candidates may be able to demonstrate in their responses. That is, if an item is shown as targeting Bands E2 – E3, it indicates that candidates who demonstrate performance equivalent to the Band E2 descriptions should be able to score some marks on the item, while those who perform at Band E3 or above could reasonably be expected to gain high marks. In the case of one-mark items, candidates who demonstrate performance at or above the bands shown generally could be expected to answer the item correctly.

Question	Marks	Content	Syllabus outcomes	Targeted performance bands
1(a)	2	Integration	H8	E2 – E3
1(b)	2	Integration	HE6	E2 – E3
1(c)	3	Integration	H8, E4	E2 – E3
1(d)(i)	2	Integration	E4	E2 – E3
1(d)(ii)	2	Integration	E8	E2 – E3
1(e)	4	Integration	E8	E2 – E3
2(a)(i)	1	Complex Numbers	E3	E2 – E3
2(a)(ii)	2	Complex Numbers	E3	E2 – E3
2(b)(i)	2	Complex Numbers	E3	E2 – E3
2(b)(ii)	2	Complex Numbers	E3	E2 – E3
2(c)	2	Complex Numbers	E3	E2 – E3
2(d)(i)	1	Polynomials	E2	E2 – E3
2(d)(ii)	1	Polynomials	E4	E2 – E3
2(e)(i)	2	Complex Numbers	E2, E3, E9	E2 – E3
2(e)(ii)	2	Complex Numbers	E3, E9	E3 – E4
3(a)(i)	2	Graphs	E6	E2 – E3
3(a)(ii)	2	Graphs	E6	E2 – E3
3(a)(iii)	2	Graphs	E6	E3 – E4
3(b)(i)	2	Conics	E4	E2 – E3
3(b)(ii)	2	Conics	E3	E2 – E3
3(b)(iii)	2	Conics	E2, E3, E4	E3 – E4
3(b)(iv)	3	Conics	E2, E3, E4	E3 – E4
4(a)	4	Volumes	E7	E2 – E4
4(b)(i)	2	Polynomials	E2, E4	E2 – E3
4(b)(ii)	2	Polynomials	E4	E2 – E3
4(b)(iii)	2	Polynomials	E4	E2 – E3
4(c)(i)	1	Graphs	E6	E2 – E3
4(c)(ii)	2	Graphs	E6	E2 – E3
4(c)(iii)	2	Graphs	E6	E3 – E4
5(a)(i)	2	Polynomials	E4	E3 – E4
5(a)(ii)	1	Polynomials	E4	E2 – E3
5(b)(i)	1	Mechanics	E9	E2 – E3
5(b)(ii)	3	Mechanics	E2, E5, E9	E2 – E3
5(c)(i)	2	Harder 3 Unit Topics	HE3, E2	E2 – E3
5(c)(ii)	3	Harder 3 Unit Topics	HE3, E2, E4	E3 – E4
5(c)(iii)	3	Harder 3 Unit Topics	HE3, HE7, E2, E4	E3 – E4

6(a)(i)	3	Harder 3 Unit Topics	HE2, E2, E4, E9	E2 – E4
6(a)(ii)	1	Harder 3 Unit Topics	E2, E9	E3 – E4
6(b)(i)	2	Mechanics	E2, E5	E2 – E3
6(b)(ii)	2	Mechanics	E2, E4, E5	E3 – E4
6(b)(iii)	2	Mechanics	E2, E4, E5	E3 – E4
6(b)(iv)	2	Mechanics	E2, E4	E3 – E4
6(b)(v)	2	Mechanics	E2, E4, E9	E3 – E4
6(b)(vi)	1	Mechanics	E2, E9	E3 – E4
7(a)(i)	2	Graphs	P4, H9	E2 – E3
7(a)(ii)	2	Graphs	H5, E2, E9	E2 – E3
7(a)(iii)	2	Harder 3 Unit Topics; Graphs	E2, E6, E9	E3 – E4
7(a)(iv)	2	Harder 3 Unit Topics; Graphs	E2, E6, E9	E3 – E4
7(a)(v)	2	Harder 3 Unit Topics	E2, E4, E9	E3 – E4
7(b)(i)	1	Harder 3 Unit Topics	H5, E2	E3 – E4
7(b)(ii)	2	Harder 3 Unit Topics	H5, E2	E3 – E4
7(b)(iii)	2	Harder 3 Unit Topics	H5, E2	E3 – E4
8(a)(i)	1	Complex Numbers; Polynomials	E2, E4, E9	E3 – E4
8(a)(ii)	2	Complex Numbers; Polynomials	E2, E3, E9	E3 – E4
8(a)(iii)	3	Polynomials	E2, E4, E9	E3 – E4
8(a)(iv)	2	Complex Numbers; Polynomials	E2, E3, E9	E3 – E4
8(b)(i)	2	Harder 3 Unit Topics	PE3, E2, E9	E3 – E4
8(b)(ii)	1	Harder 3 Unit Topics	PE3, E2, E9	E3 – E4
8(b)(iii)	2	Harder 3 Unit Topics	PE3, E2, E9	E3 – E4
8(b)(iv)	2	Harder 3 Unit Topics	PE3, E2, E9	E3 – E4

BOARD OF STUDIES
NEW SOUTH WALES

Sample marking guidelines for Mathematics Extension 2

The following marking guidelines have been developed for selected questions from the 2001 HSC Specimen Examination in Mathematics Extension 2. These guidelines indicate the approach that would be taken to marking questions.

For each question, the following are typically included:

1. The syllabus outcomes that are targeted by the question.
2. The assessment rubric from the specimen paper, where there is one, listing the set of general criteria that are used to assess responses.
3. The marking guidelines, which show the criteria to be applied to responses along with the marks to be awarded in line with the quality of the responses. For extended-response questions, performance is described at a number of levels of performance, each covering a range of marks.
4. A sample answer or some points that answers might include. Sample answers indicate the scope and depth of treatment expected, and are not intended to be prescriptive. Similarly, the points that could be included in answers are not intended to be an exhaustive list, but rather an indication of the considerations that students could include in their responses.

Marking guidelines will generally require some refinement at the Marking Centre to take account of unanticipated responses that students present. For essay-type questions, the standard described at each mark range will be made clear during pilot-marking by the selection of sample scripts.

In a standards-referenced framework, examination questions are closely linked to syllabus content and outcomes. Expectations of the question are to be clear in the wording of the question. Marking guidelines will be developed at the same time as the examination questions, by examination committees. The development of marking guidelines will be guided by the Board's *Principles for Developing Marking Guidelines in a Standards-Referenced Framework*, published in Board Bulletin Volume 9 Number 3 (May 2000).

Question 5 (15 marks) **Marks**

(a) The roots of $x^3 + 5x^2 + 11 = 0$, are α, β and γ.

(i) Find the polynomial equation whose roots are α^2, β^2 and γ^2. **2**

Outcome assessed: E4

MARKING GUIDELINES

Criteria	Marks
• Gives correct polynomial equation $y^3 - 25y^2 - 110y - 121 = 0$	2
• Makes partial progress towards the solution, such as making the substitution $x = \sqrt{y}$ or $\alpha^2\beta^2\gamma^2 = 121$	1

Sample answer:

$x^3 + 5x^2 + 11 = 0$ ________ ①

Let $y = x^2$

$x = \sqrt{y}$

Substituting into ① gives

$$\left(\sqrt{y}\right)^3 + 5\left(\sqrt{y}\right)^2 + 11 = 0$$

$$y\sqrt{y} = -5y - 11$$

Squaring both sides:

$$y^3 = 25y^2 + 110y + 121$$

The required polynomial equation is $y^3 - 25y^2 - 110y - 121 = 0$

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **1**

Outcome assessed: E4

MARKING GUIDELINES

Criteria	Marks
• Calculates the correct value of $\alpha^2 + \beta^2 + \gamma^2$ directly, or obtains a value consistent with their cubic polynomial in part (i)	1

Marks

Sample answer:

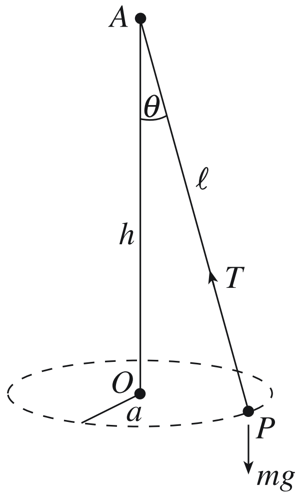
α^2, β^2 and γ^2 are the roots of
$y^3 - 25y^2 - 110y - 121 = 0$ found in part (i).

So, using sum of roots result:

$$\alpha^2 + \beta^2 + \gamma^2 = -\frac{b}{a}$$
$$= 25$$

OR $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$
$$= (-5)^2 - 2(0)$$
$$= 25$$

(b)



A conical pendulum consists of a bob P of mass m kg and a string of length ℓ metres. The bob rotates in a horizontal circle of radius a and centre O at a constant angular velocity of ω radius per second. The angle OAP is θ and $OA = h$ metres. The bob is subject to a gravitational force of mg newtons and a tension in the string of T newtons.

(i) Write down the magnitude, in terms of ω, of the force acting on P towards centre O. **1**

Outcome assessed: E9

MARKING GUIDELINES

Criteria	Marks
• Writes $ma\omega^2$ or $\frac{mv^2}{a}$ as the force acting on P towards centre O.	1

Marks

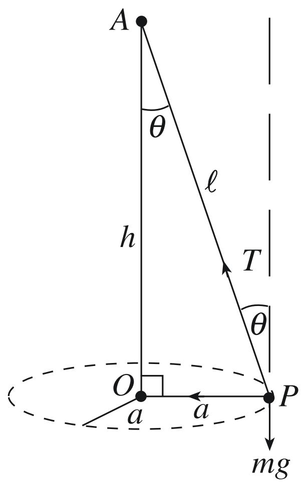
(ii) By resolving forces, show that $\omega^2 = \frac{g}{h}$. **3**

Outcomes assessed: E2, E5, E9

MARKING GUIDELINES

Criteria	Marks
• Resolves both vertical and radial components of force • Combines these results to deduce $\omega^2 = \frac{g}{h}$	3
• Resolves both vertical and radial components of force	2
• Resolves either the vertical or the radial component of force	1

Sample answer:



$$\text{Resolving vertically: } T\cos\theta - mg = 0$$
$$T\cos\theta = mg$$
$$\text{Resolving radially: } ma\omega^2 = T\sin\theta$$

$$\therefore \frac{T\sin\theta}{T\cos\theta} = \frac{ma\omega^2}{mg}$$
$$\tan\theta = \frac{a\omega^2}{g} = \frac{a}{h} \text{ (from } \Delta AOP)$$
$$\therefore \omega^2 = \frac{g}{h}$$

Marks

(c) At time t a wasp population consists of $w(t)$ workers and $r(t)$ reproductives. For the first s days of the wasp season the population produces workers only and after s days the population produces reproductives only.

(i) For $0 \le t \le s$, suppose that the equations determining the number of workers are **2**

$$\frac{dw}{dt} = k_1 w \quad \text{and} \quad w(0) = 1,$$

where k_1 is a positive constant.

Find an expression for $w(s)$.

Outcomes assessed: HE3, E2

MARKING GUIDELINES

Criteria	Marks
• Deduces $w(t) = Ae^{k_1 t}$ satisfies given condition • Uses initial conditions to calculate A	2
• Deduces $w(t) = Ae^{k_1 t}$ satisfies given condition	1

Sample answer:

The function $w(t) = Ae^{k_1 t}$ satisfies $\dfrac{dw}{dt} = k_1 w$

$$w(0) = 1 \Rightarrow A = 1$$

$$\therefore w(t) = e^{k_1 t}$$

and so $w(s) = e^{k_1 s}$

(ii) For $t \ge s$, suppose that the equations determining the number of reproductives are **3**

$$\frac{dr}{dt} = k_2 w(s) \quad \text{and} \quad r(s) = 0,$$

where k_2 is a positive constant.

Show that $r(t) = k_2 e^{k_1 s}(t - s)$ for $t \ge s$.

Outcomes assessed: HE3, E2, E4

MARKING GUIDELINES

Criteria	Marks
• Uses the techniques of calculus to show $r(t) = k_2 e^{k_1 s}(t - s)$ OR • Shows that the given expression satisfies $\frac{dr}{dt} = k_2 w(s)$ and $r(s) = 0$	3
• Substitutes their expression for $w(s)$ into $\frac{dr}{dt} = k_2 w(s)$ and integrates correctly, or equivalent by alternative method • Shows that the given expression satisfies $\frac{dr}{dt} = k_2 w(s)$	2
• Substitutes their expression for $w(s)$ into $\frac{dr}{dt} = k_2 w(s)$ OR • Shows that the given expression satisfies $r(s) = 0$	1

Sample answer:

$$\frac{dr}{dt} = k_2\ w(s)$$

$$\frac{dr}{dt} = k_2 e^{k_1 s}$$

$$\therefore\ r = \int k_2 e^{k_1 s}\ dt$$

$$r = k_2 e^{k_1 s} t + C$$

As $r(s) = 0,\quad 0 = k_2 e^{k_1 s} s + C$

$$\therefore C = -k_2 e^{k_1 s} s$$

$$\therefore r = k_2 e^{k_1 s} t - k_2 e^{k_1 s} s$$

$$r = k_2 e^{k_1 s}(t - s)$$ as required.

Marks

(iii) If $k_1 = 0.04$, find the value of s which maximises $r(100)$. **3**

Outcomes assessed: HE3, HE7, E2, E4

MARKING GUIDELINES

Criteria	Marks
• Substitutes $t = 100$ into $r(t)$ • Differentiates the resulting expression correctly • Solves $\frac{dR}{ds} = 0$ correctly where $R = r(100)$	3
• Substitutes $t = 100$ into $r(t)$ and differentiates the resulting expression correctly OR • Substitutes $t = 100$ into $r(t)$ and solves their (incorrect) $\frac{dR}{ds} = 0$ correctly, with $e^{0.04s}$ handled correctly	2
• Substitutes $t = 100$ into $r(t)$	1

Sample answer:

$$k_1 = 0.04$$

Let $R = r(100)$

ie $R = k_2 e^{0.04s}(100 - s)$

$$\frac{dR}{ds} = k_2 e^{0.04s}(-1) + k_2(100 - s)0.04e^{0.04s}$$

$$= k_2 e^{0.04s}(-1 + 4 - 0.04s)$$

$$= k_2 e^{0.04s}(3 - 0.04s)$$

Maximum occurs when $\frac{dR}{ds} = 0$

ie $k_2 e^{0.04s}(3 - 0.04s) = 0$

$$0.04s = 3 \qquad \left(k_2 e^{0.04s} \neq 0\right)$$

$$s = \frac{3}{0.04}$$

$$s = 75$$

Check maximum:

s	<75	75	>75
$\frac{dR}{ds}$	+	0	—
R	/	—	\

$\therefore$ $s = 27$ maximises $r(100)$.

BOARD OF STUDIES
NEW SOUTH WALES

2001
HIGHER SCHOOL CERTIFICATE
SPECIMEN EXAMINATION

Mathematics Extension 2

Total marks (**120**)
- Attempt Questions 1 – 8
- All questions are of equal value

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using blue or black pen
- Board-approved calculators may be used
- A table of standard integrals is provided on page 14
- All necessary working should be shown in every question

Total marks (120)
Attempt Questions 1 – 8
All questions are of equal value

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

Marks

Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate $\int_0^1 xe^{-x^2}\,dx$. **2**

(b) Using the substitution $u = e^x$, or otherwise, find $\int \frac{e^x dx}{\sqrt{1-e^{2x}}}$. **2**

(c) Find $\int \frac{4x^3 - 2x^2 + 1}{2x - 1}\,dx$. **3**

(d) (i) Find constants a, b and c such that **2**

$$\frac{x^2 + 2x}{(x^2+4)(x-2)} = \frac{ax+b}{x^2+4} + \frac{c}{x-2}.$$

(ii) Hence find $\int \frac{x^2 + 2x}{(x^2+4)(x-2)}\,dx$. **2**

(e) Use integration by parts to evaluate $\int_0^{\frac{\pi}{2}} x^2 \sin x\,dx$. **4**

Marks

Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 3 + 2i$ and $w = -1 + i$. Express the following in the form $a + ib$, where a and b are real numbers:

(i) zw **1**

(ii) $\dfrac{2}{iw}$. **2**

(b) Let $\alpha = 1 + i\sqrt{3}$.

(i) Find the exact value of $|\alpha|$ and $\arg \alpha$. **2**

(ii) Find the exact value of α^{11} in the form $a + ib$, where a and b are real numbers. **2**

(c) Sketch the region in the Argand diagram where the two inequalities $|z - i| \le 2$ and $0 \le \arg(z + 1) \le \dfrac{\pi}{4}$ both hold. **2**

(d) Consider the equation $2z^3 - 3z^2 + 18z + 10 = 0$.

(i) Given that $1 - 3i$ is a root of the equation, explain why $1 + 3i$ is another root. **1**

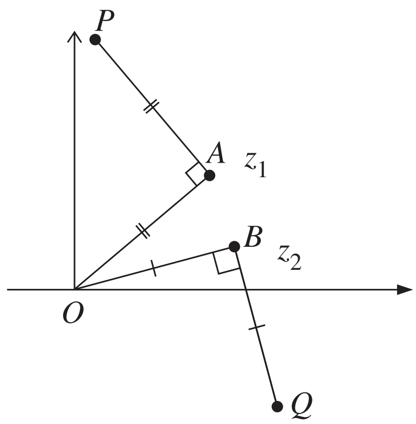
(ii) Find all roots of the equation. **1**

Question 2 continues on page 4

Marks

Question 2 (continued)

(e)



The points A and B in the complex plane correspond to complex numbers z_1 and z_2 respectively. Both triangles OAP and OBQ are right-angled isosceles triangles.

(i) Explain why P corresponds to the complex number $(1 + i)z_1$. **2**

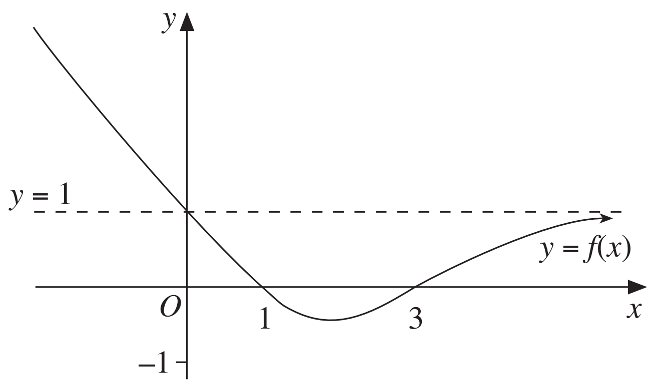
(ii) Let M be the midpoint of PQ. What complex number corresponds to M? **2**

End of Question 2

Marks

Question 3 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows the graph of the function $y = f(x)$. The graph has a horizontal asymptote at $y = 1$.

Draw separate half-page sketches of the graphs of the following functions:

(i) $y = |f(x)|$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \ln f(x)$. **2**

(b) Consider the ellipse $\mathcal{E}$ with equation

$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

and let $P = (x_0, y_0)$ be an arbitrary point on $\mathcal{E}$.

(i) Calculate the eccentricity of $\mathcal{E}$. **2**

(ii) Find the coordinates of the foci of $\mathcal{E}$ and the equations of the directrices of $\mathcal{E}$. **2**

(iii) Show that the equation of the tangent at P is **2**

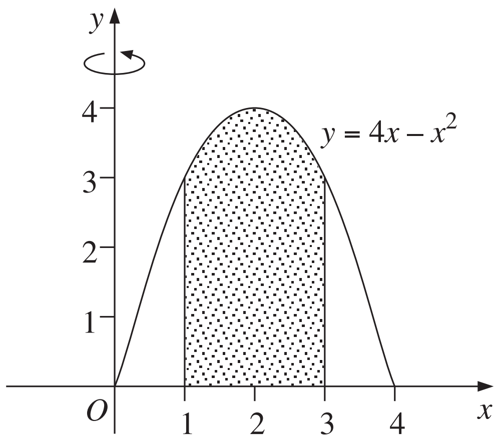
$$\frac{x_0 x}{5^2} + \frac{y_0 y}{3^2} = 1 .$$

(iv) Let the tangent at P meet a directrix at a point L. Show that $\angle PFL$ is a right angle where F is the corresponding focus. **3**

Marks

Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) **4**



The shaded area shown on the diagram between the curve $y = 4x - x^2$, the x axis, $x = 1$ and $x = 3$, is rotated about the y axis to form a solid. Use the method of cylindrical shells to find the volume of the solid.

(b) (i) Suppose the polynomial $P(x)$ has a double root at $x = \alpha$. Prove that $P'(x)$ also has a root at $x = \alpha$. **2**

(ii) The polynomial $A(x) = x^4 + ax^2 + bx + 36$ has a double root at $x = 2$. Find the values of a and b. **2**

(iii) Factorise the polynomial $A(x)$ of part (ii) over the real numbers. **2**

(c) (i) Determine the domain of the function $\sin^{-1}(3x + 1)$. **1**

(ii) Sketch the graph of the function $y = \sin^{-1}(3x + 1)$. **2**

(iii) Solve $\sin^{-1}(3x + 1) = \cos^{-1}x$. **2**

Marks

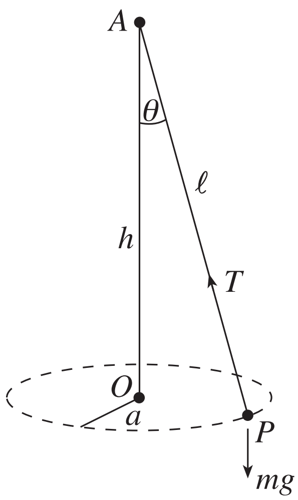
Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The roots of $x^3 + 5x^2 + 11 = 0$, are α, β and γ.

(i) Find the polynomial equation whose roots are α^2, β^2 and γ^2. **2**

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$. **1**

(b)



A conical pendulum consists of a bob P of mass m kg and a string of length ℓ metres. The bob rotates in a horizontal circle of radius a and centre O at a constant angular velocity of ω radians per second. The angle OAP is θ and $OA = h$ metres. The bob is subject to a gravitational force of mg newtons and a tension in the string of T newtons.

(i) Write down the magnitude, in terms of ω, of the force acting on P towards centre O. **1**

(ii) By resolving forces, show that $\omega^2 = \dfrac{g}{h}$ **3**

Question 5 continues on page 8

Marks

Question 5 (continued)

(c) At time t a wasp population consists of $w(t)$ workers and $r(t)$ reproductives. For the first s days of the wasp season the population produces workers only and after s days the population produces reproductives only.

(i) For $0 \le t \le s$, suppose that the equations determining the number of workers are **2**

$$\frac{dw}{dt} = k_1 w \quad \text{and} \quad w(0) = 1 ,$$

where k_1 is a positive constant.

Find an expression for $w(s)$.

(ii) For $t \ge s$, suppose that the equations determining the number of reproductives are **3**

$$\frac{dr}{dt} = k_2 w(s) \quad \text{and} \quad r(s) = 0 ,$$

where k_2 is a positive constant.

Show that $r(t) = k_2 e^{k_1 s}(t - s)$ for $t \ge s$.

(iii) If $k_1 = 0.04$, find the value of s which maximises $r(100)$. **3**

End of Question 5

Marks

Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Let x be a fixed, non-zero number satisfying $x > -1$. Use the method of mathematical induction to prove that **3**

$$(1 + x)^n > 1 + nx$$

for $n = 2, 3, \ldots$.

(ii) Deduce that $\left(1 - \frac{1}{2n}\right)^n > \frac{1}{2}$ for $n = 2, 3, \ldots$. **1**

(b) A ball of unit mass is projected vertically upwards from ground level with initial speed U. Assume that air resistance is kv, where v is the ball's speed and k is a positive constant.

We wish to consider the ball's motion as it falls back to ground level. Let y be the displacement of the ball measured *vertically downwards* from the point of maximum height, t be the time elapsed after the ball has reached maximum height, and g be the acceleration due to gravity.

(i) Explain why $v(0) = 0$, and $\frac{dv}{dt} = g - kv$ while the ball is in motion. **2**

(ii) Deduce that $v = \frac{g}{k}\left(1 - e^{-kt}\right)$ for $t \geq 0$. **2**

(iii) By writing $\frac{dv}{dt} = v\frac{dv}{dy}$, deduce from part (i) that **2**

$$\frac{g}{k}\log_e\left(\frac{g - kv}{g}\right) + v = -ky.$$

(iv) Using parts (ii) and (iii), deduce that $t = \frac{v + ky}{g}$ **2**

Question 6 continues on page 10

Marks

Question 6 (continued)

(v) You are given that the ball reaches maximum height **2**

$$h = \frac{1}{k}\left(U - \frac{g}{k}\log_e\left(\frac{g + kU}{g}\right)\right)$$

in time $t_h = \frac{1}{k}\log_e\left(\frac{g + kU}{g}\right)$.

(Do NOT prove these results.)

Deduce that the total time T that the ball is in the air is $T = \frac{U + V}{g}$, where V is the final speed that the ball reaches when returning to ground level.

(vi) If air resistance is ignored, the total time T_0 that the ball is in the air is $T_0 = \frac{U + V_0}{g}$, where V_0 is the final speed the ball then reaches when returning to ground level. By considering V_0 and V, determine which is larger: T or T_0. **1**

End of Question 6

Marks

Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Graph $y = \ln x$ and draw the tangent to the graph at $x = 1$. **2**

(ii) By considering the appropriate area under the tangent, deduce that **2**

$$\int_1^{\frac{3}{2}} \ln x \, dx \le \frac{1}{8}.$$

(iii) By considering the graph of $y = \ln x$, explain why **2**

$$\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \ln x \, dx \le \ln k \quad \text{for} \quad k = 2, 3, 4, \ldots .$$

(iv) Deduce that **2**

$$\int_1^n \ln x \, dx \le \frac{1}{8} + \ln 2 + \ln 3 + \cdots + \ln(n-1) + \frac{1}{2}\ln n \quad \text{for} \quad n = 2, 3, 4, \ldots .$$

(v) Assuming that $\int_1^n \ln x \, dx = n\ln n - n + 1$, deduce that **2**

$$n! \ge e^{\frac{7}{8}} n^n \sqrt{n}\, e^{-n} \quad \text{for} \quad n = 2, 3, 4, \ldots .$$

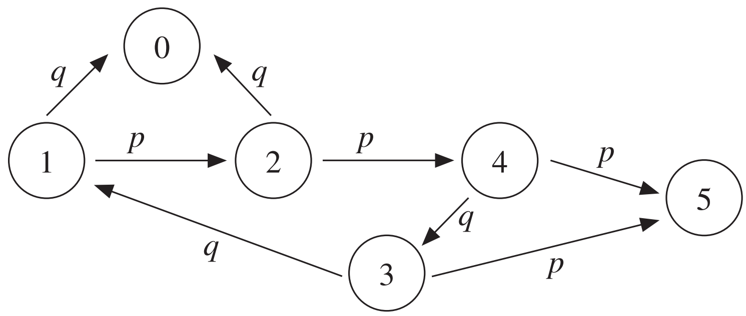
Question 7 continues on page 12

Marks

Question 7 (continued)

(b) A player has one token and needs exactly five tokens to win a prize. He plays a game where he can vary the number of tokens he bets. At each stage he either doubles the number of tokens he bets or loses the tokens he bets. The probability that he doubles the number of tokens he bets is p and the probability that he loses the number of tokens he bets is $q = 1 - p$. His strategy is to reach his goal of exactly five tokens as quickly as possible.

The diagram shows the possible outcomes in terms of number of tokens and the probabilities associated with each stage.



(i) Starting with one token, what is the probability that he loses all of his tokens without ever having four tokens? **1**

(ii) What is the probability that he obtains four tokens once and then loses all of his tokens without ever having four tokens again? **2**

(iii) If $p = \frac{1}{2}$, find the probability that he wins a prize. **2**

End of Question 7

Marks

Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Let $\rho = \cos\dfrac{2\pi}{7} + i\sin\dfrac{2\pi}{7}$. The complex number $\alpha = \rho + \rho^2 + \rho^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real.

(i) Prove that $1 + \rho + \rho^2 + \ldots + \rho^6 = 0$. **1**

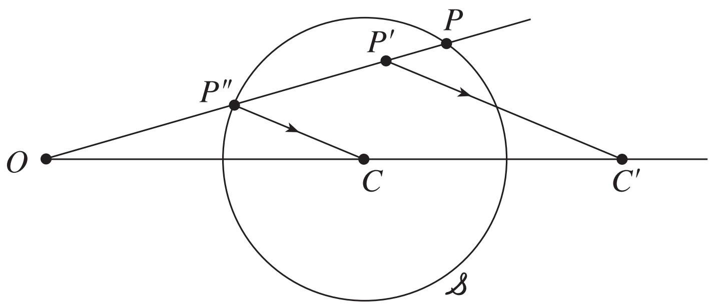
(ii) The second root of the quadratic equation is β. Express β in terms of positive powers of ρ. Justify your answer. **2**

(iii) Find the values of the coefficients a and b. **3**

(iv) Deduce that **2**

$$-\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2} .$$

(b)



In the diagram, $\mathcal{S}$ is a circle, centre C, and O is a fixed point outside the circle. The point P is a variable point on $\mathcal{S}$ and P'' is the other point of intersection of OP with $\mathcal{S}$. The point P' is on OP such that $OP \cdot OP' = k^2$ where k is a constant. The point C' is on OC and $P''C \parallel P'C'$.

(i) Explain why $OP \cdot OP''$ is a constant. **2**

(ii) Deduce that $\dfrac{OP''}{OP'}$ is a constant. **1**

(iii) Show that C' is a fixed point. **2**

(iv) Describe fully the locus of P'. **2**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$