BOARD OF STUDIES
NEW SOUTH WALES

Mathematics Extension 1

Stage 6

Draft Syllabus

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Contents

1 The Higher School Certificate Program of Study

The purpose of the Higher School Certificate program of study is to:

- provide a curriculum structure which encourages students to complete secondary education

- foster the intellectual, social and moral development of students, in particular developing their:
  - knowledge, skills, understanding and attitudes in the fields of study they choose
  - capacity to manage their own learning
  - desire to continue learning in formal or informal settings after school
  - capacity to work together with others
  - respect for the cultural diversity of Australian society

- provide a flexible structure within which students can prepare for:
  - further education and training
  - employment
  - full and active participation as citizens

- provide formal assessment and certification of students' achievements

- provide a context within which schools also have the opportunity to foster students' physical and spiritual development.

2 Rationale for Mathematics Advanced, Mathematics Extension 1 and Mathematics Extension 2 in the Stage 6 Curriculum

Mathematics is deeply embedded in modern society. From the numeracy skills required to manage personal finances, to making sense of data in various forms, to leading-edge technologies in the Sciences and Engineering, Mathematics provides the framework for interpreting, analysing and predicting, and the tools for effective participation in an increasingly complex society.

The need to interpret the large volumes of data made available through technology draws on skills in logical thought and in checking claims and assumptions in a systematic way. Mathematics is the appropriate training ground for the development of these skills. The thinking required to enhance further the power and usefulness of technology in real-world applications requires advanced mathematical training. The rapid advances in technology experienced in recent years have driven, and been driven by, advances in the discipline of Mathematics.

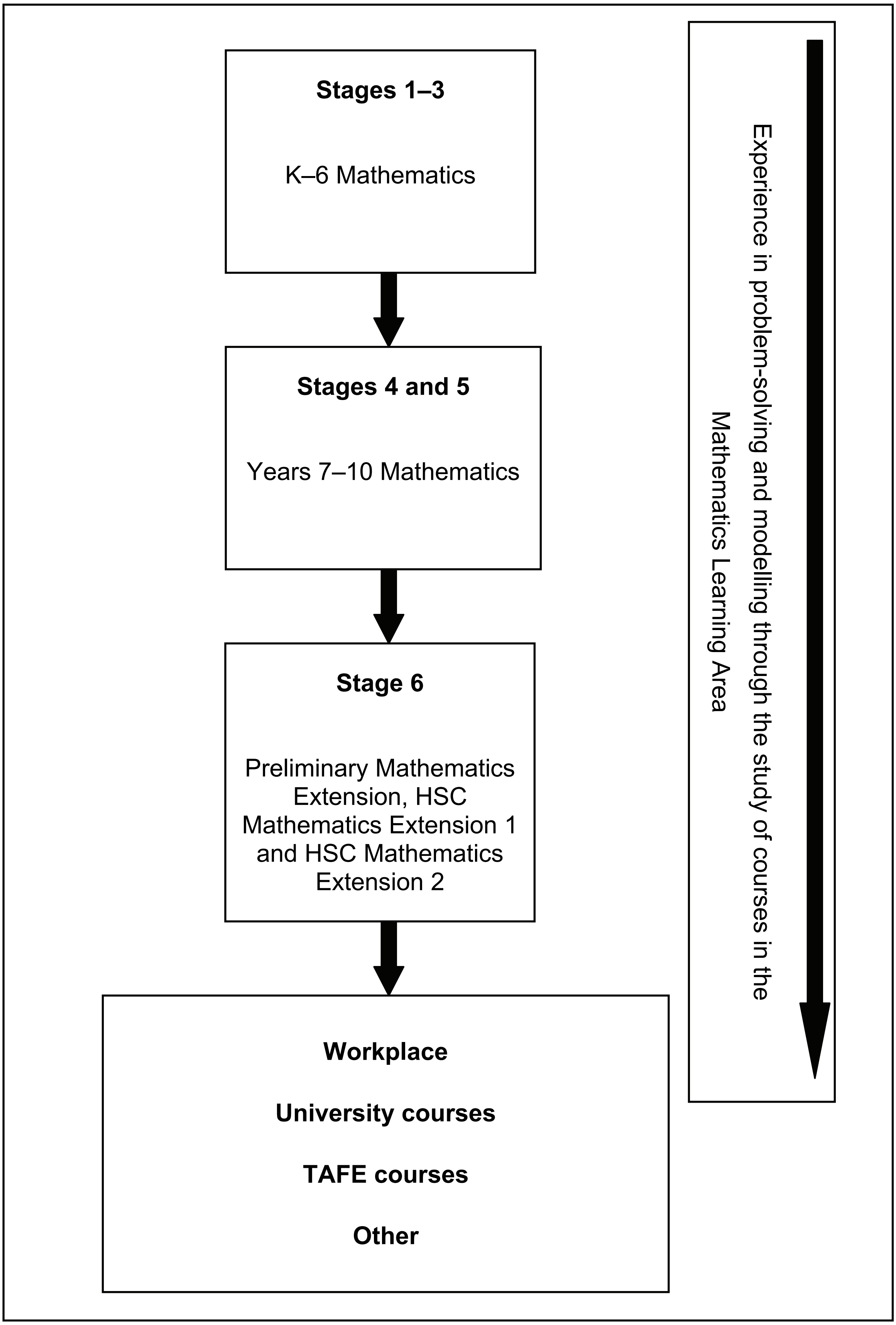
The development of Mathematics throughout history has been catalysed by its utility in explaining real-world phenomena and its inherent beauty. In this way, the discipline has continued to evolve through a process of observation, conjecture, proof and application.

The Mathematics Advanced, Mathematics Extension 1 and Mathematics Extension 2 courses provide the opportunity for students to acquire knowledge, skills and understanding in relation to important concepts within areas of Mathematics that have applications in an increasing number of contexts. These concepts and applications are appropriate to the students' continued experience of Mathematics as a coherent, interrelated, interesting and intrinsically valuable study that forms a basis for future learning. The introductory concepts and techniques of differential and integral calculus form a strong basis of the courses, and are developed and utilised across the courses, through a range of applications.

Students develop an appreciation of Mathematics as a study with high levels of internal structure that provide opportunities for the development of logical and disciplined thought. Through the learning experiences within the courses, students are able to progress from a knowledge and understanding of facts, procedures and applications in idealised contexts to facility in the use of mathematical models that situate the Mathematics in context and provide information on the behaviour of real-world systems, and to more advanced generalisations based on deductive and inductive reasoning processes. This involves the development and use of an increasingly sophisticated level of communication and literacy.

The courses provide students with the opportunity to study applications of Mathematics in a range of contexts relevant to contemporary professional practice, including examples from the Mathematics, Science, Engineering, Technology, Education, Business and Finance areas.

3 Continuum of Learning for Stage 6 Mathematics Extension 1 Students



4 Mathematics in Stage 6

There are five Board-developed Mathematics courses of study for the Higher School Certificate: (in increasing order of difficulty) Mathematics General 1, Mathematics General 2, Mathematics Advanced, Mathematics Extension 1, and Mathematics Extension 2.

Students of the Mathematics General 1 and Mathematics General 2 courses study a common Preliminary course, Preliminary Mathematics General, leading to the HSC Mathematics General 1 and HSC Mathematics General 2 courses.

Mathematics Advanced consists of the courses Preliminary Mathematics Advanced and HSC Mathematics Advanced. Students studying one or both Extension courses study Preliminary Mathematics Extension course before undertaking the study of HSC Mathematics Extension 1, or HSC Mathematics Extension 1 and HSC Mathematics Extension 2.

The following assumptions and recommendations regarding learning from Stage 5 Mathematics, typically undertaken by students in Years 9 and 10, are provided in relation to the study of the suite of Stage 6 courses. It is assumed that students who intend to study the Stage 6 Mathematics General 1 course have experienced all of the Stage 5.1 content. For students who intend to study the Stage 6 Mathematics General 2 course, it is recommended that they experience at least some of the Stage 5.2 content, particularly the *Patterns and Algebra* topics and *Trigonometry*, if not all of the content. For students who intend to study the Stage 6 Mathematics Advanced course, it is recommended that they experience the topics *Real Numbers*, *Algebraic Techniques* and *Coordinate Geometry* as well as at least some of *Trigonometry* and *Deductive Geometry* from 5.3 (identified by §), if not all of the content. For students who intend to study the Stage 6 Mathematics Extension 1 course, it is recommended that they experience the optional topics (identified by #) *Curve Sketching and Polynomials*, *Functions and Logarithms*, and *Circle Geometry*.

The Preliminary and HSC course components undertaken by students who study Mathematics General 1, Mathematics General 2, or Mathematics Advanced, and by students who study Stage 6 Mathematics to Mathematics Extension 1 or Mathematics Extension 2 level, are illustrated on the following pages.

Mathematics General 1 – Preliminary and HSC course components

Preliminary Mathematics General

Units: 2
Indicative hours: 120

→

HSC Mathematics General 1

Units: 2
Indicative hours: 120

Total indicative hours: 240

Mathematics General 2 – Preliminary and HSC course components

Preliminary Mathematics General

Units: 2
Indicative hours: 120

→

HSC Mathematics General 2

Units: 2
Indicative hours: 120

Total indicative hours: 240

Mathematics Advanced – Preliminary and HSC course components

Preliminary Mathematics Advanced

Units: 2
Indicative hours: 120

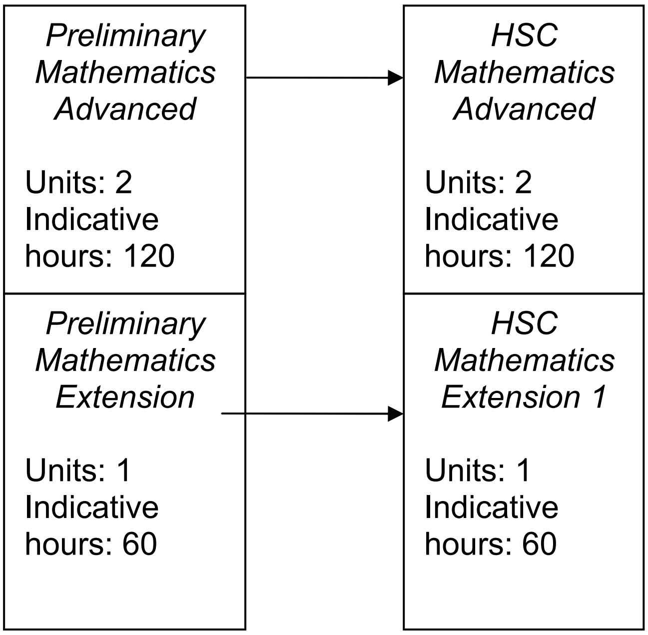
→

HSC Mathematics Advanced

Units: 2
Indicative hours: 120

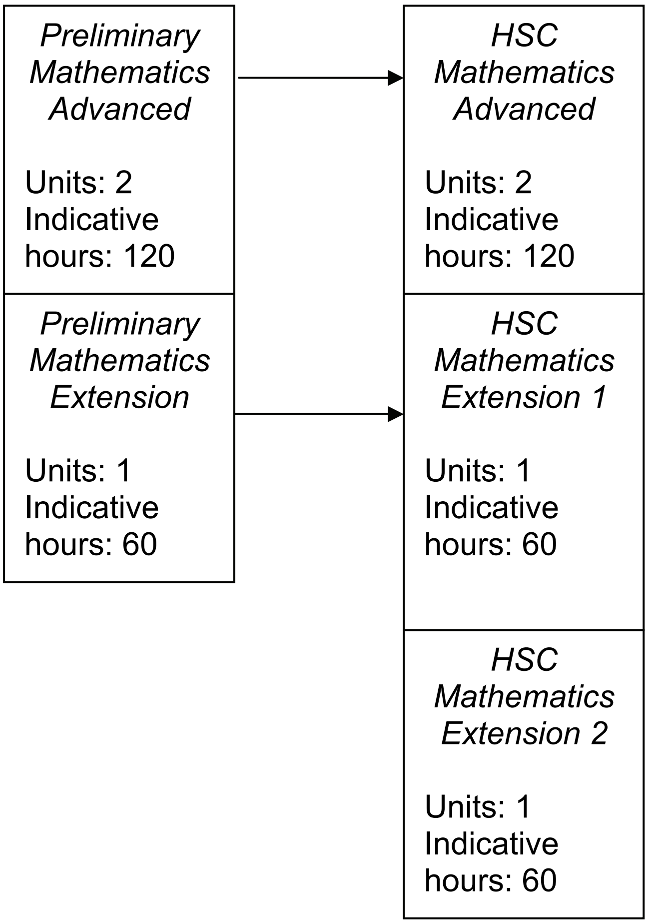
Total indicative hours: 240

Preliminary and HSC course components undertaken by students studying Stage 6 Mathematics to Mathematics Extension 1 level



Total indicative hours: 360

Preliminary and HSC course components undertaken by students studying Stage 6 Mathematics to Mathematics Extension 2 level



Total indicative hours: 420

5 Aim (Mathematics Advanced, Mathematics Extension 1, Mathematics Extension 2 courses)

The calculus-based Mathematics Advanced, Mathematics Extension 1 and Mathematics Extension 2 courses are designed to promote the development of knowledge, skills and understanding in relation to important concepts within areas of Mathematics that have applications in an increasing number of contexts. This includes the development of deductive and inductive reasoning skills and the ability to construct, solve and interpret mathematical models.

Students will learn to use a range of techniques and tools, including relevant technologies, in order to develop solutions to a wide variety of problems relating to their present and future needs and aspirations.

6 Objectives (Mathematics Advanced, Mathematics Extension 1, Mathematics Extension 2 courses)

Knowledge, understanding and skills

Students will develop the ability to:

- apply deductive and inductive reasoning, and use appropriate language, in the construction of proofs and mathematical arguments
- use concepts and techniques, including technology, in the solution of problems
- interpret and use mathematical models in a range of contexts
- interpret solutions to problems and communicate Mathematics in appropriate forms.

Values and attitudes

Students will develop:

- appreciation of the scope, usefulness, power and elegance of Mathematics.

7 Course Structure

The following schematic view illustrates the structure of the Mathematics Extension 1 course, through its Preliminary and HSC course components.

Preliminary Mathematics Extension Course	HSC Mathematics Extension Course
PMX1 Circle geometry	**MX1 Binomial theorem**
PMX2 Further algebra	**MX2 Further polynomials**
PMX3 Transformations of graphs	**MX3 Further trigonometry**
PMX4 Polynomials	**MX4 Methods of integration**
PMX5 Mathematical induction	**MX5 Inverse functions and the inverse trigonometric functions**
PMX6 Elementary difference equations and the discrete logistic growth model	**MX6 Further applications of calculus**

8 Objectives and Outcomes

8.1 Table of Objectives and Outcomes

	Mathematics Advanced	
Objectives	Preliminary Outcomes	HSC Outcomes
Students will develop the ability to:	**A student:**	**A student:**
apply deductive reasoning, and use appropriate language, in the construction of proofs and mathematical arguments	PA1 provides reasoning to support conclusions appropriate to the context	HA1 constructs arguments to prove and justify results

Mathematics Extension 1		**Mathematics Extension 2**
Preliminary Outcomes	HSC Outcomes	HSC Outcomes
A student:	**A student:**	**A student:**
PX1 uses deductive reasoning to solve problems and prove results in circle geometry		HXX1 constructs arguments and proofs in concrete and abstract settings
PX2 uses algebraic techniques to solve inequalities and prove results and identities	HX1 uses the binomial theorem and algebraic and calculus techniques to prove identities	HXX2 applies algebraic, graphical and calculus techniques in the construction of proofs involving inequalities

	Mathematics Advanced		**Mathematics Extension 1**		**Mathematics Extension 2**
Objectives	Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes	HSC Outcomes
Students will develop the ability to:	**A student:**	**A student:**	**A student:**	**A student:**	**A student:**
				HX2 uses mathematical induction in the construction of proofs	HXX3 constructs proofs involving inequalities
use concepts and techniques, including technology, in the solution of problems (Note: further outcomes related to this objective on pages 15–18)	PA2 uses algebraic and graphical concepts in the solution of problems involving, functions and coordinate geometry PA3 uses counting strategies in the solution of problems involving ordered and unordered selections	HA2 manipulates algebraic expressions and solves problems involving logarithmic and exponential functions	PX3 uses the relationship between the algebraic and geometric representations of a function in the solution of problems	HX3 uses the concept of inverse functions in the solution of problems HX4 demonstrates understanding of the significance of the binomial coefficients in counting, expansion of algebraic expressions, and probability calculations	HXX4 combines the ideas of algebra and calculus to determine features of graphs

	Mathematics Advanced		Mathematics Extension 1		Mathematics Extension 2
Objectives	Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes	HSC Outcomes
Students will develop the ability to:	**A student:**	**A student:**	**A student:**	**A student:**	**A student:**
	PA4 uses the concepts of complementary, mutually exclusive and independent events				
use concepts and techniques, including technology, in the solution of problems	PA5 uses concepts and techniques in trigonometry in the solution of problems			HX5 uses the properties of trigonometric functions in the derivation of formulae, construction of proofs, and solution of problems, including problems involving trigonometry in three dimensions	HXX5 performs arithmetic operations on complex numbers and uses De Moivre's theorem in the solution of problems involving powers and roots
(Note: further outcomes related to this objective on pages 14, 16–18)	PA6 uses the concept of circular angle measure in the solution of problems				HXX6 uses the relationship between algebraic and geometric representations of complex numbers

	Mathematics Advanced		**Mathematics Extension 1**		**Mathematics Extension 2**
Objectives	Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes	HSC Outcomes
Students will develop the ability to:	**A student:**	**A student:**	**A student:**	**A student:**	**A student:**
use concepts and techniques, including technology, in the solution of problems (Note: further outcomes related to this objective on pages 14,15,17–18			PX4 solves problems using concepts from the theory of polynomial functions	HX6 uses algebraic and calculus techniques in the investigation of the properties of polynomial functions	HXX7 uses concepts from the theory of polynomial functions and complex numbers in the solution of problems involving factors, roots and coefficients of polynomial functions

	Mathematics Advanced		Mathematics Extension 1		Mathematics Extension 2
Objectives	Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes	HSC Outcomes
Students will develop the ability to:	**A student:**	**A student:**	**A student:**	**A student:**	**A student:**
use concepts and techniques, including technology, in the solution of problems	PA7 determines the derivatives of functions through the routine application of the rules of differentiation	HA3 uses the graphical relationship between a function and its derivative in curve sketching and in the solution of problems	PX5 uses the theory of difference equations in the solution of problems		HXX8 solves first-order and second-order ordinary linear differential equations
(Note: further outcomes related to this objective on pages 14–16, 18)		HA4 applies techniques of differentiation and integration to logarithmic, exponential and trigonometric functions			
	PA8 uses concepts and techniques from descriptive statistics to present and interpret data	HA5 uses the concept of a z-score, standardises normal random variables, and solves related probability problems			

	Mathematics Advanced		Mathematics Extension 1		Mathematics Extension 2
Objectives	Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes	HSC Outcomes
Students will develop the ability to:	**A student:**	**A student:**	**A student:**	**A student:**	**A student:**
use concepts and techniques, including technology, in the solution of problems	PA9 derives general results for arithmetic and geometric series, and applies the results in the solution of problems	HA6 applies series techniques to the solution of financial problems and interprets results			
(Note: further outcomes related to this objective on pages 14–17)	PA10 uses the relationship between the primitive and derivative of a function and determines primitives for functions involving powers of x	HA7 uses techniques of integration to calculate definite integrals, areas and volumes		HX7 evaluates integrals using given substitutions and trigonometric identities	HXX9 uses integral calculus in the solution of problems requiring the use of integration tables, identification and use of appropriate substitutions, partial fractions, integration by parts, and recurrence formulae

	Mathematics Advanced		Mathematics Extension 1		Mathematics Extension 2
Objectives	Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes	HSC Outcomes
Students will develop the ability to:	**A student:**	**A student:**	**A student:**	**A student:**	**A student:**
use concepts and techniques, including technology, in the solution of problems			PX6 uses differential and integral calculus to interpret mathematical models of projectile motion, modified exponential growth and decay, and financial situations		HXX10 uses the techniques of slicing, cylindrical shells and similar cross-sections to calculate volumes
interpret and use mathematical models in a range of contexts	PA11 uses algebraic, numerical and graphical representations of linear and quadratic functions in mathematical modelling situations and interprets results in context	HA8 uses differential and integral calculus to interpret mathematical models of linear motion, exponential growth and decay, and financial situations		HX8 applies differential and integral calculus to mathematical modelling situations involving related rates, linear and projectile motion, and modified exponential growth and decay	HXX11 formulates and solves ordinary differential equations arising in mathematical modelling situations

	Mathematics Advanced		Mathematics Extension 1		Mathematics Extension 2
Objectives	Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes	HSC Outcomes
Students will develop the ability to:	**A student:**	**A student:**	**A student:**	**A student:**	**A student:**
interpret solutions to problems and communicate Mathematics in appropriate forms	PA12 interprets and uses mathematical language	HA9 interprets solutions to problems and communicates using mathematical language, notation, diagrams and graphs	PX7 communicates making comprehensive use of mathematical language, notation, diagrams and graphs	HX9 interprets solutions to problems and communicates the solutions in appropriate forms	HXX12 communicates abstract ideas and relationships using appropriate notation and logical argument

Values and attitudes **Student will develop:**	**A student:**	**A student:**	**A student:**	**A student:**	**A student:**
appreciation of the scope, usefulness, power and elegance of Mathematics	PA/VA demonstrates confidence in using Mathematics to obtain realistic solutions to problems	HA/VA seeks to apply mathematical techniques in the solution of problems	PX/VA demonstrates confidence in extending known concepts to derive results with broader applicability	HX/VA appreciates interrelationships between ideas drawn from different areas of Mathematics	HXX/VA appreciates the power and elegance of Mathematics in the solution of a broad range of problems

8.2 Key Competencies

Mathematics Extension 1 provides a context within which to develop general competencies considered essential for the acquisition of effective, higher-order thinking skills necessary for further education, work and everyday life.

Key competencies are embedded in the *Mathematics Extension 1 Stage 6 Syllabus* to enhance student learning. The key competencies are developed through the methodologies of the syllabus and through classroom pedagogy. The key competencies of collecting, analysing and organising information and communicating ideas and information, reflect core processes of mathematical inquiry undertaken by students as they engage with the various syllabus topics. Students work as individuals and as members of groups to engage with applications and modelling tasks. Through this, the key competencies of planning and organising activities and working with others and in teams are developed. At all levels of the course, students are developing the key competency of using mathematical ideas and techniques. Through the advice provided on the selection and use of appropriate technology, students can develop the key competency of using technology. Finally, students' continual involvement with seeking solutions to problems, both large and small, contributes towards their development of the key competency of solving problems.

Presentation of Content

The course content for the Mathematics Extension 1 course is presented in twelve topics, of which six are studied in the Preliminary course and six in the HSC course. Within each topic, the material is divided into cohesive subtopics, each of which contributes to the students' achievement of one or more of the course outcomes. It is intended that the prescribed knowledge, skills and understanding be developed through the study of appropriate tasks and applications that clearly demonstrate the need for such skills.

The course content for the Mathematics Extension 1 course is presented in the following format:

1. Initial facing pages for topic

Name of topic

A brief summary of the content/purpose of the topic.

Assumed Stage 5.3 outcomes [This section appears only for topics in the Preliminary course.]

Outcomes from Stage 5.3 of *Mathematics Years 7–10 Syllabus,* that students should have achieved to engage successfully with the topic.

Assumed Mathematics Advanced outcomes [This section appears only for topics in the Preliminary course.]

Outcomes from the Preliminary course of *Mathematics Advanced Stage 6 Syllabus,* that students should have achieved to engage successfully with the topic.

Outcomes addressed

A list of the course outcomes addressed in the study of the topic.

Content summary

A list of the subtopic(s) studied within the topic.

Terminology

A list of key words and/or phrases that may be met in the topic, some of which may be new to students.

Use of technology

Advice about the nature and use of technology that is appropriate to the topic.

Topic notes

Notes relevant to teaching particular aspects of the topic.

2. Subsequent facing pages for topic

Name of subtopic

A brief summary of the content/purpose of the subtopic.

Outcomes addressed

A list of the course outcomes addressed in the study of the subtopic.

Students develop the following knowledge, skills and understanding

The mathematical content to be addressed in the subtopic.

Applications and considerations

The provision of examples indicating the range and style of applications used to introduce and illustrate the mathematical content of the subtopic, as well as important considerations for learning and teaching the subtopic.

Use of technology

(a) in learning and teaching, and school-based assessment

The appropriateness, viability and level of use of different types of technology in the learning and teaching of courses within the Mathematics Key Learning Area are decisions for students, teachers and schools. However, the use of technology is encouraged in the learning and teaching, and school-based assessment, where appropriate, of courses within the learning area. In accordance with the Broad Directions from the first phase of the development of the revised Stage 6 Mathematics courses, the use of technology with capabilities beyond the level of scientific calculators is encouraged in the learning and teaching, and school-based assessment, of the courses ie Mathematics General 1, Mathematics General 2, Mathematics Advanced, Mathematics Extension 1, and Mathematics Extension 2.

Each of the five Stage 6 syllabuses contain advice and suggestions in relation to the use of a range of technology in the ‘Use of technology’ and ‘Applications and considerations’ sections within the course content. The *Mathematics Extension 1 Syllabus* provides a range of opportunities for the use of calculators and computer software packages in learning and teaching. This includes opportunities to utilise the graphing functions and financial and statistical capabilities of calculators, spreadsheets, and dynamic geometry and statistics software packages.

(b) in the HSC examinations

In the HSC examinations for the revised Stage 6 Board-developed Mathematics courses, candidates will be permitted to use only calculators manufactured to meet a clear set of Board-prescribed calculator functions and capabilities. These functions and capabilities will be consistent with and support the knowledge and skills that students should be able to demonstrate after completing a course, or courses. The list of functions and capabilities are being determined in parallel with the development of the content for the courses and will be completed in conjunction with the finalisation of the courses following consultation on the draft syllabuses.

9 Preliminary Mathematics Extension Course Content

PMX1 Circle geometry

PMX1.1 Definition of terms related to circles, simple angle properties of a circle
PMX1.2 Derivation of further angle, chord and tangent results, applications to numerical and theoretical problems requiring one or more steps of reasoning

PMX2 Further algebra

PMX2.1 General theory of quadratic equations
PMX2.2 Identity of two quadratic expressions, equations reducible to quadratics, algebraic results concerning cubes

PMX3 Transformations of graphs

PMX3.1 Stretching graphs horizontally and vertically, sketching graphs of reciprocals of known functions

PMX4 Polynomials

PMX4.1 Definitions of polynomial, degree, polynomial equation; graphs of simple polynomials
PMX4.2 The remainder and factor theorems

PMX5 Mathematical induction

PMX5.1 The principle of mathematical induction
PMX5.2 Use of mathematical induction to prove results relating to series and divisibility

PMX6 Elementary difference equations and the discrete logistic growth model

PMX6.1 Introduction to difference equations: terminology and simple examples
PMX6.2 Methods of solution of difference equations: first-order and second-order equations
PMX6.3 The logistic growth equation: equilibrium, periodic and chaotic solutions

Total indicative hours	**60 hours**

PMX1: Circle geometry

The principal focus of this topic is the use of standard results to solve problems in circle geometry.

Assumed Stage 5.3 outcomes

SGS5.3.1, SGS5.3.2, SGS5.3.3

Assumed Mathematics Advanced outcomes

PA1

Outcomes addressed

A student:

PX1 uses deductive reasoning to solve problems and prove results in circle geometry.

PX7 communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Content summary

PMX1.1 Definition of terms related to circles, simple angle properties of a circle

PMX1.2 Derivation of further angle, chord and tangent results, applications to numerical and theoretical problems requiring one or more steps of reasoning.

Terminology

arc
chord
concyclic points
radii
subtend
tangent

Use of technology

Many of the results in this topic can be explored with dynamic geometry software.

Topic notes

The value of this topic is mainly to be found in the problem solving that requires several steps of reasoning. Searching for a path through a harder problem in geometry involves strategies such as clearly stating the goal, working backwards, trying an easier case first, listing known information, and so on.

PMX1.1: Definition of terms related to circles, simple angle properties of a circle

In this subtopic, students consolidate and extend their knowledge and understanding of Euclidean geometry developed in Stage 5.

Outcomes addressed

PX1, PX7

Students develop the following knowledge, skills and understanding

- identifying and naming parts of a circle from their definitions
- identifying common tangents
- using the assumption that equal arcs on circles of equal radii subtend equal angles at the centre, and its converse
- using the results:
    (1) Equal angles at the centre stand on equal chords, and its converse
    (2) The angle at the centre is twice the angle at the circumference subtended by the same arc
    (3) The tangent to a circle is perpendicular to the radius drawn to the point of contact, and its converse.

Applications and considerations

- Definitions of circle, centre, radius, diameter, arc, sector, segment, chord, tangent, common tangent, secant, concyclic points, cyclic quadrilateral, an angle subtended by an arc or chord at the centre and at the circumference, and of an arc subtended by an angle, should be given.
- Two circles touch if they have a common tangent at the point of contact.
- The results (1), (2) and (3) should be discussed and proofs given. Reproduction of memorised proofs of these three results will not be required.

PMX1.2: Derivation of further angle, chord and tangent results, applications to numerical and theoretical problems requiring one or more steps of reasoning

In this subtopic, students develop their knowledge, skills and understanding in relation to setting out formal proofs in circle geometry, and solve problems requiring one or more steps of reasoning.

Outcomes addressed

PX1, PX7

Students develop the following knowledge, skills and understanding

- proving the following results:
  - (4) The perpendicular from the centre of a circle to a chord bisects the chord.
  - (5) The line from the centre of a circle to the midpoint of a chord is perpendicular to the chord.
  - (6) Equal chords in equal circles are equidistant from the centres.
  - (7) Chords in a circle which are equidistant from the centre are equal.
  - (8) Any three non-collinear points lie on a unique circle whose centre is the point of concurrency of the perpendicular bisectors of the intervals joining the points.
  - (9) Angles in the same segment are equal.
  - (10) The angle in a semicircle is a right angle.
  - (11) Opposite angles of a cyclic quadrilateral are supplementary.
  - (12) The exterior angle at a vertex of a cyclic quadrilateral equals the interior opposite angle.
  - (13) If the opposite angles in a quadrilateral are supplementary then the quadrilateral is cyclic (also a test for four points to be concyclic).
  - (14) If an interval subtends equal angles at two points on the same side of it then the end points of the interval and the two points are concyclic.
  - (15) The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.
  - (16) Tangents to a circle from an external point are equal.
  - (17) The products of the intercepts of two intersecting chords are equal.
  - (18) The square of the length of the tangent from an external point is equal to the product of the intercepts of the secant passing through this point.
  - (19) When circles touch, the line of centres passes through the point of contact.
- applying results and definitions to numerical and theoretical problems requiring one or more steps of reasoning.

Applications and considerations

- In application to problems, any of the definitions given or results (1) to (19) may be used without proof provided a specific reference is made to each result used. If a proof is required for any of the results (4) – (19) this will be clearly indicated in the question.
- Examples of the types of problems to be discussed can be found in the support material.

PMX2: Further algebra

In this topic, students develop further their algebraic skills and their knowledge and understanding of algebraic techniques. They will need to draw on the material in this topic as they complete more complex work in other topics, particularly concerning polynomials.

Assumed Stage 5.3 outcomes

PAS5.3.1, PAS5.3.2, PAS5.3.3, PAS5.3.4

Assumed Mathematics Advanced outcomes

PA2

Outcomes addressed

A student:

PX2 uses algebraic techniques to solve inequalities and prove results and identities.

PX7 communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Content summary

PMX2.1 General theory of quadratic equations

PMX2.2 Identity of two quadratic expressions, equations reducible to quadratics, algebraic results concerning cubes.

Terminology

algebraic identity
completing the square
cube of a difference
cube of a sum
difference of cubes
discriminant
equations reducible to quadratics
indefinite
negative definite
positive definite
root of an equation
sum of cubes

Use of technology

Dynamic graphing software is a useful tool for exploring the impact on a graph of a quadratic polynomial of changing the coefficients in the formula.

Topic notes

Some of these results may be known to students. The emphasis here is on a careful exploration of the quadratic results as a preliminary to later work on general polynomials.

Students should be reminded of the meaning of the word ‘discriminate’ in everyday language.

PMX2.1: General theory of quadratic equations

In this subtopic, students revise and extend their knowledge and understanding of the quadratic equation. This provides a platform for later work on polynomials. Connections are made between the quadratic equation $ax^2 + bx + c = 0$ and features of the graph of $y = ax^2 + bx + c$.

Outcomes addressed

PX2, PX7

Students develop the following knowledge, skills and understanding

- describing the quadratic expression $ax^2 + bx + c$ as a polynomial of second degree
- using the terms 'root of the quadratic equation' and 'zero of the quadratic polynomial' and finding these by suitable algebraic methods
- graphing the parabola $y = ax^2 + bx + c$ by locating its roots and intercept on the y-axis where appropriate
- graphing the parabola $y = ax^2 + bx + c$ in the case where there are no real roots
- solving quadratic inequalities
- using the methods of factorising and completing the square to solve quadratic equations
- using the method of completing the square in the derivation of the quadratic formula
- establishing the relationships between the roots of a quadratic equation and the coefficients
- identifying a quadratic expression as positive definite, negative definite or indefinite, depending on the value of the discriminant and the leading coefficient
- sketching the graph of $f = ax^2 + bx + c \quad (a \neq 0)$ by finding the turning point and zeros (if any) of f.

Applications and considerations

- The quadratic expression $ax^2 + bx + c$ is called a quadratic polynomial or a polynomial of the second degree. In all quadratic polynomials to be studied, the coefficients will be rational (usually integers) and the domain of x will be the set of real numbers. The quadratic function will be expressed as $y = ax^2 + bx + c$.
- When graphing quadratic functions, note that (1) for large values of x the term ax^2 effectively determines the value of the function; (2) the graph crosses the x-axis at the roots of the quadratic equation $ax^2 + bx + c = 0$. Examples should include cases where the graph has respectively two points, one point, and no points in common with the x-axis.
- Solution by 'completing the square' should be carried out for simple particular cases. It will be noted that, applied to an equation such as $x^2 + 2x + 2 = 0$ the method leads to $(x + 1)^2 + 1 = 0$, showing that no (real) value of x can be found which will make $x^2 + 2x + 2$ equal to zero. The traditional formula is derived by applying the method of completing the square to the general quadratic. The discriminant is to be defined and used to determine the condition for real, equal or rational roots.
- By actually solving the general quadratic, an important existence theorem has been established: A quadratic equation may have two (real) roots, one root or no roots. It does not have more than two roots. The relation $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$ between the roots α, β of a quadratic equation and its coefficients a, b, c, should be derived.
- To establish the conditions for positive definite, negative definite and indefinite quadratic expressions, consider the roots of the expression $f(x) = ax^2 + bx + c \quad (a \neq 0)$.

  (i) Suppose the discriminant $\Delta = b^2 - 4ac < 0$. Then $f(x)$ cannot be zero. Therefore if $\Delta < 0$ and $a > 0$, then $f(x) > 0$ for all values of x, and is called positive definite. If $\Delta < 0$ and $a < 0$, then $f(x) < 0$ for all values of x, and is called negative definite.

  (ii) If $\Delta > 0$, then $f(x) = 0$ for two distinct values of x, say x_1 and x_2. The greatest or least value of $f(x)$ occurs at $x = \frac{(x_1 + x_2)}{2}$ and $f(x)$ takes both positive and negative values.

  (iii) If $\Delta = 0$, then $f(x) = 0$ for *one* value of x, at $x = -\frac{b}{2a}$. Then $f(x) \geq 0$ if $a > 0$, and $f(x) \leq 0$ if $a < 0$, for all values of x.

  (iv) The turning point of $f(x)$ is at $\frac{df}{dx} = 0$, ie at $x = -\frac{b}{2a}$.
- Example: Find the range of values of k for which the expression $x^2 - 2x + (3 - 2k)$ is positive definite.

PMX2.2: Identity of two quadratic expressions, equations reducible to quadratics, algebraic results concerning cubes

In this subtopic, students deepen their knowledge and understanding of the behaviour of quadratic expressions and equations. Special results concerning cubes are derived or verified, and then applied to problems. This provides the opportunity for the further development of students' algebraic skills.

Outcomes addressed

PX2, PX7

Students develop the following knowledge, skills and understanding

- solving problems using the theorem concerning the identity of two quadratic expressions
- solving equations that are reducible to quadratic equations
- verifying the results for factorising the sum of two cubes and for factorising the difference of two cubes
- verifying the results for expanding the cube of a sum and the cube of a difference
- applying the results form the previous two dot points in the simplification of algebraic expressions
- solving inequalities with the unknown in the denominator.

Applications and considerations

- Theorem:
  If $a_1x^2 + b_1x + c_1 = a_2x^2 + b_2x + c_2$ for more than two values of x, then $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$.

  The proof reduces to a discussion of the equation $ax^2 + bx + c = 0$ with $a = a_1 - a_2$, $b = b_1 - b_2$, and $c = c_1 - c_2$. We know from our work on quadratic equations that $ax^2 + bx + c = 0$ can vanish for *at most* two values of x. There is one exception: if $a = b = c = 0$, the expression vanishes for *all* values of x. If it is given that $ax^2 + bx + c = 0$ for more than two values of x, we must conclude that $a = b = c = 0$. Otherwise the data presents us with a contradiction.
- Examples should include the expression of a quadratic polynomial $ax^2 + bx + c$ in the form $Ax(x - 1) + Bx + C$, where $C = c, A = a, B = a + b$; the fitting of a quadratic to three given function values; and similar identities.
- Examples of the following kinds of equations reducible to quadratics should be discussed:
  $x^4 - 4x^2 - 12 = 0,\ (x+1)^2 = 4x^2,\ 9^x - 4(3)^x + 3 = 0,$
  $\left(x+\frac{1}{x}\right)^2 - 5\left(x+\frac{1}{x}\right) + 6 = 0.$
- Examples of questions concerning cubes include: Simplify $\frac{a^3+b^3}{a^2-b^2}$,
  Expand $\left(3x+\frac{1}{2}\right)^3$.
- Examples of inequalities to be solved: $\frac{x^2-1}{x} > 0,\quad \frac{2t+1}{t-2} > 0$.

PMX3: Transformations of graphs

The principal focus of this topic is the way that graphs of functions change when the functions are altered in a systematic way.

Assumed Stage 5.3 outcomes

PAS5.3.4

Assumed Mathematics Advanced outcomes

PA1, PA2

Outcomes addressed

A student:

PX3 uses the relationship between the algebraic and geometric representations of a function in the solution of problems.

PX7 communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Content summary

PMX3.1 Stretching graphs horizontally and vertically, sketching graphs of reciprocals of known functions.

Terminology

parameter
reciprocal
scale factor
stretching
transformation
translation

Use of technology

Many of the results in this topic can be explored with graphing software.

Topic notes

The value of this topic is in the linking of the algebraic representation and the geometric representation of graphs, and knowing how certain kinds of new functions are related to known functions.

PMX3.1: Stretching graphs horizontally and vertically, sketching graphs of reciprocals of known functions

In this subtopic, students build 'new functions from old' by using scaling properties. They also sketch functions by using the graphs of reciprocals of functions.

Outcomes addressed

PX3, PX7

Students develop the following knowledge, skills and understanding

- for the graph of $y = f(x)$, comparing the features of the graphs $y = kf(x)$ and $y = f(kx)$ (k is a constant)
- given the graph of $y = f(x)$ without being given the function rule, sketching the graph of $y = kf(x)$ on the same axes
- given the graph of $y = f(x)$, without being given the function rule, sketching the graph of $y = f(kx)$ on the same axes
- commenting on the differences to transformed graphs when k is (a) negative (b) positive and smaller than 1
- extending curve sketching skills developed in PMA3 in relation to the vertical and horizontal translation of graphs
- using the features of stretched graphs to sketch functions derived from polynomial, exponential, logarithmic and trigonometric functions
- establishing the features of the pairs of graphs $y = f(x),\ y = \dfrac{1}{f(x)}$
- given the graph of $y = f(x)$ without being given the function rule, sketching the graph of $y = \dfrac{1}{f(x)}$ on the same axes
- using the features of graphs of reciprocals of functions to sketch functions derived from polynomial, exponential, logarithmic and trigonometric functions.

Applications and considerations

- Students should be encouraged to state in words what they expect a transformed graph to look like, then use graphing software to draw it, and comment on their conclusions.
- Tables of values assist in establishing 'what happens' when graphs are stretched in this way.
- Care is needed when using graphing calculators or graphing software that automatically selects a scale to suit the viewing window. Sometimes the graph of $y = kf(x)$ will appear to be the same as the graph of $y = f(x)$. Attention needs to be paid to the scale on the axes.
- Examples include: Sketch on the same axes: $f(x) = e^x$, $g(x) = e^{6x}$, $h(x) = 6e^x$.
- Combinations of transformations should consist of at most two transformations from a standard function. For example, $f(x) = 4\sin\left(\frac{x}{2}\right)$, $g(x) = 5 + \ln(3x)$.
- Tables of values assist in establishing 'what happens' when graphs are transformed.
- This is an opportunity to revise the graphs of the reciprocal trigonometric functions, $f(x) = \csc x$, (cosec), $f(x) = \sec x$, and $f(x) = \cot x$.
- Example: Sketch the graph of the function $f(x) = x^2 - 9$ and hence sketch the graph of $g(x) = \frac{1}{x^2 - 9}$.

PMX4: Polynomials

In this subtopic, students investigate polynomial functions and their graphs and solve problems related to the application of the remainder theorem and the factor theorem.

Assumed Stage 5.3 outcomes

PAS5.3.1, PAS5.3.2, PAS5.3.4

Assumed Mathematics Advanced outcomes

PA2

Outcomes addressed

A student:

PX4 solves problems using concepts from the theory of polynomial functions.

PX7 communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Content summary

PMX4.1 Definitions of polynomial, degree, polynomial equation; graphs of simple polynomials

PMX4.2 The remainder and factor theorems.

Terminology

cubics (cubic polynomials)
degree
dividend
divisor
factor theorem
leading term
long division
monic polynomial
polynomial
quartics (quartic polynomials)
quotient
remainder theorem

Use of technology

Dynamic graphing software is a useful way to explore graphs of polynomials.

Topic notes

Some of these results may be known to students who have studied the relevant option topic in Stage 5.3.

PMX4.1: Definitions of polynomial, degree, polynomial equation; graphs of simple polynomials

In this subtopic, definitions related to polynomials are introduced, and students use knowledge, skills and understanding from previous topics to sketch polynomials, including cases involving repeated roots.

Outcomes addressed

PX4, PX7

Students develop the following knowledge, skills and understanding

- using appropriate terminology to describe and classify polynomials
- using graph sketching techniques, including calculus, to sketch graphs of polynomials
- defining the term 'multiple root of a polynomial'
- identifying the order (multiplicity) of a root
- solving simple problems involving multiple roots of a polynomial
- describing the shape of the graphs of polynomials near single and multiple roots.

Applications and considerations

- This topic builds on previous work with quadratic equations and quadratic functions, which could usefully be revised here. Quadratic functions are special cases of polynomials.
- A real polynomial function $P(x)$ of degree n is a function of the form $P(x) = a_0 + a_1x + a_2x^2 + \dots a_{n-1}x^{n-1} + a_nx^n,\ (a_n \neq 0)$ where the real numbers $a_0, \dots a_n$ are called the coefficients of $P(x)$ and for convenience will usually be chosen to be integers. The terms 'polynomial expression' or 'polynomial' may be used to refer to $a_0 + a_1x + a_2x^2 + \dots a_{n-1}x^{n-1} + a_nx^n,\ (a_n \neq 0)$.
- The degree of $P(x)$ is that of the highest power of x occurring with non-zero coefficient.
- $P(x)$ is defined for all real x and is a continuous and differentiable function of x. The equation $P(x) = 0$ is called a polynomial equation of degree n, and those real numbers x which satisfy the equation are called real roots of the equation or real zeros of the corresponding polynomial.
- Examples should be given illustrating cases where one or more real roots occur and where none occur.
- Graphs of simple polynomials should be drawn, using all the techniques available. The following useful facts should be noted. (Proofs are not required, although teachers may choose to present proofs, depending on the level of interest of their classes.)
    - (i) For very large $|x|$, $P(x) \approx p_nx^n$.
    - (ii) A polynomial of odd degree always has at least one real zero.
    - (iii) At least one maximum or minimum value of P occurs between any two distinct real zeros.
- The convention of double and multiple roots should be explained. For example, the polynomial $(x-1)^3$ is said to have three roots, all of which are equal (rather than saying it has one root).
- Students should be able to explain why a graph 'touches' the x-axis at a repeated root, but cuts the x-axis at a single root. For example, in $f(x) = (x-3)^2(x-5)$ consider the *sign* of $(x-3)^2$ near $x = 3$, and how this affects the sign of $(x-3)^2(x-5)$. Repeat at $x = 5$.

PMX4.2 The remainder and factor theorems

The principal focus of this subtopic is the application of the remainder theorem and the factor theorem.

Outcomes addressed

PX4, PX7

Students develop the following knowledge, skills and understanding

- describing the way that a division operation with integers can be re-written as a multiplication and a sum: that is, $\frac{a}{b} = c + \frac{d}{b}$, where a, b, c, d are integers, can be written as $a = bc + d$, where a is called the 'dividend', b the 'divisor', c the 'quotient', and d the 'remainder'. Describing and giving examples of this with polynomials.
- describing the condition on the degree of the remainder $R(x)$ when a polynomial $P(x)$ is expressed as $P(x) = A(x)Q(x) + R(x)$ and also as $\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)}$
- performing polynomial long division or synthetic division with polynomials
- describing how the remainder theorem and the factor theorem are derived from the relationships above
- stating and using results that follow from the remainder theorem and the factor theorem in order to factorise polynomials
- solving problems related to the application of the remainder theorem and the factor theorem.

Applications and considerations

- The long division of one polynomial by another should be discussed and illustrated by examples using linear or quadratic divisors. The division process should be expressed as an identity: $P(x) = A(x)Q(x) + R(x)$ where $A(x)$ is the divisor, $Q(x)$ the quotient and $R(x)$ the remainder. The degree of $R(x)$ must be less than that of $A(x)$. With this condition satisfied, it may be stated that $Q(x)$ and $R(x)$ are then unique.
- Synthetic division is a shorthand way of recording the results of long division. Students may use either method in assessment.
- Rational functions should be defined as ratios of polynomials and the division process also expressed in the form $\frac{P(x)}{A(x)} = Q(x) + \frac{R(x)}{A(x)}$.
- The remainder theorem, which states that the remainder when $P(x)$ is divided by $x - a$ is $P(a)$, and the factor theorem, which states that if $P(a) = 0$ then $x - a$ is a factor of $P(x)$, both follow from the identity and the condition on $R(x)$. Students should understand how this is done and be able to answer questions about it, rather than reproduce a complete proof.
- The following results should be obtained:
  - (1) If $P(x)$ has k distinct real zeros $a_1, \dots a_k$, then $(x - a_1), \dots (x - a_k)$ is a factor of $P(x)$.
  - (2) If $P(x)$ has degree n and n distinct real zeros $a_1, \dots a_n$ then $P(x) = p_n(x - a_1) \dots (x - a_n)$.
  - (3) A polynomial of degree n cannot have more than n distinct real zeros.
  - (4) A polynomial equation of degree n has at most n real roots (and may have none).
  - (5) A polynomial of degree at most n, which has more than n distinct real zeros, is the zero polynomial (ie the polynomial in which $p_0 = p_1 = \dots p_n = 0$).
  - (6) If two polynomials of degree n are equal for more than n distinct values of x, then the coefficients of like powers of x are equal (ie the polynomials are equal for all values of x).
- Students will not be asked to reproduce proofs of results (1) – (6) but may be asked to state and to use them.
- Since all coefficients and roots will be integers in this topic, if $x - a$ is a factor of $P(x)$, then a must be an integer factor of the constant term in $P(x)$.

PMX5: Mathematical induction

In this topic the proof technique, mathematical induction, is introduced and used to prove results relating to series and divisibility.

Assumed Stage 5.3 outcomes

PAS5.3.1, PAS5.3.2

Assumed Mathematics Advanced outcomes

PA1

Outcomes addressed

A student:

PX2 uses algebraic techniques to solve inequalities and prove results and identities.

PX7 communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Content summary

PMX5.1 The principle of mathematical induction

PMX5.2 Use of mathematical induction to prove results relating to series and divisibility.

Terminology

assumption
divisibility
identity
principle of mathematical induction
sequences
series

Use of technology

Spreadsheets can be used to look for patterns in sequences of numbers.

Topic notes

Examples are to be limited to problems on sums and on divisibility. Problems in geometry will not be examined in the HSC examination, but could be discussed.

PMX5.1: The principle of mathematical induction

In this subtopic, the basic principle of mathematical induction, and a slight variation, are introduced.

Outcomes addressed

PX2, PX7

Students develop the following knowledge, skills and understanding

- describing the basic principle of mathematical induction
- identifying errors in false 'proofs by induction', such as cases where only one of the required two steps of a proof by induction is true, and understanding that this means that the statement has not been proved
- describing the variation that is required to prove results that are true for all integers greater than a certain integer (other than one).

Applications and considerations

- The method of proof known as 'proof by induction' makes use of a test for a set to contain the set of positive integers. This test, called the principle of mathematical induction, is an assumption concerning the positive integers, and may be stated as follows: 'If a set of positive integers (a) contains the positive integer 1, and (b) can be proved to contain the positive integer k + 1 whenever it contains the positive integers 1, 2, …, k, then the set contains all positive integers'.
- The use of this method of proof is often suggested when a problem of the following kind arises. From given information and perhaps by experiment, we obtain a statement $S(n)$ depending on the positive integer n, which we wish to prove true for every positive integer n. We let S denote the set of positive integers n for which $S(n)$ is true. We now try to prove: (i) that S contains 1 (ie that $S(1)$ is true), and (ii) that if S contains 1, 2, …, k, then S contains k + 1 (ie if $S(1)$, $S(2) \ldots S(k)$ are true, then $S(k+1)$ is true). If we manage to prove (i) and (ii), then by our test, S contains all positive integers (ie $S(n)$ is true for every positive integer n).
- It frequently happens that we may be able to prove (ii) by using only the assumption that $S(k)$ is true, instead of the full assumption that $S(1)$, $S(2)$, ... $S(k)$ are all true. Sometimes, we may guess that $S(n)$ is true only for positive integers $n \geq M$, a given positive integer. In that case we replace (i) by 'S contains M' and (ii) by 'if S contains $M, M+1, \ldots, k,$ then S contains $k+1$'. The test enables us to conclude that S contains every positive integer greater than or equal to M.
- It is important to realise that both steps of the method of induction must be verified, before the proof is valid. This can be illustrated vividly by 'proofs' of false results. For example, the 'proof by induction' that all successive integers are equal to each other (let $S(n)$ be the statement $n = n + 1$, then $S(k+1)$ follows logically from $S(k)$, but $S(1)$ is not true).

PMX5.2: Use of mathematical induction to prove results relating to series and divisibility

In this subtopic, students apply the principle of mathematical induction to prove results relating to series and divisibility.

Outcomes addressed

PX2, PX7

Students develop the following knowledge, skills and understanding

- using sigma notation to express a sum
- generalising results about integers, sums, and divisibility
- using mathematical induction to prove results relating to series
- using mathematical induction to prove results relating to divisibility.

Applications and considerations

- Example

  Consider the results:

  $1 = 1^2$ We call this statement $S(1)$.
  $1 + 3 = 2^2$ We call this statement $S(2)$.
  $1 + 3 + 5 = 3^2$ We call this statement $S(3)$.
  $1 + 3 + 5 + 7 = 4^2$ We call this statement $S(4)$.
  We may now guess that the following statement $S(n)$ is true for every integer n.
  Statement $S(n): 1 + 3 + 5 + \ldots + (2n - 1) = n^2$

  The proof by induction of this statement consists of two steps and a concluding statement.

  Step 1. Verification that $S(1)$ is a true statement.
  This is easy since $S(1)$ is merely the statement $1 = 1$.
  Step 2. We assume that $S(1), S(2), \ldots, S(k)$ are true. We then attempt to deduce logically that $S(k + 1)$ must also be true. In the present instance, our assumption supposes that the following is true:
  $S(k): 1 + 3 + 5 + \ldots + (2k - 1) = k^2$, and using this we try to show that $S(k + 1)$ is true, ie that $1 + 3 + 5 + \ldots + (2(k + 1) - 1) = (k + 1)^2$.

  By adding $2k + 1$ to each side of the (by assumption) true statement $S(k)$, we obtain $1 + 3 + 5 + \ldots + (2k - 1) + (2k + 1) = k^2 + (2k + 1)$,
  That is, $1 + 3 + 5 + \ldots + (2k + 1) = (k + 1)^2$, which is $S(k + 1)$.

  Thus, from the assumption that $S(k)$ is true, we have deduced that $S(k + 1)$ is true. We have satisfied the conditions of the test for proof by induction, hence we may conclude that $S(n)$ is true for every n.

- The standard notation for the sum of a series should be introduced, and induction used to prove results such as:

$$\sum_{n=1}^{N} n^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{n=1}^{N} \frac{1}{(2n+1)(2n-1)} = \frac{N}{2N+1}.$$

- Example: Prove that $3^{2n+4} - 2^{2n}$ is divisible by 5.
- Example: Prove that $7^n + 19^n$ is divisible by 13 if n is odd.

PMX6: Elementary difference equations and the discrete logistic growth model

In this topic, students use discrete variables and solve difference equations using a variety of methods. They apply these methods in the solution of problems from a wide range of contexts including biology, agriculture, population growth, finance and investments.

Assumed Stage 5.3 outcomes

PAS5.3.1, PAS5.3.2

Assumed Mathematics Advanced outcomes

PA1, PA2, PA10, PA13

Outcomes addressed

A student:

PX5 uses the theory of difference equations in the solution of problems.

PX7 communicates making comprehensive use of mathematical language, notation, diagrams and graphs

Content summary

PMX6.1 Introduction to difference equations: terminology and simple examples

PMX6.2 Methods of solution of difference equations: first-order and second-order equations

PMX6.3 The logistic growth equation: equilibrium, periodic and chaotic solutions.

Terminology

chaotic solutions
continuous variable
closed-form solution
difference equation
discrete logistic equation
discrete variable
equilibrium
first-order
iteration
linearity
order of a difference equation
recursive
second-order

Use of technology

The connections between numerical, graphical and algebraic representations should be explored using spreadsheets and/or computer algebra systems.

Topic notes

The tools of Discrete Mathematics have wide application in the physical and biological sciences and in economics and business. The notations and ideas extend the study of sequences and series, and demonstrate the power of Mathematics to model real-world situations. In Discrete Mathematics, students are exposed to contemporary and interesting ideas that have important applications in computer science, numerical methods and chaos theory.

The theory of sequences and series developed in the Mathematics Advanced course (up to and including the sum of a geometric series) is assumed knowledge for this topic, but could be taught concurrently if required.

PMX6.1: Introduction to difference equations: terminology and simple examples

In this subtopic, students are introduced to difference equations through their familiarity with sequences.

Outcomes addressed

PX5, PX7

Students develop the following knowledge, skills and understanding

- distinguishing between discrete and continuous variables
- identifying situations which require modelling using discrete variables
- describing the difference between a sequence which is defined by an explicit formula for the nth term, and a sequence defined recursively (ie a difference equation)
- classifying difference equations by their order and linearity
- generating the terms of a sequence given a difference equation and initial values.

Applications and considerations

- In order to completely determine a sequence, initial value(s) are needed along with the recurrence relation. The same recurrence relation, with different starting values, may define different sequences.
- Many real-world phenomena experience change at regular, discrete times rather than changing in a continuous fashion. For example, an investment that grows at 4% per time interval can be modelled by the difference relation $T_{n+1} = 1.04T_n$, with $n = 1,2,3...$ or $n = 0,1,2,3...$ and a corresponding starting point (either T_1 or T_0). A system that is undergoing continuous change (for example, velocity as a function of time) may still be modelled with discrete independent variables by using the values of velocity at regular discrete times.
- The sequence $T_{n+1} = 1.04T_n$ should be recognised as a geometric sequence, with general term $T_{n+1} = (1.04)^n T_1$.
- The order of a difference equation is the number of previous terms required (implicitly or explicitly) by the definition of the nth term. For example, $x_n = 2x_{n-1} + 5$ is of order one, $x_n = 2x_{n-2} + 1$ and $x_n = 2x_{n-2} + x_{n-1} + 1$ are both of order two. It can also be regarded as the largest difference in the arguments of the terms in a recurrence relation.
- A linear difference equation is one in which x_n is defined by previous terms, which are all to the power of one: $x_n = Ax_{n-1} + Bx_{n-2} + ...$ where A, $B\ldots$are constants.
- The Fibonacci sequence (a linear, second-order difference equation) should be investigated. $F_n = F_{n-2} + F_{n-1}$, $n \geq 3$, $F_1 = F_2 = 1$.
- Students should use graphs to investigate features of sequences.
- The connections between numerical, graphical and algebraic representations should be explored. Technology useful here includes spreadsheets and computer algebra systems.

PMX6.2: Methods of solution of difference equations: first-order and second-order equations

In this subtopic, students solve a range of first-order and second-order difference equations using a variety of methods.

Outcomes addressed

PX5, PX7

Students develop the following knowledge, skills and understanding

- describing the solution of a difference equation as an integer function (a function whose domain is the set of integers) that determines the terms of the sequence defined by the difference equation and its initial values
- recognising that it may not be possible to find a closed-form solution for certain difference equations
- solving a difference equation by working backwards until a pattern emerges (iteration method)
- generating several terms of a sequence, predicting its general formula, and proving the formula by induction (induction method)
- solving first-order linear difference equations of the form $x_n - ax_{n-1} = 0$, where a is non-zero constant
- solving first-order linear difference equations of the form $x_n - ax_{n-1} = c$, where a and c are non-zero constants
- using the characteristic equation method (auxiliary equation method) for second-order linear difference equations.

Applications and considerations

- As a simple example, the difference equation $x_n = x_{n-1} + 2n - 1$, with $x_0 = 0$, gives $x_1 = 0 + 2 - 1 = 1$, $x_2 = 1 + 4 - 1 = 4$, $x_3 = 4 + 6 - 1 = 9$, so we might guess that the solution is $x_n = n^2$. (Proof using the iteration method can be found in the support material.)
- Examples of using mathematical induction to solve difference equations can be found in the support material.
- To solve $x_n - ax_{n-1} = 0$ we note that $x_2 = ax_1$, $x_3 = ax_2 = a^2x_1$ and in general $x_n = a^{n-1}x_1$.
- To solve $x_n - ax_{n-1} = c$ we write out a few terms and conclude that $x_n = a^{n-1}x_1 + c\sum_{k=0}^{n-2} a^k$, where the final term involves the sum of a geometric series.
- To solve a second-order difference equation of the form $ax_n - bx_{n-1} + cx_{n-2} = 0$ where a, b, and c are constants we assume that $x_n = \lambda^n$ is a solution (based on the form of solutions of the first-order difference equations). This leads to the need to solve $a\lambda^2 + b\lambda + c = 0$, which is called a characteristic equation or an auxiliary equation. Further details and worked examples are given in the support material.
- Examples should be given from biology, population growth, crop yields, spread of diseases, finance and investments.

PMX6.3: The logistic growth equation: equilibrium, periodic, and chaotic solutions

The principal focus of this subtopic is the investigation of population growth as modelled by the non-linear logistic difference equation. The varying behaviour of populations, as the parameter in the equation changes, is illustrated graphically.

Outcomes addressed

PX5, PX7

Students develop the following knowledge, skills and understanding

- using difference equations to describe situations of population growth, given a constant birth rate and a constant death rate
- using difference equations to describe situations of population growth, where birth and death rates include a term proportional to the population
- identifying the logistic difference equation $P_{n+1} = a(1 - P_n)P_n$
- defining the term 'equilibrium' in the context of population growth
- establishing empirically an equilibrium value for a difference equation used to model population growth, and confirming the result algebraically
- using a graphical method ('cobweb diagrams') to demonstrate the behaviour of systems, such as populations, that are modelled by plotting relevant points on the graphs of $y = a(1 - x)x$ and $y = x$
- solving problems concerning equilibrium points, periodic behaviour and chaotic behaviour.

Applications and considerations

- Examples of population models described with difference equations can be found in the support material.
- The fact that an equation that is simple to write down (ie $P_{n+1} = a(1 - P_n)P_n$) yields very different solutions as the parameter a varies, is surprising. Visual, interactive displays of this behaviour illustrate this in a striking way.

  For $1 < a \leq 2$, successive terms of P_n approach an equilibrium value

  For $2 < a < k$, where k is about 3.57, successive values of P_n oscillate periodically

  For larger values of a, P_n behaves chaotically.
- Some students will be interested to investigate further the topic 'non-linear dynamical systems'.
- Example of kinds of questions:
  A population has size P_n at the beginning of the nth year. A logistic recurrence relation for describing this is $P_n = (1.6 - 0.0002x_{n-1})x_{n-1}$. Initially, $P_1 = 2000$. Find the size of the population for the first five years and find a positive equilibrium value for the equation.

10 HSC Mathematics Extension 1 Course Content

MX1 Binomial theorem
MX1.1 Binomial expansions and Pascal's triangle, binomial identities
MX1.2 Binomial probabilities and the binomial distribution

MX2 Further polynomials
MX2.1 The roots and coefficients of a polynomial equation, behaviour of the derivatives at multiple roots, problems involving multiple roots

MX3 Further trigonometry
MX3.1 Trigonometric functions of sums and differences of angles
MX3.2 Trigonometric equations including auxiliary angle method and general solutions

MX4 Methods of integration
MX4.1 Reducing integrals to standard form using simple substitutions, primitive functions of $\sin^2 x$ and $\cos^2 x$

MX5 Inverse functions and the inverse trigonometric functions
MX5.1 General introduction to inverse functions
MX5.2 The inverse trigonometric functions and their graphs
MX5.3 Properties of the inverse trigonometric functions, their derivatives, and corresponding primitive functions

MX6 Further applications of calculus
MX6.1 Modified exponential growth and decay
MX6.2 Related rates
MX6.3 Projectile motion
MX6.4 Iterative methods for numerical estimation of the roots of an equation

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Total indicative hours **60 hours**

MX1: Binomial theorem

In this topic, the binomial expansion is introduced, Pascal's triangle is constructed, and related identities are proved. Binomial probabilities and the binomial distribution are used to model situations where one of two outcomes is possible.

Outcomes addressed

A student:

HX1 uses the binomial theorem and algebraic and calculus techniques to prove identities

HX2 uses mathematical induction in the construction of proofs

HX4 demonstrates understanding of the significance of the binomial coefficients in counting, expansion of algebraic expressions, and probability calculations.

Content summary

MX1.1 Binomial expansions and Pascal's triangle, binomial identities

MX1.2 Binomial probabilities and the binomial distribution.

Terminology

binomial distribution
binomial expansion
binomial identity
binomial probabilities
combination
Pascal's triangle

Use of technology

Spreadsheets or other computer software may be used to simulate repeated binomial trials.

Topic notes

This topic draws together aspects of probability theory and number theory.

MX1.1: Binomial expansions and Pascal's triangle, binomial identities

In this subtopic, students complete the binomial expansion $(1+x)^n$ initially for small values of n, construct Pascal's triangle from the results, and extend the triangle by observing patterns. Labelling is introduced for the coefficients in the general case, and the connection is made with combinatorial notation.

Outcomes addressed

HX1

Students develop the following knowledge, skills and understanding

- expanding by hand $(1+x)^n$ for $n = 1, 2, 3, 4$
- noting the patterns formed when the coefficients of x in the above expansions are written in triangular form.
- extending the triangle formed (the Pascal triangle, or Pascal's triangle), by continuing the patterns.
- explaining why the coefficients are combinations
- using the notations ${}^nC_r$ and $\binom{n}{r}$.
- establishing combinatorial proofs of the Pascal's triangle relations ${}^nC_0 = 1, {}^nC_n = 1;\quad {}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$ for $1 \le k \le n-1$, and ${}^nC_k = {}^nC_{n-k}$
- obtaining the general expansion of $(a+u)^n$
- using the general formula $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$ to find particular coefficients.

Applications and considerations

- Since the coefficient of x^k in the expansion of $(1 + x)^n$ is equal to the number of ways of choosing k xs from the n factors $(1 + x)$, this coefficient is ${}^nC_k$.
- Each time we select k objects from n, we discard n-k objects. Hence the number of ways of selecting k objects from n equals the number of ways of selecting n-k objects from n, and this shows that ${}^nC_k = {}^nC_{n-k}$.
- From the result $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots {}^nC_nx^n$ we can put $x = \frac{u}{a}$, then multiply both sides by a^n , to obtain $(a+u)^n = {}^nC_0a^n + {}^nC_1a^{n-1}u + {}^nC_2a^{n-2}u^2 + \ldots {}^nC_nu^n$.
- There are two separate Pascal triangle relations: (i) for the outer coefficients, and (ii) for the others. In the above expansion for $(a + u)^n$, put $a = 1$, $u = 0$ on both sides to get ${}^nC_0 = 1$, put $a = 0$, $u = 1$ on both sides to get ${}^nC_n = 1$. Next we write down $(1 + x)^{n-1}$ and below it, the same expression multiplied by x; adding them together, and collecting terms of the same degree in x, we obtain

  $$(1+x)^{n-1} + x(1+x)^{n-1} = {}^{n-1}C_0 + \left({}^{n-1}C_0 + {}^{n-1}C_1\right)x + \left({}^{n-1}C_1 + {}^{n-1}C_2\right)x^2 + \ldots \left({}^{n-1}C_{n-2} + {}^{n-1}C_{n-1}\right)x^{n-1} + {}^{n-1}C_{n-1}x^n .$$

  However, the left side is obviously equal to $(1 + x)^n$. We now use the fact that two polynomials are equal for all values of x if and only if all corresponding coefficients are equal. Comparing the right side above with the earlier expansion of $(1 + x)^n$, we immediately deduce: ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$ for $1 \le k \le n-1$. A combinatorial proof should also be given.
- Combinatorial proofs of the relation ${}^nC_k = {}^{n-1}C_{k-1} + {}^{n-1}C_k$: Consider the case of selecting k objects from n. Let us add a new object to our original pool of n objects. Now we select k objects from the new pool (as we can in ${}^{n+1}C_k$ ways). Either our selection contains the new object or it does not. In the first case we are effectively choosing $k – 1$ objects from the original n (and this arises in ${}^nC_{k-1}$ ways) while in the second case we are choosing k objects from the original n (and this occurs in ${}^nC_k$ ways). It follows that ${}^{n+1}C_k = {}^nC_{k-1} + {}^nC_k$. This is equivalent to the required result. There is only one way of choosing no objects from n objects (${}^nC_0 = 1$) and only one way of choosing all n objects from n objects (${}^nC_n = 1$).
- Example of using the general formula $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$: Find the coefficient of x^3 when $\left(x^2 - \frac{2}{x}\right)^{3N}$ is expanded in powers of x, given that N is a positive integer.
- Students should be aware that the size of the coefficients in the expansion of $(a+b)^n$ first increases and then decreases, but problems requiring finding the greatest coefficient are not in this course.

MX1.2: Binomial probabilities and the binomial distribution

The principal focus of this subtopic is the use of the binomial distribution to model particular real-world situations, and solving related problems.

Outcomes addressed

HX1, HX4

Students develop the following knowledge, skills and understanding

- describing the features of a binomial distribution
- solving problems involving binomial distributions.

Applications and considerations

- The binomial distribution is best introduced with a concrete example such as tossing a coin once, twice, three times, defining 'success' on each throw as a 'head' (H) with probability p, and 'failure' as a 'tail' (T) with probability $q = 1 - p$. Tree diagrams should be used to list all outcomes systematically and to check that for three tosses the outcomes are HHH, HHT, HTH, HTT, THH, THT, TTH, TTT with respective probabilities $p^3, p^2q, p^2q, pq^2, p^2q, pq^2, pq^2, q^3$. The number X of heads appearing in these three tosses has a frequency distribution given by P(X = 0) = q^3, P(X = 1) = $3pq^2$, P(X = 2) = $3p^2q$, P(X = 3) = p^3 and it is easy (and helpful) to draw a histogram for various values of p.
- Now generalise to the case of n tosses, where the outcomes are represented by strings of length n using the letters H, T and each string arises with probability p^rq^{n-r}, where r is the number of times H appears. Because there are $\binom{n}{r}$ strings with r occurrences of H and $n - r$ occurrences of T, P(X = r) = p^rq^{n-r}, where X denotes the number of heads which appear in n tosses. X is said to have a binomial distribution. (Observe that the statement $\sum_{r=0}^{n} P(X = r) = 1$ can be rewritten in the form $\sum_{r=0}^{n} \binom{n}{r} p^r q^{n-r} = (p+q)^n = 1$.)
- The binomial distribution can be used to model repeated trials of any experiment with precisely two outcomes. The probability p may be known in advance, eg sampling with replacement from a box with two red and three white balls (p = 2/5 if H represents drawing a red ball); or estimated from frequency considerations, eg guessing the sex of a baby from birth data.
- Example: It is known that x% of the bolts produced by a machine are faulty. What is the probability that in a random sample of four bolts:
  (a) no bolts are defective? (b) precisely one bolt is defective? (c) at most, two bolts are defective? (Express all answers in the form of 10–8$R(x)$, where R is a polynomial that need not be simplified.)
  Let $p = x/100$ denote the probability that a bolt is defective and $q = 1 - p$.
  Then the required probabilities are respectively $\binom{4}{0}q^4$, $\binom{4}{1}pq^3$, and the sum of the first two together with $\binom{4}{2}p^2q^2$. Thus we can write the answers as
- (a) $10^{-8}\,(100 - x)^4$,
  (b) 10^{-8} x $4x\,(100 - x)^3$, (c) 10^{-8} x $\{(100 - x)^4 + 4x\,(100 - x)^3 + 6x^2\,(100 - x)^2\}$.
- On the average, batters in a certain cricket team make a scoring shot on every third ball. Estimate how many six-ball overs with precisely two scoring shots occur in a thousand overs of batting by that team. We take p = 1/3 to represent the probability of a scoring shot on a given ball. The probability that a random over contains precisely two scoring shots is $\binom{6}{2}\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$.
  We multiply this by 10^3 and (round off to the nearest integer) estimate that there are 329 overs.

MX2: Further polynomials

In this topic, further features of polynomial functions, including multiple roots, are investigated.

Outcomes addressed

A student:

HX6 uses algebraic and calculus techniques in the investigation of the properties of polynomial functions.

Content summary

MX2.1 The roots and coefficients of a polynomial equation, behaviour of the derivatives at multiple roots, problems involving multiple roots.

Terminology

coefficients
factor
multiple roots
multiplicity
polynomial equations

Use of technology

Dynamic graphing software is a useful way to explore graphs of polynomials.

Topic notes

This topic builds on topic PMX4 (Polynomials). Calculus is used to explore further features of polynomial functions.

MX2.1: The roots and coefficients of a polynomial equation, behaviour of the derivatives at multiple roots, problems involving multiple roots

In this subtopic, students extend their knowledge and understanding of quadratic equations to cubics and to the general case.

Outcomes addressed

HX6

Students develop the following knowledge, skills and understanding

- deriving the relationship between the roots and coefficients of quadratic, cubic, and quartic equations
- recognising that the relationship between the roots and coefficients of any polynomial can be obtained in a similar way
- solving problems using the relationship between the roots and coefficients of quadratic, cubic, and quartic equations. For example:
  - using these relationships to form a polynomial equation given its roots
  - forming an equation, whose roots are a multiple of the roots of a given equation
  - forming an equation, whose roots are the reciprocals of the roots of a given equation
  - forming an equation, whose roots differ by a constant from the roots of a given equation
  - forming an equation, whose roots are the squares of the roots of a given equation.
- showing that if a is a multiple root of the polynomial equation $f(x)=0$ then $f(a)=f'(a)=0$
- proving that if $P(x)=(x-a)^r S(x)$, where $r>1$ and $S(a)\neq 0$, then $P'(x)$ has a root a of multiplicity $(r-1)$
- solving problems involving polynomial equations with multiple roots.

Applications and considerations

- The relation between the coefficients and the roots (if they exist) of the quadratic equation $ax^2+bx+c=0$ should be derived using the identity $ax^2+bx+c=a(x-\alpha)(x-\beta)$. The corresponding relations for cubic equations should also be derived and the general result indicated.
- Particular examples should not involve polynomial equations of degree four.
- It may happen that $P(x)=(x-a)Q(x)$ and that $(x-a)$ is a factor of $Q(x)$. The number a is then called a repeated or multiple root of $P(x)$.
- If $P(x)=(x-a)^r S(x)$, where *r* is a positive integer and $S(a)\neq 0$, then a is a root of $P(x)$ of order (or multiplicity) r. The binomial $(x-a)$ is called a factor of $P(x)$ of order r. A simple root corresponds to a factor of order 1.
- The simplest approach to forming an equation whose roots are related to the roots of a given equation often does not involve using the relationship between the roots and coefficients. An equation, whose roots are m times those of a polynomial equation $P(x)=0$, is $P\left(\frac{x}{m}\right)=0$. For example, a cubic equation $ax^3+bx^2+cx+d=0$ may have roots α, β, γ. Then an equation with roots $m\alpha, m\beta, m\gamma$ is $a\left(\frac{x}{m}\right)^3+b\left(\frac{x}{m}\right)^2+c\left(\frac{x}{m}\right)+d=0$. An equation, whose roots are reciprocals of those of $P(x)=0$, is $P\left(\frac{1}{x}\right)=0$. Thus a cubic whose roots are $\frac{1}{\alpha}, \frac{1}{\beta}, \frac{1}{\gamma}$ is $\frac{a}{x^3}+\frac{b}{x^2}+\frac{c}{x}+d=0$ or $a+bx+cx^2+dx^3=0$.

  An equation, whose roots are all *k* less than those of $P(x)=0$, is $P(x+k)=0$. Thus, a cubic with roots $\alpha-k, \beta-k, \gamma-k$ is $a(x+k)^3+b(x+k)^2+c(x+k)+d=0$.

  An equation, whose roots are the squares of those of $P(x)=0$, is $P(\sqrt{x})=0$ (converted to a polynomial in x). Thus, an equation with roots $\alpha^2, \beta^2, \gamma^2$ is $\sqrt{x}\,(ax+c)+bx+d=0$, or $x(ax+c)^2=(bx+d)^2$.

MX3: Further trigonometry

In this topic, students learn techniques for solving a limited range of differential equations, and also see the connections between particular functions of a real variable through the use of complex numbers.

Outcomes addressed

A student:

HX5 uses the properties of trigonometric functions in the derivation of formulae, construction of proofs, and solution of problems, including problems involving trigonometry in three dimensions

HX9 interprets solutions to problems and communicates the solutions in appropriate forms.

Content summary

MX3.1 Trigonometric functions of sums and differences of angles

MX3.2 Trigonometric equations including auxiliary angle method and general solutions.

Terminology

amplitude
auxiliary angle method
circular measure
general solution
phase angle

Use of technology

The use of graphing technology to provide quick sketches of relevant functions is strongly recommended. For example, it is easy to demonstrate the result that when two sine waves of the same frequency are added together, the result is a ‘shifted’ sine wave of different amplitude.

Topic notes

This topic reveals the power of calculus to model real-world systems. Examples are to be done using circular measure.

MX3.1: Trigonometric functions of sums and differences of angles

The principal focus of this subtopic is the derivation of trigonometric expansions and the use of double angle results to prove identities and to solve equations.

Outcomes addressed

HX5, HX9

Students develop the following knowledge, skills and understanding

- deriving the expansions for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$
- using the expansions for $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$ to derive the results for double angles
- proving identities and solving equations.

Applications and considerations

- An easy, yet quite general, method of approach starts by drawing two points P and Q on the unit circle, P at an angle A from the positive x-axis, Q at an angle B. Let d be the distance from P to Q. We then compute the square of d in two ways:
  - (i) From the cosine rule, $d^2 = 1 + 1 - 2\cos(A - B)$.
  - (ii) From the cartesian coordinates of P and Q,

    $d^2 = (\cos A - \cos B)^2 + (\sin A - \sin B)^2$.

  Equating these two results, we obtain $\cos(A - B) = \cos A\cos B + \sin A\sin B$.

  We now use this basic formula to obtain all the other relations. By letting B be a negative angle, say $B = -C$, and by using $\cos(-C) = \cos C$, $\sin(-C) = -\sin C$, we obtain $\cos(A + C) = \cos A\cos C - \sin A\sin C$.

  Next in the formula for $\cos(A - B)$, let $A = \dfrac{\pi}{2}$ and obtain $\sin B = \cos\left(\dfrac{\pi}{2} - B\right)$. We then write $\sin(A + B)$ in the form $\sin(A + B) = \cos\left(\dfrac{\pi}{2} - (A + B)\right) = \cos\left(\left(\dfrac{\pi}{2} - A\right) - B\right)$ to derive $\sin(A + B) = \sin A\cos B + \cos A\sin B$.

  Sin $(A - C)$ is obtained by the substitution $B = -C$. The sum and difference formulae for the tangent ratio are now obtained from its definition and by use of the above formulae.
- The formulae for $\cos 2A$, $\sin 2A$ and $\tan 2A$ should be obtained explicitly as particular cases.
- Denoting $\tan\dfrac{\theta}{2}$ by t, the addition formula for the tangent gives

  $\tan\theta = \dfrac{2t}{1 - t^2}\quad (t \neq \pm 1)$. The expressions for $\cos\theta$ and $\sin\theta$ should also be derived.
- The use of the symbol $\equiv$ (is identically equal to) should be understood.

MX3.2: Trigonometric equations including auxiliary angle method and general solutions

The principal focus of this subtopic is to extend students' skills in solving trigonometric equations from those developed in the Mathematics Extension 1 Preliminary course.

Outcomes addressed

HX5, HX9

Students develop the following knowledge, skills and understanding

- solving trigonometric equations requiring factorising
- solving trigonometric equations requiring the application of sums and differences of angles
- converting expressions of the form $a\cos\theta + b\sin\theta$ to the form $A\cos(\theta - \alpha)$
- sketching graphs of trigonometric functions
- solving trigonometric equations using half-angle results
- finding general solutions of trigonometric equations
- applying trigonometric equations to practical situations.

Applications and considerations

- Expressions of the form $f(\theta) = a\cos\theta + b\sin\theta$ (1)
  arise quite commonly in the solutions of modelling problems (eg in MXX6 with differential equations). Sketching the graph of such expressions is simple once the following result is proved.

  Show that if $a \neq 0$, then (1) is expressible in the form $f(\theta) = A\cos(\theta - \alpha)$ where $A = \sqrt{a^2 + b^2}$ and $\alpha = \tan^{-1}\left(\frac{b}{a}\right)$. We call A the amplitude of the periodic function $f(\theta)$ and α is called its phase angle (sometimes also called the subsidiary angle).

  Exercise: Sketch the graph of $f(\theta) = 3\cos\theta - 4\sin\theta$ for $0 \leq \theta \leq 6\pi$.

- The following examples illustrate the types of problems to be treated:

  (i) Show that $\sin(A + B)\sin(A - B) = \sin^2 A - \sin^2 B$.

  (ii) Find all angles θ for which $\sin 2\theta = \cos\theta$

  (The solution is $\frac{\pi}{6} + 2n\pi$ or $\frac{5\pi}{6} + 2n\pi$ where n is an arbitrary integer.)

  (iii) Find all values of x in the range of $0 \leq x \leq 2\pi$ for which $4\cos x + 3\sin x = 1$

- The 'half-angle' results assist in the solution of certain kinds of equations. By putting $t = \tan\left(\frac{\theta}{2}\right)$ we can obtain expressions without surds for $\sin\theta$, $\cos\theta$, etc.

MX4: Methods of integration

In this topic, students' skills in finding integrals are extended through the use of the method of substitution. Primitive functions for the squares of some trigonometric functions are found through the use of trigonometric identities.

Outcomes addressed

A student:

HX7 evaluates integrals using given substitutions and trigonometric identities

Content summary

MX4.1 Reducing integrals to standard form using simple substitutions, primitive functions of $\sin^2 x$ and $\cos^2 x$.

Terminology

integration by substitution
limits of integration
primitive function
standard form
trigonometric identities

Use of technology

Computer algebra systems perform integration quickly and easily, but care is needed in interpreting output. In some cases, the output may not be in standard form. This software is useful for checking.

Topic notes

Although technology such as computer algebra systems is available, and has reduced the need for students to become expert at finding integrals of many functions, the basic method of substitution remains an important and useful skill.

The required substitutions will be given in all cases.

MX4.1: Reducing integrals to standard form using simple substitutions, primitive functions of $\sin^2 x$ and $\cos^2 x$

In this subtopic, students develop skills in reducing integrals to standard form through the use of simple substitutions.

Outcomes addressed

HX7

Students develop the following knowledge, skills and understanding

- using given substitutions to convert indefinite integrals to standard forms
- evaluating definite integrals by using given substitutions to convert to standard form
- obtaining the identity $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ from the expansion of $\cos(x + y)$
- obtaining the identity $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$ from the expansion of $\cos(x + y)$
- finding the integrals $\int \sin^2 x \, dx$ and $\int \cos^2 x \, dx$
- solving problems involving the integrals $\int \sin^2 x \, dx$ and $\int \cos^2 x \, dx$.

Applications and considerations

- The required substitutions will be given in all cases.
- It may be of interest to investigate the requirement that the given substituted function is one-to-one on the relevant interval.
- Care is needed to change the limits of integration appropriately.
- Examples of questions:

  Find $\int x\sqrt{1+x^2}\,dx$, using the substitution $u = x^2 + 1$.

  Use the substitution $t = u^2 - 1$ to evaluate $\int_0^1 \frac{t}{\sqrt{1+t}}\,dt$.

- Applications may include areas and volumes. For example, find $\int_0^2 \sqrt{4-x^2}\,dx$ using the substitution $x = 2\sin\theta$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, and give a geometric interpretation of the result.

MX5: Inverse functions and the inverse trigonometric functions

In this topic, the idea of inverse functions is introduced. The important examples, the inverse trigonometric functions, and their graphs and properties, are explored.

Outcomes addressed

A student:

HX3	uses the concept of inverse functions in the solution of problems
HX5	uses the properties of trigonometric functions in the derivation of formulae, construction of proofs, and solution of problems, including problems involving trigonometry in three dimensions
HX7	evaluates integrals using given substitutions and trigonometric identities
HX9	interprets solutions to problems and communicates the solutions in appropriate forms.

Content summary

MX5.1	General introduction to inverse functions
MX5.2	The inverse trigonometric functions and their graphs
MX5.3	Properties of the inverse trigonometric functions, their derivatives, and corresponding primitive functions.

Terminology

differentiable function
inverse cosine
inverse sine
inverse tangent
inverse trigonometric function
one-to-one function

Use of technology

Graphing technology is very useful in this topic to illustrate the functions being studied, and to pose and investigate 'what if' questions. Animations and simulations are available on various websites and some examples are given in the support material.

Topic notes

The inverse trigonometric functions are important for providing a mathematical context for the development of algebraic and trigonometric skills. The study of inverse functions is an important topic in Pure Mathematics. However, this topic also provides integrals that are very useful in a range of contexts. These contexts may involve optimisation of angles, but also may not involve angles.

For example, the fact that we can evaluate integrals of the form $\int \frac{1}{1+x^2}\,dx$ is very useful in a wide range of Applied Mathematics.

MX5.1: General introduction to inverse functions

The principal focus of this subtopic is the concept of an inverse function, and familiarisation with the notations and derivatives involved.

Outcomes addressed

HX3

Students develop the following knowledge, skills and understanding

- describing the conditions under which a function has an inverse
- using the notation $f^{-1}(x)$ for the inverse of $f(x)$
- finding inverse functions
- sketching the inverse of a given function
- using the relationship $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ to find derivatives of inverse functions
- limiting the domain of a function to obtain a new function that is one-to-one
- solving problems based on the relationship that $y = \log_a x$ and $y = a^x$ are inverse functions.

Applications and considerations

- Suppose that $y = f(x)$ is a continuous increasing function in the domain $a \le x \le b$ (ie $f(\beta) > f(\alpha)$ for all $\beta > \alpha$ in this domain). To each value y in $f(a) \le y \le f(b)$ there corresponds a *unique* value (namely, the x such that $y = f(x)$) and we may write $x = g(y)$ where the function g has domain $f(a) \le y \le f(b)$.
- The two relations $y = f(x)$ and $x = g(y)$ are equivalent and are represented by the same graph in the x, y plane. These two relations are mutually inverse functions in the sense that

  for $a \le x \le b$, $g(f(x)) = g(y) = x$,

  for $f(a) \le y \le f(b)$, $y = f(x) = f(g(y))$;

  accordingly, the notation f^{-1} is commonly used for the function inverse to f.
- If we express the function f^{-1} in the conventional form $y = f^{-1}(x)$, its graph is obtained from that of $y = f(x)$ by reflection in the line $y = x$. The domain of $y = f^{-1}(x)$ is the range of $y = f(x)$ and vice versa.
- Care must be taken to distinguish $f^{-1}(x)$ from $\dfrac{1}{f(x)} = \left(f(x)\right)^{-1}$.
- Simple examples of mutually inverse functions, which could be used to introduce this topic, are:

  (i) $y = x^3$ all real x; , $y = x^{1/3}$ all real x.

  (ii) $y = e^x$, all real x; $y = \log_e x$, $x > 0$.
- The problem of defining an inverse function when the equation $y = f(x)$ has more than one solution x for a given y should be discussed; the case $y = x^2$ is a useful illustration.
- If in addition f is a differentiable function of x then (since a tangent to $y = f(x)$ is also a tangent to $x = g(y)$) g is a differentiable function of y, and (since the relevant angles of inclination are complementary) $\dfrac{dy}{dx} = \dfrac{1}{\frac{dx}{dy}}$ or

  $\dfrac{dy}{dx} \cdot \dfrac{dx}{dy} = 1$, which may be formally obtained from the definition of the derivative.

MX5.2 The inverse trigonometric functions and their graphs

The principal focus of this subtopic is the construction of the inverse trigonometric functions, the use of relevant relationships, and the general solution of trigonometric equations.

Outcomes addressed

HX3, HX5

Students develop the following knowledge, skills and understanding

- illustrating with graphs the need to restrict the domain of the function $f(x) = \sin x$ in order to obtain a new graph that is either increasing throughout its domain or decreasing throughout its domain
- illustrating graphically the shape of the inverse function obtained by reflection in the line $y = x$
- defining formally the function $g(x) = \sin^{-1} x$
- repeating the above for the inverse cosine and inverse tangent functions
- using the relationships $\sin(\sin^{-1} x) = x$ and $\sin^{-1}(\sin x) = x$ and explaining why they are valid for particular values of x; similarly for inverse cosine and inverse tangent
- finding general solutions to equations of the form (i) $\sin\theta = b,\ |b| < 1$,

  (ii) $\cos\theta = b,\ |b| < 1$, and (iii) $\tan\theta = b$.

Applications and considerations

- By convention the part of the sine function that is used is $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$; the part of the cosine function that is used is $0 \le x \le \pi$; and the part of the tangent function that is used is $-\frac{\pi}{2} < x < \frac{\pi}{2}$.
- Formal definitions: The function $g(x) = \sin^{-1} x$ is defined on the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$, and has the property $\sin(\sin^{-1} x) = x.$ In words, 'The inverse sine of x is the number in the interval $-\frac{\pi}{2} \le x \le \frac{\pi}{2}$ whose sine is x'.
- Students should be aware that $\sin^{-1} x \ne \frac{1}{\sin x}$, and should know the notation 'ArcSin'. Similarly for inverse cosine and inverse tangent.
- The general solution to the equation $\sin\theta = b,\ |b| < 1$, is $\theta = n\pi + (-1)^n \sin^{-1} b$, and can be obtained graphically. Similarly for the related equations involving inverse cosine and inverse tangent.

MX5.3: Properties of the inverse trigonometric functions, their derivatives, and corresponding primitive functions

The principal focus of this subtopic is the derivation and use of properties of the inverse trigonometric functions.

Outcomes addressed

HX3, HX5, HX7, HX9

Students develop the following knowledge, skills and understanding

- deriving and using the properties $\sin^{-1}(-x) = -\sin^{-1} x$, $\cos^{-1}(-x) = \pi - \cos^{-1} x$, $\tan^{-1}(-x) = -\tan^{-1} x$, $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$
- deriving and using the derivatives of the inverse sine function, inverse cosine function, and inverse tangent function.

Applications and considerations

- The symmetry properties of the inverse trigonometric functions may be derived graphically (using symmetry properties of the relevant graphs), or by using the derivatives (after they have been established). For example, to show that $\cos^{-1}(-x) = \pi - \cos^{-1} x$, first show that the derivative of $\cos^{-1}(-x) + \cos^{-1} x$ is zero (for all x in the domain of $\cos^{-1} x$). This means that $\cos^{-1}(-x) + \cos^{-1} x$ is equal to a constant. The value of this constant can be obtained by evaluating $\cos^{-1}(-x) + \cos^{-1} x$ at any convenient value of x. Noting that $\cos^{-1}(-0) + \cos^{-1} 0 = \pi$ we have the result we need.
- Practice in numerical exercises should include examples such as finding $\sin(\cos^{-1} 0.6)$ with and without a calculator.
- Example: show that $\sin(\cos^{-1} t) = \sqrt{1 - t^2}$.
- The derivatives of $\sin^{-1} x, \cos^{-1} x, \tan^{-1} x$ should be obtained. Students do not need to be able to reproduce the derivations in full, but may be given the steps and asked to comment on how they are obtained, that is, on how each line follows from the previous one.
- The derivatives of $\sin^{-1}\left(\frac{x}{a}\right), \cos^{-1}\left(\frac{x}{a}\right), \tan^{-1}\left(\frac{x}{a}\right)$ lead to useful forms of the integrals.
- Examples should be given of using the product rule, quotient rule, and chain rule in conjunction with these new functions.
- Examples should be given of using these new functions in questions about area under a curve and volumes of solids of revolution.
- Practical examples include questions where an angle is to be maximised or minimised. For example, deciding where to take a kick for goal from the sideline of a football match, and finding the spot where the viewing angle of a screen at the movies (or a picture in a gallery) is maximised.
- Practical examples also include questions where the relevant integral arises in a non-trigonometric context.

MX6: Further applications of calculus

In this topic, students build on their knowledge and understanding of exponential functions from Mathematics Advanced course topics, and work with functions that describe projectile motion. They are introduced to related rates, and Newton's method for solving equations, which are examples of some of the many ways that calculus is applied.

Outcomes addressed

A student:

HX8 applies differential and integral calculus to mathematical modelling situations involving related rates, linear and projectile motion, and modified exponential growth and decay

HX9 interprets solutions to problems and communicates the solutions in appropriate forms.

Content summary

MX6.1 Modified exponential growth and decay

MX6.2 Related rates

MX6.3 Projectile motion

MX6.4 Iterative methods for numerical estimation of the roots of an equation.

Terminology

halving the interval
horizontal component
implicit differentiation
iteration
Newton's law of cooling
Newton's method
parametric representation
related rates
vertical component

Use of technology

Graphing technology is very useful in this topic to illustrate the functions and processes being studied. Animations and simulations are available on various websites and some examples are given in the support material.

Topic notes

This topic provides starting points for some of the important applications of calculus: modelling with differential equations, numerical methods for solving equations, applications to physics and science in general, applications to engineering. Teachers may find that some students are interested in reading more widely about these applications.

MX6.1: Modified exponential growth and decay

The principal focus of this subtopic is the use of a modified exponential function to model situations where a population (or other physical quantity) varies so that its rate is proportional to the difference between the size of the quantity and a constant.

Outcomes addressed

HX8, HX9

Students develop the following knowledge, skills and understanding

- describing with the equation $\frac{dN}{dt} = k(N - P)$, the rate of change in a quantity N which varies directly with the difference between N and a constant, P
- verifying (by substitution) that a solution to the differential equation $\frac{dN}{dt} = k(N - P)$ is $N = P$, and that $N = P + Ae^{kt}$ is another solution, for an arbitrary constant A
- for a given situation, determining A and k from initial conditions
- solving problems involving situations that are modelled with these equations, and sketching graphs of solutions of such problems
- noting that whenever $k < 0$, the quantity N tends to the limit P as $t \to \infty$, irrespective of the initial conditions.

Applications and considerations

- Applications include 'Newton's Law of Cooling'. It should be noted that it also applies to bodies that are placed in surroundings that are warmer than the initial temperature of the body.
- Note that when $P > 0$, $k < 0$ and $A < 0$, the initial value of N (which is $P + A$) will be less than P. The graph of N approaches the limiting value of P from below.
- The question arises of why some populations $N(t)$ would grow according to the rule $\frac{dN}{dt} = kN$ and others according to the rule $\frac{dN}{dt} = k(N - P)$. The role of the environment could be considered. In the latter case, P might represent a natural 'carrying capacity' for the ecosystem, and as N approaches P, the lack of space and food limits the growth of the population and the rate of change becomes smaller.

MX6.2: Related rates

The principal focus of this subtopic is the use of the chain rule to solve problems involving related rates.

Outcomes addressed

HX8, HX9

Students develop the following knowledge, skills and understanding

- solving problems that require the construction of equations involving rates, and then using the chain rule
- communicating solutions to related rates problems clearly in words.

Applications and considerations

- Examples:
  (i) A spherical balloon is being deflated so that the radius decreases at a constant rate of 10 mm per second. Calculate the rate of change of volume when the radius of the balloon is 100 mm.
  (ii) A spherical bubble is expanding so that its volume increases at the constant rate of 70 mm^3 per second. What is the rate of increase of its surface area when the radius is 10 mm?
- A hot air balloon is at a constant height of 160 metres above the ground, and moving parallel to the ground at a speed of 20 metres per minute. Find the rate at which the balloon is moving away from an observer on the ground at the time when the distance from the observer to the balloon is 400 metres.

  Note that the equation to set up is $r^2 = 160^2 + x^2$, where r is the straight line distance from the observer to the balloon, and x is the horizontal distance of the observer to the balloon. It is easier now to use implicit differentiation to get $2r\frac{dr}{dt} = 2x\frac{dx}{dt}$ than to make r the subject and differentiate: $r = \sqrt{160^2 + x^2}$ then $\frac{dr}{dt} = \frac{1}{2}\left(160^2 + x^2\right)^{-1/2} 2x\frac{dx}{dt}$.

MX6.3: Projectile motion

In this subtopic, students develop their knowledge and understanding of calculus through an application to physics that demonstrates the use of parametric representations of Cartesian equations. The idea of resolving motion in two dimensions into its vertical and horizontal parts is a valuable introduction to dynamics.

Outcomes addressed

HX8, HX9

Students develop the following knowledge, skills and understanding

- from observations, describing in words the path of a projectile under a variety of conditions
- choosing a frame of reference in order to construct axes for the horizontal and vertical movement of a projectile
- deriving the equations of motion for a particle projected vertically upwards
- deriving the equations of motion for a particle projected at an angle to the horizontal
- determining the maximum height reached by a projectile
- determining the time of flight of a projectile
- solving problems relating to the path of a projectile
- communicating solutions to problems relating to the path of a projectile clearly in words.

Applications and considerations

- Many students will benefit from a physical introduction to this topic, involving throwing objects in the air, dropping objects, aiming to hit targets while walking at a steady pace, using data loggers to record displacement data or viewing and making movies of projectiles.
- Examples should not be limited to destructive projectiles. Other applications include water jets (fire-fighting and irrigation), dropping food to stranded stock, sports such as shot-put, cricket, softball, javelin, even long jump (where the athlete is the projectile).
- The equations of motion of a particle projected vertically upwards should be derived.
- The two-dimensional motion of a projectile with the initial conditions that at $t = 0, x = y = 0$, $u = \frac{dx}{dt} = V\cos\alpha$, $v = \frac{dy}{dt} = V\sin\alpha$, results in the expressions that, at time t, $x = Vt\cos\alpha$, $y = Vt\sin\alpha - \frac{1}{2}gt^2$ (ignoring air resistance). This pair of equations gives a *parametric representation* of the 'flight parabola'. Here V is the initial speed and α the angle of projection. The Cartesian equation of the flight parabola is $y = x\tan\alpha - \frac{1}{2}gx^2/(V^2\cos^2\alpha)$.
- The range should be derived for a projectile fired on a horizontal plane. The maximum range on a horizontal plane is V^2/g when $\alpha = 45^\circ$.

MX6.4: Iterative methods for numerical estimation of the roots of an equation

In this subtopic, students learn two methods for finding numerical estimates for the roots of an equation. Both of these are iterative procedures. The emphasis is on understanding how and why these methods work, rather than on laborious calculations.

Outcomes addressed

HX9

Students develop the following knowledge, skills and understanding

- using the method of *halving the interval* to estimate the root of a continuous function
- describing how Newton's method works
- using Newton's method to estimate the root of a continuous, differentiable function
- identifying simple examples of difficulties with Newton's method, depending on the kind of function, and location of its roots and turning points.

Applications and considerations

- Suppose we have two values of x, say $x = x_1$ and $x = x_2$, such that the continuous function $P(x)$ is positive for $x = x_1$, ie $P(x_1) > 0$, and is negative for $x = x_2$, ie $P(x_2) < 0$. Since $P(x)$ is a continuous function, there is a root of $P(x)$ in the interval $x_1 < x < x_2$. Now compute the midpoint $x_3 = (x_1 + x_2)$ and the corresponding function value $P(x_3)$. If $P(x_3) = 0$, x_3 is the desired root. If $P(x_3) > 0$, we replace x_1 by x_3 and repeat the process using x_3 and x_2. If $P(x_3) < 0$, we replace x_2 by x_3 and repeat the process.
- Newton's method. Suppose z is close to a root of $P(x) = 0$. The tangent to $y = P(x)$ at $x = z$ has the equation $y - P(z) = P'(z)(x - z)$. This tangent intersects the x-axis at $x = z - P(z)/P'(z)$. If the original value of z was sufficiently close to the desired root, and if certain other conditions are satisfied, the new value x is even closer. We repeat the process to converge in general to the desired root. Newton's method is in principle faster (requires fewer steps for a given accuracy) than halving the interval, but some care must be exercised in applying it. A check, that the values obtained do appear to be approaching a root, should be made by calculating the corresponding function values.
- Clearly, Newton's method will fail if $x = z - P(z)/P'(z)$ cannot be calculated because of a 'divide by zero' error.
- Examples will require at most three applications of these methods by hand, more if technology is being used.
- Students should be made aware that numerical approximations to roots are found quickly by computer programs, which may also check for the convergence of the method used.

11 Course Requirements

The *Mathematics Extension 1 Stage 6 Syllabus* includes a Preliminary course of 60 (indicative) hours and an HSC course of 60 (indicative) hours.

The Preliminary Mathematics Extension course is constructed on the assumption that students have experienced all of the Stage 5.3 content of the *Mathematics Years 7–10 Syllabus*. It is recommended that they also experience the Stage 5.3 optional topics (identified by #) *Curve Sketching and Polynomials*, *Functions and Logarithms*, and *Circle Geometry*.

Completion of the Preliminary Mathematics Extension course is a prerequisite for the study of the HSC Mathematics Extension 1 course.

Students may not study the Mathematics Extension 1 course in conjunction with the Mathematics General 1 course or the Mathematics General 2 course.

12 Post-school Opportunities

The study of Mathematics Extension 1 provides students with knowledge, skills and understanding that form a valuable foundation for a range of courses at university and other tertiary institutions.

In addition, the study of Mathematics Extension 1 assists students to prepare for employment and full and active participation as citizens. In particular, there are opportunities for students to gain recognition in vocational education and training. Teachers and students should be aware of these opportunities.

Recognition of Student Achievement in Vocational Education and Training (VET)

Wherever appropriate, the skills and knowledge acquired by students in their study of HSC courses should be recognised by industry and training organisations. Recognition of student achievement means that students who have satisfactorily completed HSC courses will not be required to repeat their learning in courses in TAFE NSW or other Registered Training Organisations (RTOs).

Registered Training Organisations, such as TAFE NSW, provide industry training and issue qualifications within the Australian Qualifications Framework (AQF).

The degree of recognition available to students in each subject is based on the similarity of outcomes between HSC courses and industry training packages endorsed within the AQF. Training packages are documents that link an industry’s competency standards to AQF qualifications. More information about industry training packages can be found on the National Training Information Service (NTIS) website (www.ntis.gov.au).

Recognition by TAFE NSW

TAFE NSW conducts courses in a wide range of industry areas, as outlined each year in the TAFE NSW Handbook. Under current arrangements, the recognition available to students of Mathematics in relevant courses conducted by TAFE is described in the HSC/TAFE Credit Transfer Guide. This guide is produced by the Board of Studies and TAFE NSW and is distributed annually to all schools and colleges. Teachers should refer to this guide and be aware of the recognition that may be available to their students through the study of Mathematics Extension 1. This information can be found on the TAFE NSW website (www.tafensw.edu.au/mchoice).

Recognition by other Registered Training Organisations

Students may also negotiate recognition into a training package qualification with another Registered Training Organisation. Each student will need to provide the RTO with evidence of satisfactory achievement in Mathematics Extension 1 so that the degree of recognition available can be determined.

13 Assessment and Reporting

13.1 Requirements and Advice

The information in this section of the syllabus relates to the Board of Studies' requirements for assessing and reporting achievement in the Preliminary and HSC courses for the Higher School Certificate.

Assessment is the process of gathering information and making judgements about student achievement for a variety of purposes.

In the Preliminary and HSC courses those purposes include:
- assisting student learning
- evaluating and improving teaching and learning programs
- providing evidence of satisfactory achievement and completion in the Preliminary course
- providing the Higher School Certificate results.

Reporting refers to the Higher School Certificate documents that are used by the Board to report to students both the internal and external measures of achievement.

Higher School Certificate results comprise:
- an assessment mark derived from the mark submitted by the school and produced in accordance with the Board's requirements for the internal assessment program
- an examination mark derived from the HSC external examinations
- an HSC mark, which is the average of the assessment mark and the examination mark
- a performance band, determined by the HSC mark.

Results will be reported using a course report containing a performance scale with bands describing standards of achievement in the course.

The use of both internal assessment and external examination of student achievement allows measurements and observations to be made at several points and in different ways throughout the HSC Mathematics Extension 1 course. Taken together, the external examination and internal assessment marks provide a valid and reliable assessment of the achievement of the knowledge, understanding and skills described for each course.

The Board of Studies uses a standards-referenced approach to assessing and reporting student achievement in the Higher School Certificate.

The standards in the HSC are:
- the knowledge, skills and understanding expected to be learnt by students – the *syllabus standards*
- the levels of achievement of the knowledge, skills and understanding – the *performance standards*.

Both *syllabus standards* and *performance standards* are based on the aims, objectives, outcomes and content of a course. Together they specify what is to be learnt and how well it is to be achieved.

Teacher understanding of standards comes from the set of aims, objectives, outcomes and content in each syllabus together with:

- the performance descriptions that summarise the different levels of performance of the course outcomes
- HSC examination papers and marking guidelines
- samples of students' achievement, collected in the Standards Packages.

13.2 Internal Assessment

The internal assessment mark submitted by the school will provide a summation of each student's achievements measured at points throughout the course. The marks for each course group at a school should reflect the rank order of students and relative differences between students' achievements.

Internal assessment provides a measure of a student's achievement based on a wider range of syllabus content and outcomes than may be covered by the external examination alone. The assessment components and weightings to be applied to internal assessment are identified on pages 107–108. They ensure a common focus for internal assessment in the course across schools, while allowing for flexibility in the design of tasks. A variety of tasks should be used to give students the opportunity to demonstrate outcomes in different ways and to improve the validity and reliability of the assessment.

13.3 External Assessment

In Mathematics Extension 1, the external examination consists of a written examination. The specifications for the HSC examination in Mathematics Extension 1 are on page 109.

The external examination provides a measure of student achievement in a range of syllabus outcomes that can be reliably measured in an examination setting.

The external examination and its marking and reporting will relate to syllabus standards by:

- providing clear links to syllabus outcomes
- enabling students to demonstrate the levels of achievement outlined in the course performance scales
- applying marking guidelines based on established criteria.

13.4 Board Requirements for the Internal Assessment Mark in Board Developed Courses

The Board requires schools to submit an assessment mark for each candidate in the Mathematics Extension 1 HSC course. The Board requires that the assessment tasks used to determine the internal assessment mark must comply with the components and weightings specified in the table on page 108.

The collection of information for the HSC internal assessment mark must not begin before the completion of the Preliminary course.

Schools are required to develop an internal assessment program that:

- specifies the various assessment tasks and the weightings allocated to each task
- provides a schedule of the tasks designed for the whole course.

The standards-referenced approach to assessment for the HSC involves schools ensuring that in the design and marking of tasks:

- assessment tasks are designed to focus on outcomes
- the types of assessment tasks are appropriate for the outcomes being assessed
- students are given the opportunity to demonstrate their level of achievement of the outcomes in a range of different task types
- tasks reflect the weightings and components specified in the relevant syllabus
- students know the assessment criteria before they begin a task
- marking guidelines for each task are linked to the standards by including the wording of syllabus outcomes and relevant performance descriptions
- marks earned on individual tasks are expressed on a scale sufficiently wide to reflect adequately the relative differences in student performances.

In feedback and reporting:

- students receive meaningful feedback about what they are able to do and what they need to do in order to improve their level of performance
- the ranking and relative differences between students result from different levels of achievement of the specified standards
- marks submitted to the Board for each course are on a scale sufficiently wide to reflect adequately the relative differences in student performances.

Note that:

- measures of objectives and outcomes that address values and attitudes should not be included in school-based assessments of students' achievements. As these objectives are important elements of any course, schools may decide to report on them separately to students and parents, perhaps using some form of descriptive statements
- measures that reflect student conduct should not be included.

13.5 Assessment Components, Weightings and Tasks

Preliminary Course

The suggested components and weightings for the Preliminary course are set out below.

Preliminary Course

Component	Description	Weighting	Suggested tasks
Concepts and techniques	Use of concepts and techniques in the solution and interpretation of mathematical problems	50%	• assignments • examination-style questions • multimedia-based tasks • open-book tasks • oral or written reports • practical investigations or projects • practical tasks such as measurement activities • student's written explanation of problem solutions
Reasoning and communication	Application of reasoning and communication in appropriate forms in the construction of mathematical proofs and arguments and the interpretation and use of mathematical models	50%	
	Total:	**100**	

HSC Course

The mandatory components and weightings for the HSC course are set out below. The internal HSC assessment mark is to be based mainly on the Mathematics Extension 1 HSC course, and assessment tasks will focus on the course objectives and HSC outcomes. The Preliminary Mathematics Extension course will be assumed knowledge. Assessment tasks, while focusing on the Mathematics Extension 1 HSC outcomes, may relate to knowledge, skills and understanding from the Preliminary Mathematics Extension course.

Teachers can use their discretion in determining the manner in which they allocate tasks within course content. While the allocation of weightings to the various tasks set for the HSC course is left to individual schools, the percentages allocated to each syllabus component must be maintained.

It is suggested that three or four tasks are sufficient to assess the Mathematics Extension 1 HSC course. The range of tasks comprising the school-based assessment schedule should be varied and address the range of outcomes. One task may be used to assess several components.

HSC Course

Component	Description	Weighting	Suggested tasks
Concepts and techniques	Use of concepts and techniques in the solution and interpretation of mathematical problems	50%	• assignments • examination-style questions • multimedia-based tasks • open-book tasks • practical investigations or projects • practical tasks such as measurement activities • student's written explanation of problem solutions
Reasoning and communication	Application of reasoning and communication in appropriate forms in the construction of mathematical proofs and arguments and the interpretation and use of mathematical models	50%	
	Total:	**100**	

13.6 HSC External Examination Specifications

The Mathematics Extension 1 HSC examination will consist of a written examination paper of two hours duration (plus five minutes reading time) containing two sections with a total mark value of 70 marks. All questions in the examination are compulsory.

The examination will be based mainly on the Mathematics Extension 1 HSC course and will focus on the course objectives and HSC outcomes. The Preliminary Mathematics Extension and Mathematics Advanced courses will be assumed knowledge for this examination. Questions focusing on Mathematics Extension 1 HSC outcomes may relate to knowledge, skills and understandings from the Mathematics Extension Preliminary and Mathematics Extension 1 courses.

A formula sheet, including standard integrals, will be provided with the examination paper.

In addition to basic examination equipment, a pair of compasses, set squares, a protractor and a mathematical curve-drawing template* may be used.

Calculators that are Board-approved for the Mathematics Advanced, Mathematics Extension 1 and Mathematics Extension 2 HSC examinations may be used.

Section I (10 marks)

- Questions in this section will be in objective–response format, such as multiple-choice, multiple correct/incorrect, or other constrained response questions.
- The mark value of these questions will be one or more marks, depending on the length and demands of the question.

Section II (60 marks)

- There will be FOUR questions.
- All questions will be worth 15 marks.
- Each question will consist of a number of parts requiring free-response answers. These parts may be stand-alone questions, or may consist of several related sub-parts.

Note: Sample questions for this examination may be accessed on the Board's website (www.boardofstudies.nsw.edu.au).

* Students may take into any Mathematics examination a curve-drawing template, provided the template contains no printed formulae other than equations of simple curves (such as $y = x^2$) that may be drawn using the template.

13.7 Summary of Internal and External Assessment

Internal Assessment	Weighting	External Assessment	Weighting
Concepts and techniques	50	A written examination consisting of a range of item types.	
Reasoning and communication	50		100
	100		100

13.8 Reporting Student Performance Against Standards

Student performance in an HSC course will be reported against standards on a course report. The course report includes a performance scale for the course describing levels (bands) of achievement, an HSC mark located on the performance scale, an internal assessment mark and an examination mark. It will also show, graphically, the statewide distribution of examination marks of all students in the course.

Each band on the performance scale (except for Band 1) includes descriptors that summarise the attainments typically demonstrated in that band.

The distribution of marks will be determined by students' performance against the standards and not scaled to a predetermined pattern of marks.