

**B O A R D O F S T U D I E S**  
NEW SOUTH WALES

# **Mathematics Extension 2**

**Stage 6**

**Draft  
Syllabus**

**2008**

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# 1 The Higher School Certificate Program of Study

The purpose of the Higher School Certificate program of study is to:

- provide a curriculum structure which encourages students to complete secondary education
- foster the intellectual, social and moral development of students, in particular developing their:
  - knowledge, skills, understanding and attitudes in the fields of study they choose
  - capacity to manage their own learning
  - desire to continue learning in formal or informal settings after school
  - capacity to work together with others
  - respect for the cultural diversity of Australian society
- provide a flexible structure within which students can prepare for:
  - further education and training
  - employment
  - full and active participation as citizens
- provide formal assessment and certification of students' achievements
- provide a context within which schools also have the opportunity to foster students' physical and spiritual development.

## **2 Rationale for Mathematics Advanced, Mathematics Extension 1 and Mathematics Extension 2 in the Stage 6 Curriculum**

Mathematics is deeply embedded in modern society. From the numeracy skills required to manage personal finances, to making sense of data in various forms, to leading-edge technologies in the Sciences and Engineering, Mathematics provides the framework for interpreting, analysing and predicting, and the tools for effective participation in an increasingly complex society.

The need to interpret the large volumes of data made available through technology draws on skills in logical thought and in checking claims and assumptions in a systematic way. Mathematics is the appropriate training ground for the development of these skills. The thinking required to enhance further the power and usefulness of technology in real-world applications requires advanced mathematical training. The rapid advances in technology experienced in recent years have driven, and been driven by, advances in the discipline of Mathematics.

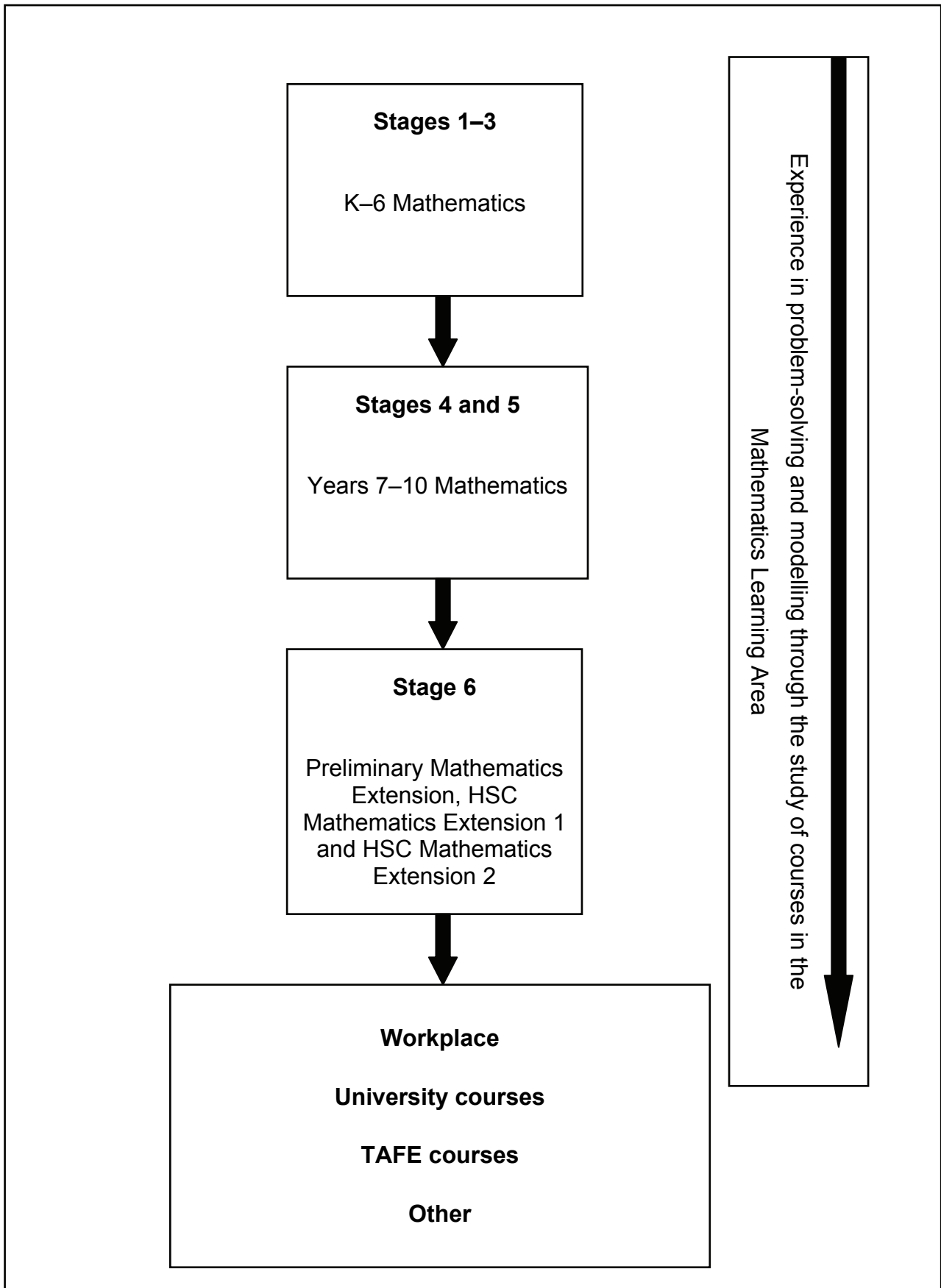
The development of Mathematics throughout history has been catalysed by its utility in explaining real-world phenomena and its inherent beauty. In this way, the discipline has continued to evolve through a process of observation, conjecture, proof and application.

The Mathematics Advanced, Mathematics Extension 1 and Mathematics Extension 2 courses provide the opportunity for students to acquire knowledge, skills and understanding in relation to important concepts within areas of Mathematics that have applications in an increasing number of contexts. These concepts and applications are appropriate to the students' continued experience of Mathematics as a coherent, interrelated, interesting and intrinsically valuable study that forms a basis for future learning. The introductory concepts and techniques of differential and integral calculus form a strong basis of the courses, and are developed and utilised across the courses, through a range of applications.

Students develop an appreciation of Mathematics as a study with high levels of internal structure that provide opportunities for the development of logical and disciplined thought. Through the learning experiences within the courses, students are able to progress from a knowledge and understanding of facts, procedures and applications in idealised contexts to facility in the use of mathematical models that situate the Mathematics in context and provide information on the behaviour of real-world systems, and to more advanced generalisations based on deductive and inductive reasoning processes. This involves the development and use of an increasingly sophisticated level of communication and literacy.

The courses provide students with the opportunity to study applications of Mathematics in a range of contexts relevant to contemporary professional practice, including examples from the Mathematics, Science, Engineering, Technology, Education, Business and Finance areas.

### 3 Continuum of Learning for Stage 6 Mathematics Extension 2 Students



## 4 Mathematics in Stage 6

There are five Board-developed Mathematics courses of study for the Higher School Certificate: (in increasing order of difficulty) Mathematics General 1, Mathematics General 2, Mathematics Advanced, Mathematics Extension 1, and Mathematics Extension 2.

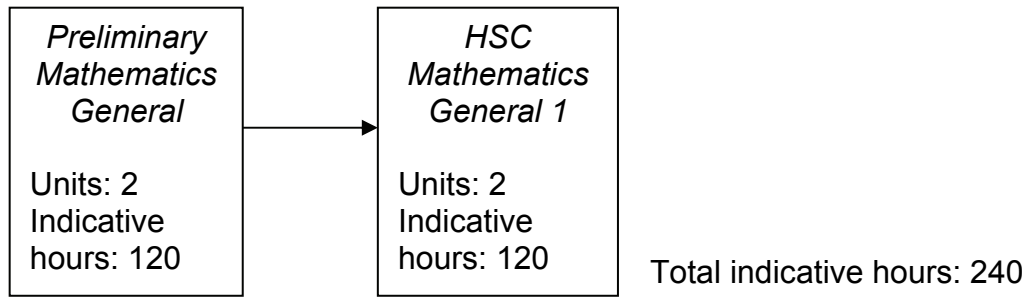
Students of the Mathematics General 1 and Mathematics General 2 courses study a common Preliminary course, Preliminary Mathematics General, leading to the HSC Mathematics General 1 and HSC Mathematics General 2 courses.

Mathematics Advanced consists of the courses Preliminary Mathematics Advanced and HSC Mathematics Advanced. Students studying one or both Extension courses study Preliminary Mathematics Extension course before undertaking the study of HSC Mathematics Extension 1, or HSC Mathematics Extension 1 and HSC Mathematics Extension 2.

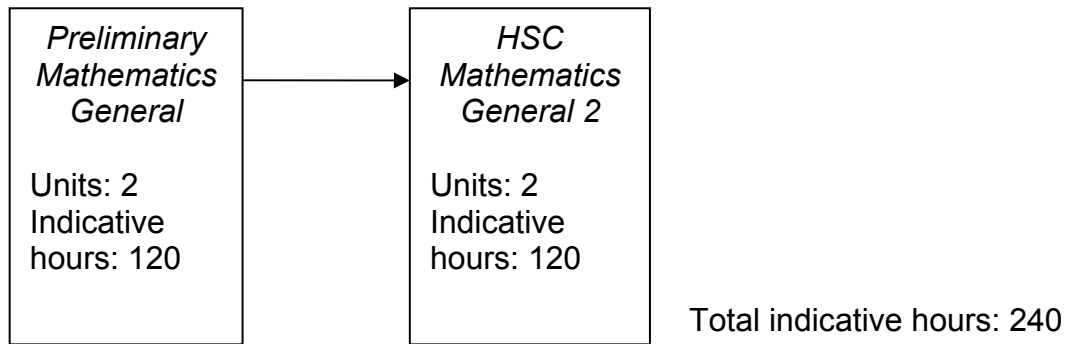
The following assumptions and recommendations regarding learning from Stage 5 Mathematics, typically undertaken by students in Years 9 and 10, are provided in relation to the study of the suite of Stage 6 courses. It is assumed that students who intend to study the Stage 6 Mathematics General 1 course have experienced all of the Stage 5.1 content. For students who intend to study the Stage 6 Mathematics General 2 course, it is recommended that they experience at least some of the Stage 5.2 content, particularly the *Patterns and Algebra* topics and *Trigonometry*, if not all of the content. For students who intend to study the Stage 6 Mathematics Advanced course, it is recommended that they experience the topics *Real Numbers*, *Algebraic Techniques* and *Coordinate Geometry* as well as at least some of *Trigonometry* and *Deductive Geometry* from 5.3 (identified by §), if not all of the content. For students who intend to study the Stage 6 Mathematics Extension 1 course, it is recommended that they experience the optional topics (identified by #) *Curve Sketching and Polynomials*, *Functions and Logarithms*, and *Circle Geometry*.

The Preliminary and HSC course components undertaken by students who study Mathematics General 1, Mathematics General 2, or Mathematics Advanced, and by students who study Stage 6 Mathematics to Mathematics Extension 1 or Mathematics Extension 2 level, are illustrated on the following pages.

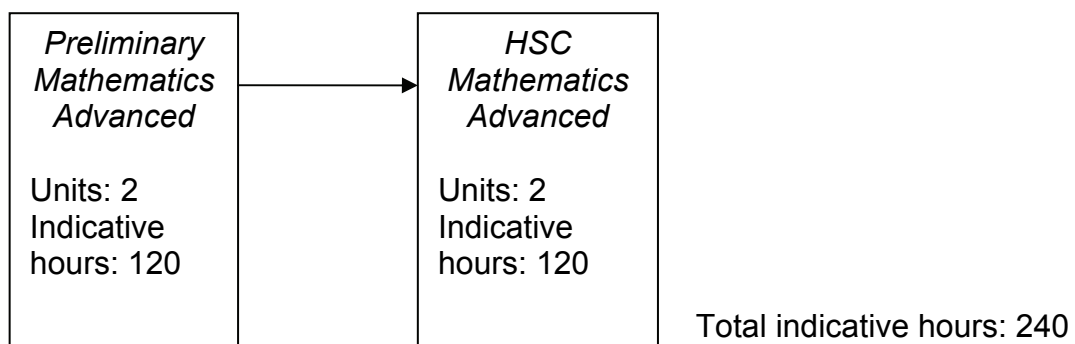
**Mathematics General 1 – Preliminary and HSC course components**



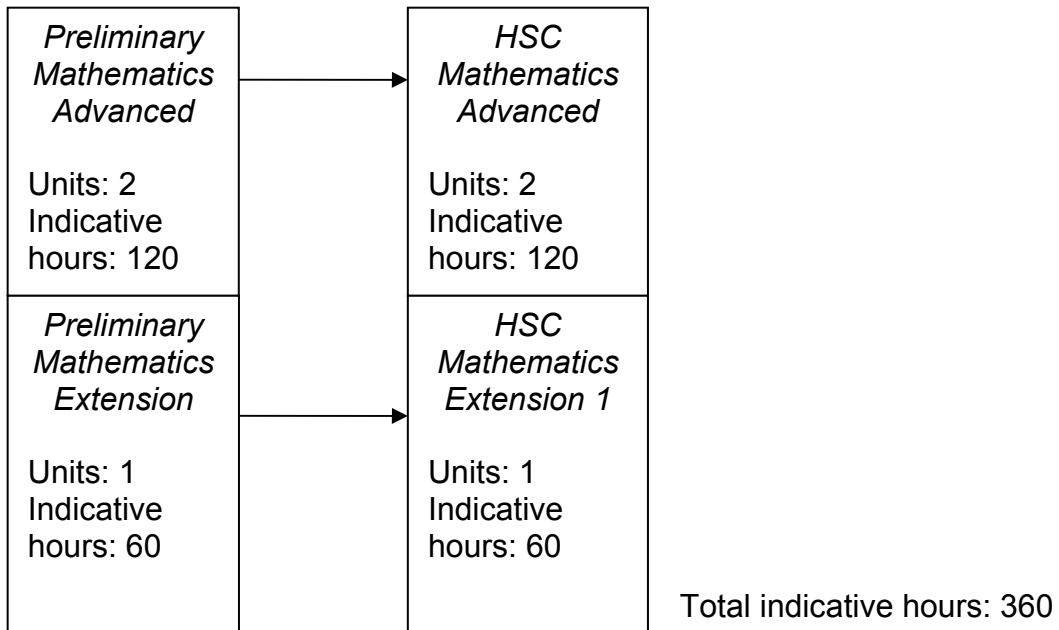
**Mathematics General 2 – Preliminary and HSC course components**



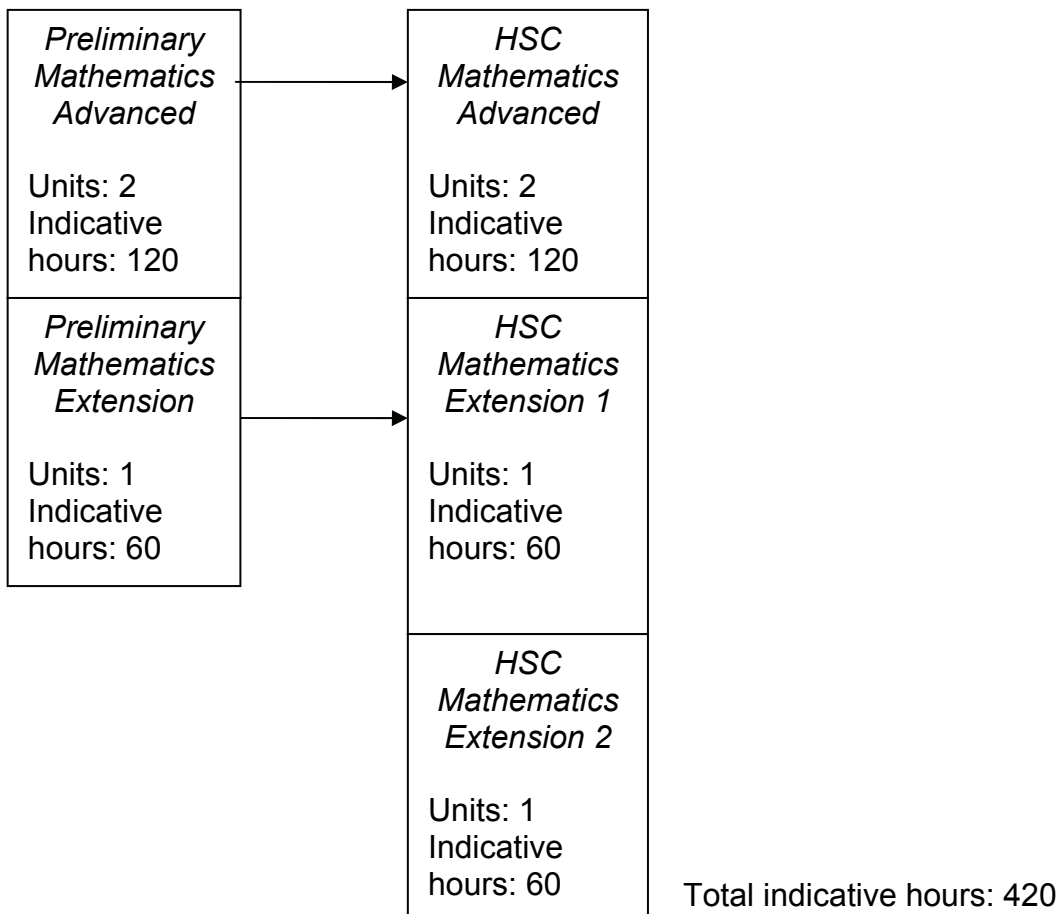
**Mathematics Advanced – Preliminary and HSC course components**



**Preliminary and HSC course components undertaken by students studying Stage 6 Mathematics to Mathematics Extension 1 level**



**Preliminary and HSC course components undertaken by students studying Stage 6 Mathematics to Mathematics Extension 2 level**



## **5 Aim (Mathematics Advanced, Mathematics Extension 1, Mathematics Extension 2 courses)**

The calculus-based Mathematics Advanced, Mathematics Extension 1 and Mathematics Extension 2 courses are designed to promote the development of knowledge, skills and understanding in relation to important concepts within areas of Mathematics that have applications in an increasing number of contexts. This includes the development of deductive and inductive reasoning skills and the ability to construct, solve and interpret mathematical models.

Students will learn to use a range of techniques and tools, including relevant technologies, in order to develop solutions to a wide variety of problems relating to their present and future needs and aspirations.

## **6 Objectives (Mathematics Advanced, Mathematics Extension 1, Mathematics Extension 2 courses)**

### **Knowledge, understanding and skills**

Students will develop the ability to:

- apply deductive and inductive reasoning, and use appropriate language, in the construction of proofs and mathematical arguments
- use concepts and techniques, including technology, in the solution of problems
- interpret and use mathematical models in a range of contexts
- interpret solutions to problems and communicate Mathematics in appropriate forms.

### **Values and attitudes**

Students will develop:

- appreciation of the scope, usefulness, power and elegance of Mathematics.

## 7 Course Structure

The following schematic view illustrates the structure of the Mathematics Extension 2 course.

<b>Mathematics Extension 2 Course</b>	
<b>MXX1</b>	<b>Further inequalities</b>
<b>MXX2</b>	<b>Complex numbers and polynomials over the complex field</b>
<b>MXX3</b>	<b>Graphs</b>
<b>MXX4</b>	<b>Integration techniques</b>
<b>MXX5</b>	<b>Volumes</b>
<b>MXX6</b>	<b>Mechanics</b>
<b>MXX7</b>	<b>Modelling with functions and derivatives</b>

## 8 Objectives and Outcomes

### 8.1 Table of Objectives and Outcomes

Objectives	<b>Mathematics Extension 2</b>		<b>Mathematics Advanced</b>		<b>Mathematics Extension 1</b>	
	HSC Outcomes		Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes
<b>Students will develop the ability to:</b>	<b>A student:</b>	<b>A student:</b>	<b>A student:</b>	<b>A student:</b>	<b>A student:</b>	<b>A student:</b>
apply deductive and inductive reasoning, and use appropriate language, in the construction of proofs and mathematical arguments	HXX1 constructs arguments and proofs in concrete and abstract settings  HXX2 applies algebraic, graphical and calculus techniques in the construction of proofs involving inequalities	HA1 constructs arguments to prove and justify results	PA1 provides reasoning to support conclusions appropriate to the context	PX1 uses deductive reasoning to solve problems and prove results in circle geometry  PX2 uses algebraic techniques to solve inequalities and prove results and identities	HX1 uses the binomial theorem and algebraic and calculus techniques to prove identities	

Mathematics Extension 2	
Objectives	HSC Outcomes
<p><b>Students will develop the ability to:</b></p> <p>use concepts and techniques, including technology, in the solution of problems</p> <p>(Note: further outcomes related to this objective on pages 15–18)</p>	<p><b>A student:</b></p> <p>HXX3 constructs proofs involving inequalities</p> <p>HXX4 combines the ideas of algebra and calculus to determine the features of graphs</p>

Mathematics Advanced		Mathematics Extension 1	
Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes
<p><b>A student:</b></p> <p>PA2 uses algebraic and graphical concepts in the solution of problems involving functions and coordinate geometry</p> <p>PA3 uses counting strategies in the solution of problems involving ordered and unordered selections</p>	<p><b>A student:</b></p> <p>HA2 manipulates algebraic expressions and solves problems involving logarithmic and exponential functions</p>	<p><b>A student:</b></p> <p>PX3 uses the relationship between the algebraic and geometric representations of a function in the solution of problems</p>	<p><b>A student:</b></p> <p>HX2 uses inductive reasoning in the construction of proofs</p> <p>HX3 uses the concept of inverse functions in the solution of problems</p> <p>HX4 demonstrates understanding of the significance of the binomial coefficients in counting, expansion of algebraic expressions, and probability calculations</p>

Mathematics Extension 2	
Objectives	HSC Outcomes
<p><b>Students will develop the ability to:</b></p> <p>use concepts and techniques, including technology, in the solution of problems</p> <p>(Note: further outcomes related to this objective on pages 14, 16–18)</p>	<p><b>A student:</b></p> <p>HXX5 performs arithmetic operations on complex numbers and uses De Moivre's theorem in the solution of problems involving powers and roots</p> <p>HXX6 uses the relationship between algebraic and geometric representations of complex numbers</p>

Mathematics Advanced		Mathematics Extension 1	
Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes
<p><b>A student:</b></p> <p>PA4 uses the concepts of complementary, mutually exclusive and independent events</p> <p>PA5 uses concepts and techniques in trigonometry in the solution of problems</p> <p>PA6 uses the concept of circular angle measure in the solution of problems</p>	<p><b>A student:</b></p>	<p><b>A student:</b></p>	<p><b>A student:</b></p> <p>HX5 uses the properties of trigonometric functions in the derivation of formulae, construction of proofs, and solution of problems, including problems involving trigonometry in three dimensions</p>

<b>Objectives</b>	<b>Mathematics Extension 2</b> HSC Outcomes
<b>Students will develop the ability to:</b> use concepts and techniques, including technology, in the solution of problems  (Note: further outcomes related to this objective on pages 14, 15, 17–18)	<b>A student:</b> HXX7 uses concepts from the theory of polynomial functions and complex numbers in the solution of problems involving factors, roots and coefficients of polynomial functions HXX8 solves first-order and second-order ordinary linear differential equations

<b>Mathematics Advanced</b>		<b>Mathematics Extension 1</b>	
Preliminary Outcomes <b>A student:</b>	HSC Outcomes <b>A student:</b>	Preliminary Outcomes <b>A student:</b>	HSC Outcomes <b>A student:</b>
PA7 determines the derivatives of functions through the routine application of the rules of differentiation	HSA3 uses the graphical relationship between a function and its derivative in curve sketching and in the solution of problems	PX4 solves problems using concepts from the theory of polynomial functions	HX6 uses algebraic and calculus techniques in the investigation of the properties of polynomial functions

<b>Objectives</b>	<b>Mathematics Extension 2</b> HSC Outcomes
<b>Students will develop the ability to:</b>  use concepts and techniques, including technology, in the solution of problems  (Note: further outcomes related to this objective on pages 14–16, 18)	<b>A student:</b>

<b>Mathematics Advanced</b>		<b>Mathematics Extension 1</b>	
Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes
<b>A student</b>	<b>A student:</b>	<b>A student:</b>	<b>A student:</b>
PA8 uses concepts and techniques from descriptive statistics to present and interpret data	HA4 applies techniques of differentiation and integration to logarithmic, exponential and trigonometric functions  HA5 uses the concept of a z-score, standardises normal random variables and solves related probability problems		

<b>Objectives</b>	<b>Mathematics Extension 2</b> HSC Outcomes
<b>Students will develop the ability to:</b>  use concepts and techniques, including technology, in the solution of problems  (Note: further outcomes related to this objective on pages 14–17)	<b>A student:</b>  HXX9 uses integral calculus in the solution of problems requiring the use of integration tables, identification and use of appropriate substitutions, partial fractions, integration by parts, and recurrence formulae

<b>Mathematics Advanced</b>		<b>Mathematics Extension 1</b>	
Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes
<b>A student:</b>  PA9 derives general results for arithmetic and geometric series, and applies the results to the solution of problems	<b>A student:</b>  HA6 applies series techniques to the solution of financial problems and interprets results	<b>A student:</b>	<b>A student:</b>
PA10 uses the relationship between the primitive and derivative of a function and determines primitives for functions involving powers of $x$	HA7 uses techniques of integration to calculate definite integrals, areas and volumes		HX7 evaluates integrals using given substitutions and trigonometric identities

<b>Objectives</b>	<b>Mathematics Extension 2</b> HSC Outcomes
<b>Students will develop the ability to:</b>	<b>A student:</b>
interpret and use mathematical models in a range of contemporary contexts	HXX10 uses the techniques of slicing, cylindrical shells and similar cross-sections to calculate volumes  HXX11 formulates and solves ordinary differential equations arising in mathematical modelling situations

<b>Mathematics Advanced</b>		<b>Mathematics Extension 1</b>	
Preliminary Outcomes	HSC Outcomes	Preliminary Outcomes	HSC Outcomes
<b>A student:</b>	<b>A student:</b>	<b>A student:</b>	<b>A student:</b>
PA11 uses algebraic, numerical and graphical representations of linear and quadratic functions in mathematical modelling situations and interprets results in context	HA8 uses differential and integral calculus to interpret mathematical models of linear motion, exponential growth and decay, and financial situations	PX5 uses the theory of difference equations in the solution of problems	HX8 applies differential and integral calculus to mathematical modelling situations involving related rates, linear and projectile motion, and modified exponential growth and decay

<b>Mathematics Extension 2</b>	HSC Outcomes
<b>Students will develop the ability to:</b>	<b>A student:</b>
interpret solutions to problems and communicate Mathematics in appropriate forms	HXX12 communicates abstract ideas and relationships using appropriate notation and logical argument

<b>Mathematics Advanced</b>	<b>Mathematics Extension 1</b>
Preliminary Outcomes	Preliminary Outcomes
<b>A student:</b>	<b>A student:</b>
PA12 interprets and uses mathematical language	PX6 communicates making comprehensive use of mathematical language, notation, diagrams and graphs
HA9 interprets solutions to problems and communicates using mathematical language, notation, diagrams and graphs	HX9 interprets solutions to problems and communicates the solutions in appropriate forms

<b>Values and attitudes</b>	<b>A student</b>
<b>Students will develop:</b>	HXXVA appreciates the power and elegance of Mathematics in the solution of a broad range of problems

<b>A student</b>	<b>A student</b>	<b>A student</b>
PAVA demonstrates confidence in using Mathematics to obtain realistic solutions to problems	HAVA seeks to apply mathematical techniques in the solution of problems	HXVA appreciates interrelationships between ideas drawn from different areas of Mathematics
<b>A student</b>	<b>A student</b>	<b>A student</b>
PXVA demonstrates confidence in extending known concepts to derive results with broader applicability		

## 8.2 Key Competencies

Mathematics Extension 2 provides a context within which to develop general competencies considered essential for the acquisition of effective, higher-order thinking skills necessary for further education, work and everyday life.

Key competencies are embedded in the *Mathematics Extension 2 Stage 6 Syllabus* to enhance student learning. The key competencies are developed through the methodologies of the syllabus and through classroom pedagogy. The key competencies of collecting, analysing and organising information and communicating ideas and information, reflect core processes of mathematical inquiry undertaken by students as they engage with the various syllabus topics. Students work as individuals and as members of groups to engage with applications and modelling tasks. Through this, the key competencies of planning and organising activities and working with others and in teams are developed. At all levels of the course, students are developing the key competency of using mathematical ideas and techniques. Through the advice provided on the selection and use of appropriate technology, students can develop the key competency of using technology. Finally, students' continual involvement with seeking solutions to problems, both large and small, contributes towards their development of the key competency of solving problems.

## **Presentation of Content**

The course content for the Mathematics Extension 2 course is presented in seven topics. Within each topic, the material is divided into cohesive subtopics, each of which contributes to the students' achievement of one or more of the course outcomes. It is intended that the prescribed knowledge, skills and understanding be developed through the study of appropriate tasks and applications that clearly demonstrate the need for such skills.

The course content for the Mathematics Extension 2 course is presented in the following format:

1. Initial facing pages for topic

### **Name of topic**

A brief summary of the content/purpose of the topic.

### ***Outcomes addressed***

A list of the course outcomes addressed in the study of the topic.

### ***Content summary***

A list of the subtopic(s) studied within the topic.

### ***Terminology***

A list of key words and/or phrases met in this topic, some of which may be new to students.

### ***Use of technology***

Advice about the nature and use of technology that is appropriate to the topic.

### ***Topic notes***

Notes relevant to teaching particular aspects of the topic.

2. Subsequent facing pages for topic

**Name of subtopic**

A brief summary of the content/purpose of the subtopic.

***Outcomes addressed***

A list of the course outcomes addressed in the study of the subtopic.

***Students develop the following knowledge, skills and understanding***

The mathematical content to be addressed in the subtopic.

***Applications and considerations***

The provision of examples indicating the range and style of applications used to introduce and illustrate the mathematical content of the subtopic, as well as important considerations for learning and teaching the subtopic.

**Use of technology**

**(a) in learning and teaching, and school-based assessment**

The appropriateness, viability and level of use of different types of technology in the learning and teaching of courses within the Mathematics Key Learning Area are decisions for students, teachers and schools. However, the use of technology is encouraged in the learning and teaching, and school-based assessment, where appropriate, of courses within the learning area. In accordance with the Broad Directions from the first phase of the development of the revised Stage 6 Mathematics courses, the use of technology with capabilities beyond the level of scientific calculators is encouraged in the learning and teaching, and school-based assessment, of the courses, ie Mathematics General 1, Mathematics General 2, Mathematics Advanced, Mathematics Extension 1, and Mathematics Extension 2.

Each of the five Stage 6 syllabuses contain advice and suggestions in relation to the use of a range of technology in the 'Use of technology' and 'Applications and considerations' sections within the course content. The *Mathematics Extension 2 Syllabus* provides a range of opportunities for the use of calculators and computer software packages in learning and teaching. This includes opportunities to utilise the graphing functions and financial and statistical capabilities of calculators, spreadsheets, and dynamic geometry and statistics software packages.

**(b) in the HSC examinations**

In the HSC examinations for the revised Stage 6 Board-developed Mathematics courses, candidates will be permitted to use only calculators manufactured to meet a clear set of Board-prescribed calculator functions and capabilities. These functions and capabilities will be consistent with and support the knowledge and skills that students should be able to demonstrate after completing a course, or courses. The list of functions and capabilities are being determined in parallel with the development of the content for the courses and will be completed in conjunction with the finalisation of the courses following consultation on the draft syllabuses.

## 9 HSC Mathematics Extension 2 Course Content

### **MXX1 Further inequalities**

MXX1.1 Proofs involving inequalities – mathematical induction, algebraic, graphical and calculus techniques

### **MXX2 Complex numbers and polynomials over the complex field**

MXX2.1 Arithmetic of complex numbers and solving quadratic equations

MXX2.2 Geometric representation of a complex number as a point and as a vector

MXX2.3 Powers and roots of complex numbers, describing curves and regions

MXX2.4 Fundamental theorem of algebra, factoring polynomials

### **MXX3 Graphs**

MXX3.1 Basic curves, drawing graphs by addition and subtraction of ordinates, drawing graphs by reflection in coordinate axes

MXX3.2 Sketching functions by multiplication and division of ordinates

MXX3.3 Drawing graphs involving powers and square roots of common functions

MXX3.4 General approach to curve sketching and using graphs in problem solving

### **MXX4 Integration techniques**

MXX4.1 Integration techniques

### **MXX5 Volumes**

MXX5.1 Volumes

### **MXX6 Mechanics**

MXX6.1 Mathematical representation of motion described in physical terms and interpreting in physical terms a mathematical description of motion

MXX6.2 Resisted motion

### **MXX7 Modelling with functions and derivatives**

MXX7.1 Direction fields

MXX7.2 First-order linear differential equations

MXX7.3 Second-order linear differential equations

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**Total indicative hours**

**60 hours**

## **MX1: Further inequalities**

In this topic, students extend their knowledge, skills and understanding in relation to the logical use of algebra, and deduction and proof.

### ***Outcomes addressed***

*A student:*

- HXX1 constructs arguments and proofs in concrete and abstract settings
- HXX2 applies algebraic, graphical and calculus techniques in the construction of proofs involving inequalities
- HXX3 constructs proofs involving inequalities
- HXX12 communicates abstract ideas and relationships using appropriate notation and logical argument.

### ***Content summary***

- MX1.1 Proofs involving inequalities – mathematical induction, algebraic, graphical and calculus techniques.

### ***Terminology***

arithmetic mean  
deduction  
geometric mean

inequality  
mathematical induction  
proof

### ***Use of technology***

Many interesting results can be conjectured by observing patterns in graphs, and then proved using deductive methods.

Computer algebra systems may assist students to check their work in this topic.

### ***Topic notes***

A combination of graphical, algebraic, calculus and mathematical induction techniques may be used to solve problems involving inequalities.

**MX1.1: Proofs involving inequalities – mathematical induction, algebraic, graphical and calculus techniques**

In this subtopic, students prove inequalities using various techniques, and prove further results involving inequalities by logical use of previously obtained inequalities.

***Outcomes addressed***

HXX1, HXX2, HXX3, HXX12

***Students develop the following knowledge, skills and understanding***

- proving inequalities by using the concept that  $a > b$  if and only if  $(a - b) > 0$ , for real  $a$  and  $b$
- proving inequalities by using the property that squares of real numbers are non-negative
- establishing and using the relationship between arithmetic mean and geometric mean for non-negative numbers
- using mathematical induction to prove inequalities
- proving inequalities by using a combination of graphical and calculus techniques
- proving further results involving inequalities by logical use of previously obtained inequalities.

### **Applications and considerations**

- Simple questions, which depend on the concept that  $a > b$  if and only if  $(a - b) > 0$ , should be done by students. For example, prove that, when  $x, y, z$  are real and not all equal,  $x^2 + y^2 + z^2 > yz + zx + xy$ , and deduce that, if also  $x + y + z = 1$ , then  $yz + zx + xy < \frac{1}{3}$ .
- Establishing the relationship between the arithmetic and geometric mean often leads to an elegant solution. For example:
  - (i) Prove that  $\frac{a+b}{2} \geq \sqrt{ab}$  if  $a$  and  $b$  are positive real numbers.
  - (ii) Given that  $x + y = p$ , prove that, if  $x > 0, y > 0$ , then  $\frac{1}{x} + \frac{1}{y} \geq \frac{4}{p}$ , and
 
$$\frac{1}{x^2} + \frac{1}{y^2} \geq \frac{8}{p^2}.$$
- Examples of the application of mixed techniques can be found in the support material.

## **MXX2: Complex numbers and polynomials over the complex field**

Students learn about representations of complex numbers and methods for calculating their sums, differences, products, quotients, powers and roots. Important applications include using complex numbers to define regions and to factor polynomials.

### ***Outcomes addressed***

*A student:*

- HXX1 constructs arguments and proofs in concrete and abstract settings
- HXX5 performs arithmetic operations on complex numbers and applies De Moivre's theorem in the solution of problems involving powers and roots
- HXX6 uses the relationship between algebraic and geometric representations of complex numbers
- HXX7 uses concepts from the theory of polynomial functions and complex numbers in the solution of problems involving factors, roots and coefficients of polynomial functions
- HXX12 communicates abstract ideas and relationships using appropriate notation and logical argument.

### ***Content summary***

- MXX2.1 Arithmetic of complex numbers and solving quadratic equations
- MXX2.2 Geometric representation of a complex number as a point and as a vector
- MXX2.3 Powers and roots of complex numbers; describing curves and regions
- MXX2.4 Fundamental theorem of algebra, factoring polynomials.

### ***Terminology***

Argand diagram	imaginary part
Argand plane	intersection of regions
argument	modulus
complex conjugate	polar form
complex number	real part
factoring over the real numbers	union of regions
factoring over the complex field	vector
fundamental theorem of algebra	

### ***Use of technology***

Some computer algebra systems support complex arithmetic. Illustrations of graphs of complex valued functions can be found on the internet and show how the topic develops at a higher level.

### ***Topic notes***

Historically, complex numbers were introduced to understand results arising in the solution of cubic equations, not quadratic equations. It is intriguing that complex numbers are all that we need to represent solutions of equations of degree 2 and higher. Some students will be interested to read further in the history of Mathematics about complex numbers and other ‘numbers’ such as quaternions.

Although polar form is not explicitly mentioned in the content of subtopic MX2.2, students should know this terminology as they will encounter it in wider reading and in later studies.

An important result of this study is the realisation that in Mathematics, different representations lead to the posing and solution of new problems and investigations.

**MX2.1: Arithmetic of complex numbers and solving quadratic equations**

In this subtopic, students are introduced to complex numbers and the relevant notation.

***Outcomes addressed***

HXX1, HXX5, HXX7

***Students develop the following knowledge, skills and understanding***

- using the symbol  $i$ , where  $i^2 = -1$ , to solve quadratic equations which do not have real roots
- identifying the real part  $\text{Re}(z)$  and the imaginary part  $\text{Im}(z)$  of a complex number  $z = x + iy$
- identifying the condition for  $a + ib$  and  $c + id$  to be equal
- adding, subtracting and multiplying complex numbers written in the form  $x + iy$
- finding the complex conjugate of the number  $z = x + iy$
- finding the reciprocal of the number  $z = x + iy$
- dividing a complex number  $a + ib$  by a complex number  $c + id$
- proving that there are always two square roots of a non-zero complex number
- finding the square roots of a complex number  $a + ib$
- solving quadratic equations of the form  $ax^2 + bx + c = 0$ , where  $a, b, c$  are complex.

**Applications and considerations**

- Students could be introduced to  $i$  as a device by which quadratic equations with real coefficients could be always solvable. The arithmetic and algebra of complex numbers would then be developed and it could be shown that any non-zero complex number has two distinct square roots. This then leads to the result that a quadratic equation with complex coefficients will always have two roots.
- A diagrammatic representation of sets of numbers showing that the set of integers is a subset of the set of rationals, which in turn is a subset of the set of reals, which in turn is a subset of the set of complex numbers, may be useful. A set which is disjoint to the set of reals is the set of 'purely imaginary' numbers of the form  $ai$  where  $a \neq 0$  is real. Students sometimes initially confuse 'complex' with 'not real'.
- In finding the square roots of  $a + ib$ , the statement  $\sqrt{a + ib} = x + iy$ , where  $a, b, x, y$  are real, leads to the need to solve the equations  $x^2 - y^2 = a$  and  $2xy = b$ .

Examining graphs of these curves for various values of  $a$  and  $b$  will lead to the conclusion that two square roots will always exist for a complex number.

**MX2.2: Geometric representation of a complex number as a point and as a vector**

The principal focus of this subtopic is the introduction and application of the polar and vector forms of a complex number.

***Outcomes addressed***

HXX1, HXX6, HXX7, HXX12

***Students develop the following knowledge, skills and understanding***

- using the fact that there exists a one-to-one correspondence between the complex number  $a + ib$  and the ordered pair  $(a, b)$
- plotting the point corresponding to  $a + ib$  on an Argand diagram
- defining the modulus ( $|z|$ ) and argument ( $\arg z$ ) of a complex number  $z$
- finding the modulus and argument of a complex number
- writing  $a + ib$  in modulus-argument form
- proving basic relations involving modulus and argument
- using modulus-argument relations to carry out calculations involving complex numbers
- recognising the geometrical relationships between the point representing  $z$  and points representing  $\bar{z}$ ,  $cz$  ( $c$  real) and  $iz$
- representing complex numbers as vectors on an Argand diagram
- given the points representing  $z_1$  and  $z_2$ , finding the position of the point representing  $z$ , where  $z = z_1 + z_2$
- describing the vector representing  $z = z_1 + z_2$  as corresponding to the diagonal of a parallelogram with vectors representing  $z_1$  and  $z_2$  as adjacent sides
- given vectors  $z_1$  and  $z_2$ , constructing vectors  $z_1 - z_2$  and  $z_2 - z_1$
- given  $z_1$  and  $z_2$ , constructing the vector  $z_1 z_2$
- proving geometrically that  $|z_1 + z_2| \leq |z_1| + |z_2|$ .

### Applications and considerations

- The geometrical meaning of modulus and argument for  $z = x + iy$  should be given and the following definitions used:

$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2},$$

$\arg z$  is any value of  $\theta$  for which  $x = |z|\cos\theta$  and  $y = |z|\sin\theta$ . While  $\arg z$  is commonly assigned a value between  $-\pi$  and  $\pi$ , so that  $-\pi < \theta \leq \pi$ , the general definition is needed in order that the relations involving  $\arg z$  given below are valid.

- Students should be able to prove the following relations:

$$|z_1 z_2| = |z_1| \times |z_2|, \quad \left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad |z^n| = |z|^n,$$

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2, \quad \arg \left[ \frac{z_1}{z_2} \right] = \arg z_1 - \arg z_2, \quad \arg(z^n) = n \arg z,$$

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \quad \overline{z_1 z_2} = \overline{z_1} \overline{z_2}.$$

- The fact that multiplication by  $i$  corresponds to an anticlockwise rotation through  $\pi/2$  about  $O$ , that  $\bar{z}$  is the reflection of  $z$  in the real axis and that multiplication by a positive real number  $c$  corresponds to an enlargement about  $O$  by a factor  $c$ , should be used on simple geometrical exercises.
- Familiarity with the vector representation of a complex number is extremely useful when work on curves and loci is encountered.
- Students need to be able to interpret the expression  $|z - (a + ib)|$  as the magnitude of a vector joining  $(a, b)$  to the point representing  $z$ .
- Students need to recognise that the expression  $\arg(z - z_1)$  refers to the angle, which a vector joining the point representing  $z_1$  to the point representing  $z$ , makes with the positive direction of the real axis.

**MX2.3: Powers and roots of complex numbers, describing curves and regions**

In this subtopic, students use connections between representations of complex numbers.

***Outcomes addressed***

HXX1, HXX5, HXX6, HXX7, HXX12

***Students develop the following knowledge, skills and understanding***

- proving, by mathematical induction, that  $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$  for positive integers  $n$
- proving that  $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$  for negative integers  $n$
- finding any integer power of a given complex number
- finding the complex  $n^{\text{th}}$  roots of  $\pm 1$  in modulus-argument form
- sketching the  $n^{\text{th}}$  roots of  $\pm 1$  on an Argand diagram
- illustrating the geometrical relationship connecting the  $n^{\text{th}}$  roots of  $\pm 1$
- given equations  $\text{Re}(z) = c$ ,  $\text{Im}(z) = k$  ( $c, k$  real), sketching lines parallel to the appropriate axis
- given an equation  $|z - z_1| = |z - z_2|$ , sketching the corresponding line
- given equations  $|z| = R$ ,  $|z - z_1| = R$ , sketching the corresponding circles
- given equations  $\arg z = \theta$ ,  $\arg(z - z_1) = \theta$ , sketching the corresponding rays
- sketching regions associated with any of the above curves (eg the region corresponding to the inequality  $|z - z_1| \leq R$ )
- giving a geometrical description of any such curves or regions.

### **Applications and considerations**

- The result  $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$  for negative integers  $n$  follows algebraically from the result for positive  $n$ .
- Students should realise that points corresponding to the  $n^{\text{th}}$  roots of 1, or of  $-1$ , are equally spaced around the unit circle with centre  $O$  and so form the vertices of a regular  $n$ -sided polygon.
- Typical curves and regions are those defined by simple equations or inequalities, such as

$$\begin{array}{lll} \operatorname{Im}(z) = 4, & |z - 2 - 3i| = |z - i|, & |z - 3 + 4i| = 5, \\ 0 \leq \arg z \leq \pi/2, & 0 \leq \operatorname{Im}(z) \leq 4, & \operatorname{Re}(z) > 2. \end{array}$$

- Simple intersections, such as the region common to  $|z| = 1$  and  $0 \leq \arg z \leq \pi/4$  and corresponding unions, need to be considered.
- Examples need only involve replacing  $z$  by  $z = x + iy$  in relations such as  $2|z| = z + \bar{z} + 4$ ,  $z + \bar{z} > 0$ ,  $|z^2 - (\bar{z})^2| < 4$ .

They need not include discussion of curves such as  $w = \frac{z-i}{z+i}$ , where  $z$  lies on a unit circle.

## **MXX2.4: Fundamental theorem of algebra, factoring polynomials**

In this subtopic students apply results about polynomials to solve problems involving factors and roots.

### ***Outcomes addressed***

HXX1, HXX6, HXX12

### ***Students develop the following knowledge, skills and understanding***

- stating and applying the fundamental theorem of algebra
- deducing that a polynomial of degree  $n > 0$ , with real or complex coefficients, has exactly  $n$  complex roots, allowing for multiplicities
- recognising that a complex polynomial of degree  $n$  can be written as a product of  $n$  complex linear factors
- recognising that a real polynomial of degree  $n$  can be written as a product of real linear factors and real quadratic factors
- factoring a real polynomial into a product of real linear factors and real quadratic factors
- factoring a polynomial into a product of complex linear factors
- writing down a polynomial given a set of properties sufficient to define it
- solving polynomial equations over the real numbers
- solving polynomial equations over the complex numbers.

**Applications and considerations**

- The ‘fundamental theorem of algebra’ asserts that every polynomial  $P(x)$  of degree  $n > 0$  over the complex numbers has at least one root.
- Using this result, the factor theorem should now be used to prove (by induction on the degree) that a polynomial of degree  $n > 0$  with real (or complex) coefficients has exactly  $n$  complex roots (each counted according to its multiplicity) and is expressible as a product of exactly  $n$  complex linear factors.
- The fact that non-real roots of real polynomials occur in complex conjugate pairs leads directly to the factorisation of real polynomials over the real numbers as a product of real linear and real quadratic factors. In particular, a real polynomial of odd degree always has at least one real root.
- Students should be able to factor suitably chosen cubic and quartic polynomials over both the real and complex numbers.
- Students should be able to factor polynomials with a degree greater than 4 in cases where factors are possible to obtain by other than the remainder theorem (eg  $x^6 - 1$ ,  $z^5 + 16z$ ).
- The question as to whether formulae for solving cubic, quartic or higher degree polynomial equations, similar to the formula for solving quadratic equations, can be constructed arises naturally here and could be explored.

## **MXX3: Graphs**

In this topic, students build on their knowledge and understanding of the graphs of basic functions to draw graphs of more complex functions. The behaviour of functions and their derivatives underpins this work.

### ***Outcomes addressed***

*A student:*

HXX4 combines the ideas of algebra and calculus to determine the features of graphs.

### ***Content summary***

- MXX3.1 Basic curves, drawing graphs by addition and subtraction of ordinates, drawing graphs by reflecting functions in coordinate axes
- MXX3.2 Sketching functions by multiplication and division of ordinates
- MXX3.3 Drawing graphs involving powers and square roots of common functions
- MXX3.4 General approach to curve sketching and using graphs in problem solving.

### ***Terminology***

critical point

implicit differentiation

ordinates

rectangular hyperbola

reflection

transformations

translation

vertical tangent

### ***Use of technology***

Graphing technology is a way for teachers to demonstrate some of the principles here, and provides a platform for students to experiment with the effects on graphs that follow changes with equations of functions. Some connections are made obvious by considering a third function representation: a table of values. Spreadsheets are useful here.

### ***Topic notes***

The emphasis throughout this topic is on understanding the connections between the algebraic representation of a function and its graph. Properties of functions that are related to concepts such as derivatives, discontinuities and limits assist in the sketching of graphs, and this process also assists in learning about the concepts themselves.

**MX3.1: Basic curves, drawing graphs by addition and subtraction of ordinates, drawing graphs by reflecting functions in coordinate axes**

In this subtopic, students revise graphs of basic functions and extend their knowledge and skills to straightforward transformations of these.

***Outcomes addressed***

HXX4

***Students develop the following knowledge, skills and understanding***

- graphing a linear function ( $ax + by + c = 0$ ,  $y = mx + b$ )
- graphing a quadratic function ( $y = ax^2 + bx + c$ )
- graphing a cubic function ( $y = ax^3 + bx^2 + cx + d$ )
- graphing a quartic function ( $y = ax^4 + bx^3 + cx^2 + dx + e$ )
- graphing a rectangular hyperbola ( $xy = k$ )
- graphing a circle ( $x^2 + y^2 + 2gx + 2fy + c = 0$ )
- graphing an exponential function ( $y = a^x$  for each of the cases  $a > 1$  and  $0 < a < 1$ )
- graphing a logarithmic function ( $y = \log_a x$ )
- graphing trigonometric functions ( eg  $y = k + a \sin(b(x - c))$ ,  $y = k + a \sin(bx + c)$ )
- graphing inverse trigonometric functions ( eg  $y = a \sin^{-1} bx$ )
- graphing the functions  $y = x^{\frac{1}{2}}$  and  $y = x^{\frac{1}{3}}$
- graphing a function  $y = f(x) \pm c$  by initially graphing  $y = f(x)$
- graphing a function  $y = f(x) \pm g(x)$  by initially graphing  $y = f(x)$  and  $y = g(x)$
- graphing  $y = -f(x)$  by initially graphing  $y = f(x)$
- graphing  $y = |f(x)|$  from the graph of  $y = f(x)$
- graphing  $y = f(-x)$  by initially graphing  $y = f(x)$ .

**Applications and considerations**

- The notations  $\log_e x$  and  $\ln x$  are used to denote the natural logarithm of  $x$  and students should be familiar with both notations.
- Students will need to be able to produce quickly a neat sketch of these basic functions in order to use them in sketching further functions.
- Students need to examine the behaviour of the derivatives of  $y = x^{\frac{1}{2}}$  and  $y = x^{\frac{1}{3}}$  near  $x = 0$  and investigate the behaviour of these functions at  $x = 0$ . They must be familiar with the term 'critical point' and with the possibility of curves having vertical tangent lines at points on them.
- Typical functions involving addition of ordinates could include  $y = 1 + 3\sin 2x$  for  $-2\pi \leq x \leq 2\pi$  and  $y = \cos^{-1} x - \pi$ . Students should realise that the graph of  $y = 3\sin 2x$  can be transformed to the graph of  $y = 1 + 3\sin 2x$  by either translating the graph one unit upwards or translating the  $x$ -axis one unit in the opposite direction.
- Other types could include graphing functions such as  $f(x) = 3\sin x + x$  for  $0 < x < 4$ . This may be developed from the graphs of  $y = x$  and  $y = 3\sin x$ . The points where  $y = 3\sin x$  cuts the  $x$ -axis correspond to the points where  $y = 3\sin x + x$  cuts  $y = x$ . Once the shape of the curve has been roughed out using addition of ordinates the position of stationary points and points of inflexion may be obtained when appropriate.
- A function such as  $y = -\log_e x$  may be graphed by reflecting the graph of  $y = \log_e x$  in the  $x$ -axis. The graph of  $y = 2 - \log_e x$  may then be obtained by a suitable translation.
- The relationship between the graphs of  $y = f(x)$  and of  $y = f(x - a)$  should be discussed and used also in examples involving the reflection properties, such as, for example, the graph of  $y = |1 - \sin(x - 2)|$ .

**MXX3.2: Sketching functions by multiplication and division of ordinates**

In this subtopic, students graph functions that are related to given functions in simple ways by using properties from algebra and calculus.

***Outcomes addressed***

HXX4

***Students develop the following knowledge, skills and understanding***

- graphing a function  $y = cf(x)$  by initially graphing  $y = f(x)$
- graphing a function  $y = f(x) \times g(x)$  by initially graphing  $y = f(x)$  and  $y = g(x)$
- graphing a function  $y = \frac{1}{f(x)}$  by initially graphing  $y = f(x)$
- graphing a function  $y = \frac{f(x)}{g(x)}$  by initially graphing  $y = f(x)$  and  $y = g(x)$ .

### Applications and considerations

- A good initial idea of the behaviour of functions of the form  $f \cdot g$  may be obtained by examining the graphs of  $f$  and  $g$  independently.
- To graph  $y = xe^{-x}$  the functions  $y = x$  and  $y = e^{-x}$  may first be graphed on the same set of axes.

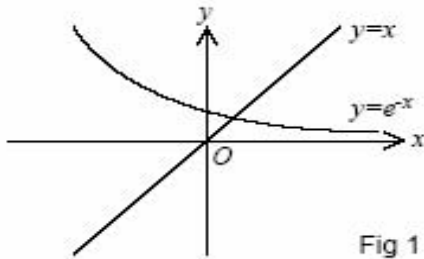


Fig 1

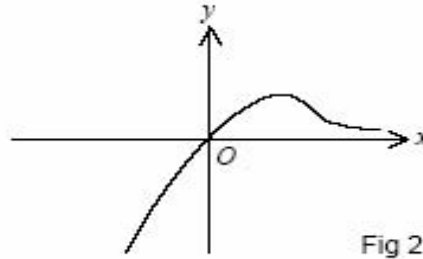


Fig 2

From Figure 1, important features of the graph of  $y = xe^{-x}$  can be obtained. These include properties that

- for  $x < 0$ ,  $xe^{-x} < 0$ ; for  $x = 0$ ,  $xe^{-x} = 0$ ; for  $x > 0$ ,  $xe^{-x} > 0$ .
- as  $x \rightarrow -\infty$ ,  $xe^{-x} \rightarrow -\infty$ ; as  $x \rightarrow \infty$ ,  $xe^{-x} \rightarrow 0$ .

This enables a rough shape to be quickly sketched (Figure 2). The exact positions of the stationary point and point of inflexion may be determined by calculus.

Teachers may prefer to introduce more formally the following results:

For  $n$  a positive integer,  $\frac{x^n}{e^x} \rightarrow 0$  as  $x \rightarrow \infty$ .

For  $n$  a positive integer,  $\frac{x^n}{\ln x} \rightarrow \infty$  as  $x \rightarrow \infty$ .

- The graph of  $y = 1/f(x)$  may be sketched by first sketching  $y = f(x)$ . Where  $f(x) = 0$ ,  $1/f(x)$  is undefined; where  $f(x) > 0$ ,  $1/f(x) > 0$ ; where  $f(x) < 0$ ,  $1/f(x) < 0$ . As well, when  $f(x)$  is increasing then  $1/f(x)$  is decreasing and vice versa.
- The graph of  $y = f(x)/g(x)$  may be sketched by initially sketching  $f(x)$  and  $g(x)$ . Where  $f(x) = 0$ ,  $f(x)/g(x) = 0$  and where  $g(x) = 0$ ,  $f(x)/g(x)$  is undefined and a discontinuity (but not necessarily a vertical asymptote) exists. Examination of the signs of  $f(x)$  and  $g(x)$  will lead to finding the intervals where  $f(x)/g(x)$  is positive and where it is negative. These results can be used together with the approach to graphs which are products of functions to obtain an idea of the shape of a function.

**MXX3.3: Drawing graphs involving powers and square roots of common functions**

In this subtopic, students graph functions that are related to given functions by using properties from algebra and calculus.

***Outcomes addressed***

HXX4

***Students develop the following knowledge, skills and understanding***

- graphing a function  $y = [f(x)]^n$ , where  $n$  is a positive integer, by first graphing  $y = f(x)$
- graphing a function  $y = \sqrt{f(x)}$ , by first graphing  $y = f(x)$ .

### **Applications and considerations**

- The graph of  $y = [f(x)]^n$ , where  $n$  is a positive integer, may be drawn by sketching  $y = f(x)$  and realising that, since its derivative is  $n[f(x)]^{n-1}f'(x)$  then all stationary points and intercepts on the  $x$ -axis of  $y = f(x)$  are stationary points of  $y = [f(x)]^n$ .  
Other features worth examining include the properties that if  $|f(x)| > 1$ , then  $|[f(x)]^n| > |f(x)|$  and that if  $0 < |f(x)| < 1$  then  $0 < |[f(x)]^n| < |f(x)|$ .  
Further, if  $n$  is even,  $[f(x)]^n \geq 0$  for all  $x$  but if  $n$  is odd,  $[f(x)]^n > 0$  for  $f(x) > 0$  and  $[f(x)]^n < 0$  for  $f(x) < 0$ .
- The graph of  $y = \sqrt{f(x)}$  may be developed from the graph of  $y = f(x)$  by noting that
  - $y = \sqrt{f(x)}$  is defined only if  $f(x) \geq 0$
  - $\sqrt{f(x)} \geq 0$  for all  $x$  in the natural domain
  - $\sqrt{f(x)} < f(x)$  if  $f(x) > 1$ ,  $\sqrt{f(x)} = f(x)$  if  $f(x) = 1$ ,  $\sqrt{f(x)} > f(x)$  if  $0 < f(x) < 1$
  - If  $y = \sqrt{f(x)}$  then  $y' = \frac{f'(x)}{2\sqrt{f(x)}}$ . This leads to the position of stationary points and the existence of critical points.
- To sketch  $y = \sqrt{\frac{x(x-1)}{x-2}}$  a rough sketch of  $y = \frac{x(x-1)}{x-2}$  can first be drawn and then the above ideas used to sketch the function.

**MXX3.4: General approach to curve sketching and using graphs in problem solving**

In this subtopic, students apply the methods of the previous subtopics to the solution of inequalities and other mathematical problems.

***Outcomes addressed***

HXX4

***Students develop the following knowledge, skills and understanding***

- using implicit differentiation to compute  $\frac{dy}{dx}$  for curves given in implicit form
- using an appropriate method to graph a given function or curve
- applying the techniques of this topic to a function whose equation is not given, but whose graph is given
- solving an inequality by sketching an appropriate graph
- finding the number of solutions of an equation by graphical considerations
- solving problems using graphs.

### ***Applications and considerations***

- The graph of a function could be obtained by using techniques which utilise basic functions together with consideration of features such as discontinuities, finding the domain of a function, investigating the behaviour of the function in the neighbourhood of  $x = 0$ , considering the behaviour of the function for  $x$  large, testing whether a function is odd or even, plotting points and deciding on positions of stationary points, critical points and points of inflexion.
- Examples of curves to be graphed include  $y = \frac{x^4}{x^2 - 1}$ ,  $y = x^2 e^{-x}$ ,  
 $y = x \ln(x^2 - 1)$ ,  $y^2 = x^2 - 9x$ ,  $x^2 + 2y^2 = 4$ ,  $y = \frac{\sin x}{x}$ , and  $y = x \cos x$ .
- Typical inequalities could include absolute values (eg  $|x| + |x-1| > 4$ ), rational functions (eg  $\frac{x^2 + 1}{x^2 - 1} < 1$ ) and trigonometric functions (eg  $\sin 2x \geq \frac{1}{2}$ ).
- The number of roots of an equation can be investigated graphically (eg find the number of solutions of  $x^3 - kx^2 + k = 0$  for varying values of  $k$ ).
- Graphs often give elegant solutions to problems which would be difficult to solve by other means.

## **MXX4: Integration techniques**

In this topic students learn a range of integration techniques and apply them to definite and indefinite integrals.

### ***Outcomes addressed***

*A student:*

- HXX9 uses integral calculus in the solution of problems requiring the use of integration tables, identification and use of appropriate substitutions, partial fractions, integration by parts, and recurrence formulae
- HXX12 communicates abstract ideas and relationships using appropriate notation and logical argument.

### ***Content summary***

MXX4.1 Integration techniques.

### ***Terminology***

distinct factors  
integration by parts  
integrand

partial fractions  
recurrence relations  
trigonometric substitution

### ***Use of technology***

Computer algebra systems are useful for checking integrals. However, the output may not be in standard form.

### ***Topic notes***

This topic includes essential skills for students who are continuing in the study of Mathematics. The study of integration techniques provides practice in manipulating common functions and in problem solving.

Various techniques for finding partial fractions are described in the support material. Note that the integration of rational functions is restricted to cases where the denominator is at most a product of simple linear or quadratic factors. (That is, cases where repeated factors occur are not included in this course.)

### **MX4.1: Integration techniques**

In this subtopic, students learn a range of integration techniques and apply them to definite and indefinite integrals.

#### ***Outcomes addressed***

HXX9, HXX12

#### ***Students develop the following knowledge, skills and understanding***

- using a table of standard integrals
- changing an integrand into an appropriate form using algebra
- evaluating integrals using algebraic substitutions
- evaluating simple trigonometric integrals
- evaluating integrals using trigonometric substitutions
- evaluating integrals using integration by parts
- deriving and using recurrence relations
- integrating rational functions by completing the square in a quadratic denominator
- integrating rational functions whose denominators have simple linear or quadratic factors.

### Applications and considerations

- Some of the results listed in the standard integrals table will need to be established as an appropriate method is developed.
- Some integrals may, through using simple algebra, be changed into a form which can be integrated, eg  $\int \left(\frac{x+1}{x}\right)^2 dx$ ,  $\int \frac{x}{\sqrt{x^2+1}} dx$ .
- Only simple substitutions are needed, eg  $u = 1+x^2$  in  $\int x(1+x^2)^4 dx$ ,  $v^2 = 1-x$  in  $\int \frac{x}{\sqrt{1-x}} dx$ . The effect on limits of integration is required, and definite integrals are to be treated.
- Include squares of all trigonometric functions, and those which can be found by a simple substitution, eg  $\int_0^{\pi/4} \sin^2 2x dx$ ,  $\int_0^{\pi/4} \sin^2 x \cos x dx$ ,  $\int_0^{\pi/4} \sin^2 \cos^3 x dx$ .
- Typical substitutions would be  $x = a \tan \theta$  and  $t = \tan \frac{\theta}{2}$  in integrals such as  $\int \frac{dx}{a^2+x^2}$ ,  $\int \frac{\sin \theta}{2+\sin \theta} d\theta$ .
- Work on integration by parts should include the integrands  $\sin^{-1} x$ ,  $e^{ax} \cos bx$ ,  $\ln x$ ,  $x^n \ln x$  ( $n$  an integer).
- Integration by parts should be extended to particular types of recurrence relations, eg  $\int x^n e^x dx$ ,  $\int_0^{\pi/2} \cos^n x dx$ . (Relations such as,  $\int_0^1 x^m (1-x)^n dx$  which involve more than one integer parameter, are excluded.)
- Examples should include cases to be integrated using a sum or difference of two squares, eg  $\int \frac{dx}{x^2-4x-1}$ ,  $\int \frac{dx}{3x^2+6x+10}$ ,  $\int \frac{3x+2}{x^2-4x+1} dx$ .
- Only rational functions, whose denominators can be broken into a product of distinct linear factors, or of a distinct quadratic factor and a linear factor, or of two distinct quadratic factors, need to be considered, eg  $\int \frac{9x-2}{2x^2-7x+3} dx$ ,  $\int \frac{3x^2-2x+1}{(x^2+1)(x^2+2)} dx$ ,  $\int \frac{2x^2+3x-1}{x^3-x^2+x-1} dx$ .  
Cases where the degree of the numerator is not less than the degree of the denominator are to be considered.

## **MXX5: Volumes**

In this topic, students learn to extend the method in the Mathematics Advanced course for finding the volume of a solid formed by revolution. Circular and annular cross-sections are used, as are cylindrical shells and cross-sections of similar shapes.

### ***Outcomes addressed***

*A student:*

HXX10 uses the techniques of slicing, cylindrical shells and similar cross-sections to calculate volumes.

### ***Content summary***

MXX5.1 Volumes.

### ***Terminology***

annular cross-section	cylindrical shells
circular cross-section	solid of revolution
cross-sections of similar shapes	

### ***Use of technology***

Simulations, animations and videos provide visual illustration of the physical situations being described.

### ***Topic notes***

The purpose of this topic is to provide practical examples of the use of a definite integral to represent a quantity (in this case, a volume) whose value can be regarded as the limit of an appropriate approximating sum. Emphasis is to be placed on understanding the various approximation methods given, deriving the relevant approximate expression for the corresponding element of volume and proceeding from this to expressing the volume as a definite integral.

The evaluations of infinite series by a definite integral, or of integrals by summation of series, are not included in this topic.

**MX5.1: Volumes**

In this subtopic, the method of finding the volume of a solid formed by revolution around an axis is revised, emphasising the formation of a sum and the evaluation of the limiting value of that sum by integration. The method is adapted for shapes with annular cross-sections, and for shapes with cross-sections that are not necessarily circular, but are similar. The method of cylindrical shells can be used for checking some results obtained by other methods, and also can be used in some situations where other methods are not appropriate.

***Outcomes addressed***

HXX10

***Students develop the following knowledge, skills and understanding***

- describing the common features of the methods used here: by dividing a solid into a number of slices or shells, whose volumes can be simply estimated, the volume of the solid is the value of the definite integral obtained as the limit of the corresponding approximating sums
- finding the volume of a solid of revolution by summing the volumes of slices with circular cross-sections
- finding the volume of a solid of revolution by summing the volumes of slices with annular cross-sections
- finding the volume of a solid of revolution by summing the volumes of cylindrical shells
- finding the volume of a solid which has parallel cross-sections of similar shapes
- choosing an appropriate method from those given above, and giving reasons for that choice.

**Applications and considerations**

- Volumes of revolution could lead, from questions involving rotation about a coordinate axis, to rotation about a line parallel to a coordinate axis, eg find the volume of the solid formed when the region bounded by  $y = 2\sqrt{x}$ , the  $x$ -axis and  $x = 4$  is rotated about the line  $x = 4$ .
- Students should be encouraged to draw a sketch of the shape of the volume to be found and a sketch of a cross-sectional slice. They should then derive an expression for the volume of a cross-sectional slice in a form which leads directly to an expression for the total volume as an integral.
- Example involving annular cross-sections:  
The region bounded by the curve  $y = (x-1)(3-x)$  and the  $x$ -axis is rotated about the line  $x = 3$  to form a solid. When the region is rotated, the horizontal line segment at height  $y$  sweeps out an annulus. Find the volume of the solid.
- A formula for summing by cylindrical shells should not be learnt. Each problem should rather be developed from first principles.
- The process of writing the limiting sum as an integral should be extended to cases where cross-sections are other than circular. These cases should only involve problems in which the geometrical shape is able to be visualised, eg prove that the volume of a pyramid of height  $h$  on a square base of side  $a$  is  $\frac{1}{3}a^2h$ .

## **MXX6: Mechanics**

In this topic, students apply techniques from calculus to model physical systems and predict the behaviour of objects that are under the influence of forces such as gravity and air resistance. A high level of problem-solving skill is developed and applications require the use of techniques from other sections of the course.

### ***Outcomes addressed***

*A student:*

- HXX1 constructs arguments and proofs in concrete and abstract settings
- HXX8 solves first-order and second-order ordinary linear differential equations
- HXX11 formulates and solves ordinary differential equations arising in mathematical modelling situations
- HXX12 communicates abstract ideas and relationships using appropriate notation and logical argument.

### ***Content summary***

- MXX6.1 Mathematical representation of motion described in physical terms, and interpreting in physical terms a mathematical description of motion
- MXX6.2 Resisted motion.

***Terminology***

horizontal component  
Newton's laws  
projectile

simple harmonic motion  
vertical component

***Use of technology***

Simulations, animations and videos provide visual illustration of the physical situations being described.

***Topic notes***

The purpose of this topic is to provide experience in applying calculus techniques to the solution of a range of physical problems. The connections between mathematical representations and physical descriptions of motion is an essential part of Applied Mathematics.

Students develop problem-solving skills as they choose a suitable representation for a physical phenomenon.

**MX6.1: Mathematical representation of motion described in physical terms and interpreting in physical terms a mathematical description of motion**

The principal focus of this subtopic is on modelling the motion of objects in situations where resistance is ignored.

***Outcomes addressed***

HXX1, HXX8, HXX11, HXX12

***Students develop the following knowledge, skills and understanding***

- deriving the equations of motion of a projectile
- using equations for horizontal and vertical components of velocity and displacement to solve problems on projectiles
- writing down equations for displacement, velocity and acceleration given that a motion is simple harmonic
- using relevant formulae and graphs to solve problems on simple harmonic motion
- using Newton's laws to obtain equations of motion of a particle in situations other than projectile motion and simple harmonic motion
- describing mathematically the motion of particles in situations other than projectile motion and simple harmonic motion.
- given  $\ddot{x} = f(x)$  and initial conditions, deriving  $v^2 = g(x)$  and describing the resultant motion
- recognising that a motion is simple harmonic, given an equation for either acceleration, velocity or displacement, and describing the resultant motion.

### ***Applications and considerations***

- Students should be able to represent mathematically, motions described in physical terms. They should be able to explain, in physical terms, features given by mathematical descriptions of motion in one or two dimensions.
- The classical statement of Newton's first and second laws of motion should be given as an illustration of the application of calculus to the physical world. Resolution of forces, accelerations and velocities in horizontal and vertical directions is to be used to obtain the appropriate equations of motion in two dimensions.
- A typical example on simple harmonic motion: The deck of a ship was 2.4 m below the level of a wharf at low tide and 0.6 m above wharf level at high tide. Low tide was at 8:30 am and high tide at 2.35 pm. Find when the deck was level with the wharf, if the motion of the tide was simple harmonic.
- Other examples may be found in the support material.

## **MX6.2: Resisted motion**

The principal focus of this subtopic is the construction of models to describe the motion of objects undergoing resistive forces.

### ***Outcomes addressed***

HXX1, HXX8, HXX11, HXX12

### ***Students develop the following knowledge, skills and understanding***

Resisted motion along a horizontal line

- deriving, from Newton's laws of motion, the equation of motion of a particle moving in a single direction under a resistance proportional to a power of the speed
- deriving an expression for velocity as a function of time (where possible)
- deriving an expression for velocity as a function of displacement (where possible)
- deriving an expression for displacement as a function of time (where possible).

Motion of a particle moving upwards in a resisting medium and under the influence of gravity

- deriving, from Newton's laws of motion, the equation of motion of a particle, moving vertically upwards in a medium, with a resistance  $R$  proportional to the first or second power of its speed
- deriving expressions for velocity as a function of time and for velocity as a function of displacement (or vice versa)
- deriving an expression for displacement as a function of time
- solving problems by using the expressions derived for acceleration, velocity and displacement.

Motion of a particle falling downwards in a resisting medium and under the influence of gravity

- deriving, from Newton's laws of motion, the equation of motion of a particle falling in a medium with a resistance  $R$  proportional to the first or second power of its speed
- determining the terminal velocity of a falling particle from its equation of motion
- deriving expressions for velocity as a function of time and for velocity as a function of displacement
- deriving an expression for displacement as a function of time
- solving problems by using the expressions derived for acceleration, velocity and displacement.

**Applications and considerations**

- Typical cases to consider include those in which the resistance is proportional to the speed and to the square of the speed.
- Analysis of the motion of a particle should include consideration of the behaviour of the particle as  $t$  becomes large. Graphs offer assistance in understanding the behaviour of the particle.
- Cases, other than where the resistance is proportional to the first or second power of the speed, are not required to be investigated.
- Students should be advised to place the origin at the point of projection.
- The maximum height reached by the particle can be obtained from the expression relating speed and displacement.
- The time taken to reach this maximum height can be obtained from the expression relating speed and time.
- Problems should include cases where the magnitude of the resistance is given (eg  $R = \frac{1}{10}v^2$ ).
- Cases, other than where the resistance is proportional to the first or second power of the speed, are not required to be investigated.
- Students should place the origin at the point from which the particle initially falls.
- If the motion of a particle both upwards and then downwards is considered then the position of the origin should be changed as soon as the particle reaches its maximum height. Care must then be taken in determining the correct initial conditions for the downward motion.
- The terminal velocity can be calculated from the equation of motion by finding  $V$  when  $\ddot{x} = 0$ .
- The time taken for the particle to reach ground level should be found.
- Problems should include a study of the complete motion of a particle, projected vertically upwards, which then returns to its starting point. For specific resistance functions, comparisons should be made between the times required for its upward and downward journeys and between the speed of projection and the speed of its return.

## **MXX7: Modelling with functions and derivatives**

In this topic students learn techniques for solving a limited range of differential equations, and also see the connections between previously unrelated functions of a real variable that are possible through the use of complex numbers.

### ***Outcomes addressed***

*A student:*

- HXX1 constructs arguments and proofs in concrete and abstract settings
- HXX8 solves first-order and second-order ordinary linear differential equations
- HXX11 formulates and solves ordinary differential equations arising in mathematical modelling situations
- HXX12 communicates abstract ideas and relationships using appropriate notation and logical argument.

### ***Content summary***

- MXX7.1 Direction fields
- MXX7.2 First-order linear differential equations
- MXX7.3 Second-order linear differential equations.

**Terminology**

complex function	non-linear DE
complex variable	second-order homogeneous DE
DE (differential equation)	second-order linear DE
direction field	
first-order linear DE	

**Use of technology**

While by-hand skills for equation solving are essential for students in this course, graphing technology is an excellent means of exploring many of the concepts studied in this topic. The use of graphing technology is encouraged in learning and teaching. For example, graphing calculators and computer or hand-held spreadsheet applications are ideal for investigating direction fields.

**Topic notes**

This topic reveals the power of calculus to model real world systems. First-order differential equations are introduced and related to previously studied examples. Some simple types capable of straightforward solution are studied.

Second-order differential equations are introduced again via some simple examples. The case of homogeneous linear second-order differential equations is then examined further. Previously studied examples are then related, via the use of complex numbers, and lead to a plausible argument suggesting how to define the function  $f(t) = e^{it}$  for a real variable  $t$ , and hence for the general complex variable  $z = x + iy$ . As a result, relations are obtained connecting the exponential and trigonometric functions. In particular, Euler's famous identity  $e^{i\pi} = -1$  is derived.

The mathematical idea of a function as a solution to a differential equation is fundamental to this course. A light treatment of direction fields will help to establish this idea. A balance of abstract ideas and applications to real-world modelling is sought.

Models from the physical and biological sciences, engineering, population studies and finance will be used to demonstrate the wide applicability of modelling with functions and derivatives.

### **MX7.1: Direction fields**

The focus of this subtopic is a graphical introduction to differential equations and their solutions through the interpretation of direction fields.

#### ***Outcomes addressed***

HXX12

#### ***Students develop the following knowledge, skills and understanding***

- recognising that an equation involving a derivative is called a differential equation, and that solving a differential equation involves finding a function which, with its derivative, satisfies the differential equation
- recognising that solutions to differential equations (if they exist) are functions that may not be unique, and that particular solutions may be found if initial conditions are given
- forming a direction field (slope field) from simple differential equations
- recognising the shape of a direction field from several alternatives, given the form of a differential equation, and vice versa
- sketching several possible solution curves on a given direction field
- sketching the graph of a particular solution given a direction field and initial conditions.

## Applications and considerations

- The general first-order differential equation

$$\frac{dy}{dx} = A(x, y), \quad (1)$$

where  $A$  is a function of the two variables  $x$  and  $y$ , may or may not have a solution  $y = f(x)$  in which the function  $f$  is expressible in terms of known functions. This topic restricts the study of differential equations to some simple important cases where  $f$  is so expressible, but initially, (1) is interpreted as describing a 'direction field' in the Cartesian plane. This leads to an understanding that, in general, the solution of (1) is a family of curves and that additional information has to be given to obtain a unique solution.

- Example: solve the DE  $\frac{dy}{dx} = 2x$ , to obtain the family of solutions  $y = x^2 + C$ . Observe that each solution is a function of  $x$  and that, for a given solution, eg  $y = x^2 + 1$ , the DE expresses the *slope* of the solution curve at the point  $(x, y)$  on it. Note that a unique solution is obtained if the value of  $y$  is specified for *one* given value of  $x$ .
- Use the DE  $\frac{dy}{dx} = 2x$ , and a starting point (eg  $(1, 1)$ ) to 'build up' the solution curve to the DE, passing through the starting point, by constructing successive small linear segments with slopes given by the DE.
- Extend this to show how a 'direction field' or 'slope field' can be imagined covering the Cartesian plane with the family of solution curves to the DE.
- Examples of direction fields to be sketched by hand include  $\frac{dy}{dx} = c$ ,  $\frac{dy}{dx} = cx$ ,  $\frac{dy}{dx} = cy$ ,  $\frac{dy}{dx} = cxy$  where  $c$  is a given constant.
- Examples of direction fields to be sketched with technology include the direction field for  $\frac{dy}{dx} = -\frac{x}{y}$ , where the solution curves are circles centre  $(0, 0)$ .

## **MX7.2: First-order linear differential equations**

The principal focus of this subtopic is recognising and solving first-order linear differential equations of two simple kinds. Students have met some of these before in other topics, and here they learn that the equations they have met before are particular kinds of a general case.

### ***Outcomes addressed***

HXX8, HXX11, HXX12

### ***Students develop the following knowledge, skills and understanding***

- recognising the features of a first-order linear differential equation
- recognising that the population models of previous topics are first-order linear differential equations, with known solutions
- investigating the solutions of first-order linear DEs
- defining appropriate independent and dependent variables in a problem situation
- constructing a differential equation to model a problem situation
- identifying initial conditions (if any) from the context of the problem
- solving the resulting DE and finding a particular solution using the initial conditions
- checking that the solution to a differential equation gives appropriate results, especially for large values of the independent variable
- if necessary, restricting the domain of the solution to give a reasonable result.

### Applications and considerations

- A first-order DE has only the first derivative appearing in it (not the second or higher derivatives).
- A linear first-order DE with solution function  $f(t)$  has  $f$  and its first derivative occurring to the first degree only. Examples include  $\frac{df}{dt} + g(t)f = c(t)$ ,  $\frac{df}{dt} + t^2 f = \sin t$ ; while  $\frac{df}{dt} + 3f^2 = 2$  is first order but not linear because of the term  $f^2$ .
- For a first-order linear DE of the form  $\frac{df}{dt} = c(t)$ , the solution is equivalent to the finding of a primitive function  $F(t)$  for  $c(t)$ : if  $F'(t) = c(t)$ , then the equation is  $\frac{df}{dt} = \frac{dF}{dt}$ , so  $\frac{d}{dt}(f - F) = 0$  and  $f(t) = F(t) + C$  is the solution for any constant  $C$ .
- If  $\frac{df}{dt} + g(t)f = c(t)$  is of the form  $\frac{df}{dt} + kf = c$ , with  $k$  and  $c$  constants, then its solution is known from previous work in population models, and is  $f(t) = Ae^{-kt} + \frac{c}{k}$ . This is to be verified by direct substitution. Methods for finding this solution are mentioned in the support material as possible stimulus material for extending students' knowledge.
- Examples of problems to be solved:
  - Manufacturers often launch a new product with a large advertising budget. However, if the product is judged by the market as being of poor quality, sales will decline as people try the product but do not continue to buy it. For a certain product the rate of weekly sales is modelled by  $S'(t) = \frac{400}{(t+1)^3} - \frac{200}{(t+1)^2}$ , where  $S$  is the number of sales in millions and  $t$  is the number of weeks since the launch of the product.
    - (a) Find the function that describes the weekly sales.
    - (b) Find the sales for the first week and for the tenth week.
  - If  $P$  represents the purchasing power (in dollars) of a pension of \$60 000 per year, then this will be decreased by inflation according to the formula  $\frac{dP}{dt} = -rP$  where  $r$  is the rate of inflation expressed as a decimal, and  $P(0) = 60\,000$ .
    - (a) Investigate this situation with direction fields for several realistic values of  $r$ .
    - (b) Find the purchasing power after 20 years for the choices of  $r$ .
- There is a general method of solution for first-order homogeneous linear DEs, which is described in the support material. While this may be used by students, its derivation or application are not included in this course.

**MX7.3: Second-order linear differential equations**

In this subtopic students learn to recognise second-order linear differential equations and solve simple types of these. The connection between exponential and trigonometric functions that is made possible with complex numbers is introduced and explored.

***Outcomes addressed***

HXX8, HXX12

***Students develop the following knowledge, skills and understanding***

- recognising the features of a second-order linear differential equation and the special case of a homogeneous second-order linear DE
- recognising that second-order DEs arise in models of physical situations involving motion because Newton's laws of motion relate the acceleration of an object to the forces acting on it
- verifying that if a second-order homogeneous linear DE has two different functions as solutions, say  $f_1$  and  $f_2$ , then the following are also solutions:  $Af_1$ ,  $Af_1 + Bf_2$ , where  $A$  and  $B$  are constants
- using the auxiliary equation method to solve DEs of the form  $f'' + bf' + cf = 0$ , where  $b$  and  $c$  are real constants, including the cases where the auxiliary equation has (i) real different roots, (ii) repeated real roots
- recognising the reasons behind the definition of the complex exponential as  $e^{it} = \cos t + i \sin t$
- showing that if  $p \pm iq$  are the complex roots of an auxiliary equation, then the solution of the relevant homogeneous DE with real initial conditions can be expressed in the form  $f = e^{pt}(C \cos qt + D \sin qt)$  where  $C$  and  $D$  are real.
- solving second-order homogeneous linear DEs and interpreting the solutions in context.

### **Applications and considerations**

- A *second-order* DE has the second derivative appearing in it (no higher derivatives), and may also have the first derivative appearing in it.
- A *linear second-order* DE with solution function  $f(t)$  has  $f$  and its first and second derivatives occurring to the first degree only. The general form of a second-order linear DE is  $\frac{d^2 f}{dt^2} + g(t)\frac{df}{dt} + h(t)f = c(t)$ , where  $g$ ,  $h$ , and  $c$  may be arbitrary functions of  $t$ .
- Examples of linear second-order DEs include  $\frac{d^2 f}{dt^2} + t\frac{df}{dt} + t^2 f = \sin t$  ;  
however  $\frac{d^3 f}{dt^3} = 3$  is third-order, and  $\frac{d^2 f}{dt^2} + t^2 f^4 = 0$  is second-order but non-linear.
- Example: Find the general solution for the DE  $f'' = 9f$  .
- Example: Find the general solution for the DE  $f'' + 6f' + 8f = 0$  .
- Example: Find the general solution for the DE  $f'' + 6f' + 13f = 0$  .
- Example: Find the particular solution for the DE  $f'' + 6f' + 13f = 0$  with initial conditions  $f(0) = 8$ ,  $f'(0) = 0$  . If  $f(t)$  represents the displacement of a particle from the origin, describe the motion and in particular what happens in the longer term.

## 10 Course Requirements

*Mathematics Extension 2 Stage 6 Syllabus* consists of an HSC course of 60 (indicative) hours.

Completion of the Preliminary Mathematics Extension course is a prerequisite for the study of the HSC Mathematics Extension 2 course. Students may study the HSC Mathematics Extension 2 course following completion of the HSC Mathematics Extension 1 course or concurrently with the HSC Mathematics Extension 1 course.

Students may not study the Mathematics Extension 2 course in conjunction with the Mathematics General 1 course or the Mathematics General 2 course.

## 11 Post-school Opportunities

The study of Mathematics Extension 2 provides students with knowledge, skills and understanding that form a valuable foundation for a range of courses at university and other tertiary institutions.

In addition, the study of Mathematics Extension 2 assists students to prepare for employment and full and active participation as citizens. In particular, there are opportunities for students to gain recognition in vocational education and training. Teachers and students should be aware of these opportunities.

### Recognition of Student Achievement in Vocational Education and Training (VET)

Wherever appropriate, the skills and knowledge acquired by students in their study of HSC courses should be recognised by industry and training organisations. Recognition of student achievement means that students who have satisfactorily completed HSC courses will not be required to repeat their learning in courses in TAFE NSW or other Registered Training Organisations (RTOs).

Registered Training Organisations, such as TAFE NSW, provide industry training and issue qualifications within the Australian Qualifications Framework (AQF).

The degree of recognition available to students in each subject is based on the similarity of outcomes between HSC courses and industry training packages endorsed within the AQF. Training packages are documents that link an industry's competency standards to AQF qualifications. More information about industry training packages can be found on the National Training Information Service (NTIS) website ([www.ntis.gov.au](http://www.ntis.gov.au)).

### Recognition by TAFE NSW

TAFE NSW conducts courses in a wide range of industry areas, as outlined each year in the TAFE NSW Handbook. Under current arrangements, the recognition available to students of Mathematics in relevant courses conducted by TAFE is described in the HSC/TAFE Credit Transfer Guide. This guide is produced by the Board of Studies and TAFE NSW and is distributed annually to all schools and colleges. Teachers should refer to this guide and be aware of the recognition that may be available to their students through the study of Mathematics Extension 2. This information can be found on the TAFE NSW website ([www.tafensw.edu.au/mchoice](http://www.tafensw.edu.au/mchoice)).

### Recognition by other Registered Training Organisations

Students may also negotiate recognition into a training package qualification with another Registered Training Organisation. Each student will need to provide the RTO with evidence of satisfactory achievement in Mathematics Extension 2 so that the degree of recognition available can be determined.

## 12 Assessment and Reporting

### 12.1 Requirements and Advice

The information in this section of the syllabus relates to the Board of Studies' requirements for assessing and reporting achievement in the Preliminary and HSC courses for the Higher School Certificate.

*Assessment* is the process of gathering information and making judgements about student achievement for a variety of purposes.

In the Preliminary and HSC courses those purposes include:

- assisting student learning
- evaluating and improving teaching and learning programs
- providing evidence of satisfactory achievement and completion in the Preliminary course
- providing the Higher School Certificate results.

*Reporting* refers to the Higher School Certificate documents that are used by the Board to report to students both the internal and external measures of achievement.

Higher School Certificate results comprise:

- an assessment mark derived from the mark submitted by the school and produced in accordance with the Board's requirements for the internal assessment program
- an examination mark derived from the HSC external examinations
- an HSC mark, which is the average of the assessment mark and the examination mark
- a performance band, determined by the HSC mark.

Results will be reported using a course report containing a performance scale with bands describing standards of achievement in the course.

The use of both internal assessment and external examination of student achievement allows measurements and observations to be made at several points and in different ways throughout the HSC Mathematics Extension 2 course. Taken together, the external examination and internal assessment marks provide a valid and reliable assessment of the achievement of the knowledge, understanding and skills described for each course.

The Board of Studies uses a standards-referenced approach to assessing and reporting student achievement in the Higher School Certificate.

The standards in the HSC are:

- the knowledge, skills and understanding expected to be learnt by students – the *syllabus standards*
- the levels of achievement of the knowledge, skills and understanding – the *performance standards*.

Both *syllabus standards* and *performance standards* are based on the aims, objectives, outcomes and content of a course. Together they specify what is to be learnt and how well it is to be achieved.

Teacher understanding of standards comes from the set of aims, objectives, outcomes and content in each syllabus together with:

- the performance descriptions that summarise the different levels of performance of the course outcomes
- HSC examination papers and marking guidelines
- samples of students' achievement, collected in the Standards Packages.

## **12.2 Internal Assessment**

The internal assessment mark submitted by the school will provide a summation of each student's achievements measured at points throughout the course. The marks for each course group at a school should reflect the rank order of students and relative differences between students' achievements.

Internal assessment provides a measure of a student's achievement based on a wider range of syllabus content and outcomes than may be covered by the external examination alone. The assessment components and weightings to be applied to internal assessment are identified on page 77. They ensure a common focus for internal assessment in the course across schools, while allowing for flexibility in the design of tasks. A variety of tasks should be used to give students the opportunity to demonstrate outcomes in different ways and to improve the validity and reliability of the assessment.

## **12.3 External Assessment**

In Mathematics Extension 2 Stage 6, the external examination consists of a written examination. The specifications for the HSC examination in Mathematics Extension 2 are on page 78.

The external examination provides a measure of student achievement in a range of syllabus outcomes that can be reliably measured in an examination setting.

The external examination and its marking and reporting will relate to syllabus standards by:

- providing clear links to syllabus outcomes
- enabling students to demonstrate the levels of achievement outlined in the course performance scales
- applying marking guidelines based on established criteria.

## **12.4 Board Requirements for the Internal Assessment Mark in Board Developed Courses**

The Board requires schools to submit an assessment mark for each candidate in the Mathematics Extension 2 course. The Board requires that the assessment tasks used to determine the internal assessment mark must comply with the components and weightings specified in the table on page 77.

Schools are required to develop an internal assessment program that:

- specifies the various assessment tasks and the weightings allocated to each task
- provides a schedule of the tasks designed for the whole course.

The standards-referenced approach to assessment for the HSC involves schools ensuring that in the design and marking of tasks:

- assessment tasks are designed to focus on outcomes
- the types of assessment tasks are appropriate for the outcomes being assessed
- students are given the opportunity to demonstrate their level of achievement of the outcomes in a range of different task types
- tasks reflect the weightings and components specified in the relevant syllabus
- students know the assessment criteria before they begin a task
- marking guidelines for each task are linked to the standards by including the wording of syllabus outcomes and relevant performance descriptions
- marks earned on individual tasks are expressed on a scale sufficiently wide to reflect adequately the relative differences in student performances.

In feedback and reporting:

- students receive meaningful feedback about what they are able to do and what they need to do in order to improve their level of performance
- the ranking and relative differences between students result from different levels of achievement of the specified standards
- marks submitted to the Board for each course are on a scale sufficiently wide to reflect adequately the relative differences in student performances.

Note that:

- measures of objectives and outcomes that address values and attitudes should not be included in school-based assessments of students' achievements. As these objectives are important elements of any course, schools may decide to report on them separately to students and parents, perhaps using some form of descriptive statements
- measures that reflect student conduct should not be included.

## 12.5 Assessment Components, Weightings and Tasks

The mandatory components and weightings for the Mathematics Extension 2 course are set out below. The internal assessment mark is to be based on the Mathematics Extension 2 course, and assessment tasks will focus on the course objectives and outcomes.

Teachers can use their discretion in determining the manner in which they allocate tasks within course content. While the allocation of weightings to the various tasks set for the HSC course is left to individual schools, the percentages allocated to each syllabus component must be maintained.

It is suggested that three or four tasks are sufficient to assess the Mathematics Extension 2 HSC course. The range of tasks comprising the school-based assessment schedule should be varied and address the range of outcomes. One task may be used to assess several components.

<b>Component</b>	<b>Description</b>	<b>Weighting</b>	<b>Suggested tasks</b>
Concepts and techniques	Use of concepts and techniques in the solution and interpretation of mathematical problems	50%	<ul style="list-style-type: none"> <li>• assignments</li> <li>• examination-style questions</li> <li>• multimedia-based tasks</li> <li>• open-book tasks</li> <li>• practical investigations or projects</li> <li>• practical tasks such as measurement activities</li> <li>• student's written explanation of problem solutions</li> </ul>
Reasoning and communication	Application of reasoning and communication in appropriate forms in the construction of mathematical proofs and arguments and the interpretation and use of mathematical models	50%	
<b>Total:</b>		<b>100</b>	

## 12.6 HSC External Examination Specifications

The Mathematics Extension 2 HSC examination will consist of a written examination paper of three hours duration (plus five minutes reading time) containing two sections with a total mark value of 100 marks. All questions in the examination are compulsory.

The examination will be based mainly on the Mathematics Extension 2 course and will focus on the course objectives and outcomes. The Mathematics Extension 1 and Mathematics Advanced courses will be assumed knowledge for this examination. Questions focusing on Mathematics Extension 2 outcomes may relate to knowledge, skills and understanding from the Mathematics Extension 1 and Mathematics Advanced courses.

A formula sheet, including standard integrals, will be provided with the examination paper.

In addition to basic examination equipment, a pair of compasses, set squares, a protractor and a mathematical curve-drawing template\* may be used.

Calculators that are Board-approved for the Mathematics Advanced, Mathematics Extension 1 and Mathematics Extension 2 HSC examinations may be used.

### Section I (10 marks)

- Questions in this section will be in objective–response format, such as multiple-choice, multiple correct/incorrect, or other constrained response questions.
- The mark value of these questions will be one or more marks, depending on the length and demands of the question.

### Section II (90 marks)

- There will be SIX questions.
- All questions will be worth 15 marks.
- Each question will consist of a number of parts requiring free-response answers. These parts may be stand-alone questions, or may consist of several related sub-parts.

**Note:** Sample questions for this examination may be accessed on the Board's website. ([www.boardofstudies.nsw.edu.au](http://www.boardofstudies.nsw.edu.au))

\* Students may take into any Mathematics examination a curve-drawing template, provided the template contains no printed formulae other than equations of simple curves (such as  $y = x^2$ ) that may be drawn using the template.

## 12.7 Summary of Internal and External HSC Assessment

Internal Assessment	Weighting	External Assessment	Weighting
Concepts and techniques	50	A written examination consisting of a range of items.	100
Reasoning and communication	50		
<b>Total:</b>	<b>100</b>	<b>Total:</b>	<b>100</b>

## 12.8 Reporting Student Performance Against Standards

Student performance in an HSC course will be reported against standards on a course report. The course report includes a performance scale for the course describing levels (bands) of achievement, an HSC mark located on the performance scale, an internal assessment mark and an examination mark. It will also show, graphically, the statewide distribution of examination marks of all students in the course.

Each band on the performance scale (except for Band 1) includes descriptors that summarise the attainments typically demonstrated in that band.

The distribution of marks will be determined by students' performance against the standards and not scaled to a predetermined pattern of marks.