DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 16.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.
QUESTION 1 Use a SEPARATE Writing Booklet.

(a) Find \( \int \frac{\cos x}{\sin^4 x} \, dx \).  

(b) Use completion of squares to find \( \int \frac{4}{x^2 + 6x + 10} \, dx \).  

(c) (i) Find the real numbers \( a, b \) and \( c \) such that \( \frac{9}{x^2(3-x)} \equiv \frac{ax+b}{x^2} + \frac{c}{3-x} \).  

(ii) Find \( \int \frac{9}{x^2(3-x)} \, dx \).  

(d) Find \( \int \sqrt{x} \ln x \, dx \).  

(e) Use the substitution \( t = \tan \frac{\theta}{2} \) to find \( \int \frac{d\theta}{1 + \sin \theta + \cos \theta} \).
QUESTION 2 Use a SEPARATE Writing Booklet.

(a) Find all pairs of integers \(x\) and \(y\) that satisfy \((x + iy)^2 = 24 + 10i\).  

(b) Consider the equation \(z^2 + az + (1 + i) = 0\). Find the complex number \(a\), given that \(i\) is a root of the equation.

(c) (i) Let \(z = \cos \frac{\pi}{6} + i \sin \frac{\pi}{6}\). Find \(z^6\).

(ii) Plot, on the Argand diagram, all complex numbers that are solutions of \(z^6 = -1\).

(d) Sketch the region in the Argand diagram that satisfies the inequality \(z\overline{z} + 2(z + \overline{z}) \leq 0\).

(e) In the Argand diagram, \(OABC\) is a rectangle, where \(OC = 2OA\). The vertex \(A\) corresponds to the complex number \(\omega\).

(i) What complex number corresponds to the vertex \(C\)?

(ii) What complex number corresponds to the point of intersection \(D\) of the diagonals \(OB\) and \(AC\)?
(a) The diagram shows the graph of the (decreasing) function \( y = f(x) \).

Draw separate one-third page sketches of the graphs of the following:

(i) \( y = |f(x)| \)

(ii) \( y = \frac{1}{f(x)} \)

(iii) \( y^2 = f(x) \)

(iv) the inverse function \( y = f^{-1}(x) \).
QUESTION 3  (Continued)

(b) The base of a solid $\mathcal{B}$ is the region in the $xy$ plane enclosed by the parabola $y^2 = 4x$ and the line $x = 4$, and each cross-section perpendicular to the $x$ axis is a semi-ellipse with the minor axis one-half of the major axis.

(i) Show that the area of the semi-ellipse at $x = h$ is $\pi h$.
(You may assume that the area of an ellipse with semi-axes $a$ and $b$ is $\pi ab$.)

(ii) Find the volume of the solid $\mathcal{B}$.

(iii) Consider the solid $\mathcal{J}$, which is obtained by rotating the region enclosed by the parabola and the line $x = 4$ about the $x$ axis. What is the relation between the volume of $\mathcal{B}$ and the volume of $\mathcal{J}$?

(c) A modern supercomputer can calculate 1000 billion (ie, $10^{12}$) basic arithmetical operations per second. Use Stirling’s formula to estimate how many years such a computer would take to calculate 100! basic arithmetical operations. Stirling’s formula states that $n!$ is approximately equal to

$$\sqrt{2\pi n^{n+\frac{1}{2}}} e^{-n}.$$  

Leave your answer in scientific notation.
The point \( P \left( cp, \frac{c}{p} \right) \), where \( p \neq \pm 1 \), is a point on the hyperbola \( xy = c^2 \), and the normal to the hyperbola at \( P \) intersects the second branch at \( Q \). The line through \( P \) and the origin \( O \) intersects the second branch at \( R \).

(i) Show that the equation of the normal at \( P \) is

\[
py - c = p^3 \left( x - cp \right).
\]

(ii) Show that the \( x \) coordinates of \( P \) and \( Q \) satisfy the equation

\[
x^2 - c \left( p - \frac{1}{p^3} \right) x - \frac{c^2}{p^2} = 0.
\]

(iii) Find the coordinates of \( Q \), and deduce that the \( \angle QRP \) is a right angle.
(b) The temperature $T_1$ of a beaker of chemical, and the temperature $T_2$ of a surrounding vat of cooler water, satisfy, in accordance with Newton’s law of cooling, the equations

$$\frac{dT_1}{dt} = -k(T_1 - T_2)$$

$$\frac{dT_2}{dt} = \frac{3}{4}k(T_1 - T_2)$$

where $k$ is a positive constant.

(i) Show, by differentiation, that $\frac{3}{4}T_1 + T_2 = C$, where $C$ is a constant.

(ii) Find an expression for $\frac{dT_1}{dt}$ in terms of $T_1$, and show $T_1 = \frac{4}{7}C + Be^{\frac{-24}{7}t}$ satisfies this differential equation for any constant $B$.

(iii) Initially, the beaker of chemical had a temperature of $120^\circ$C and the vat of water had a temperature of $22^\circ$C. Ten minutes later, the temperature of the beaker of chemical had fallen to $90^\circ$C.

Find the temperature of the beaker of chemical after a further ten minutes.
QUESTION 5 Use a SEPARATE Writing Booklet.  

(a) Consider the polynomial

\[ p(x) = ax^4 + bx^3 + cx^2 + dx + e \]

where \( a, b, c, d \) and \( e \) are integers. Suppose \( \alpha \) is an integer such that \( p(\alpha) = 0 \).

(i) Prove that \( \alpha \) divides \( e \).

(ii) Prove that the polynomial

\[ q(x) = 4x^4 - x^3 + 3x^2 + 2x - 3 \]

does not have an integer root.
A string of length $\ell$ is initially vertical and has a mass $P$ of $m$ kg attached to it. The mass $P$ is given a horizontal velocity of magnitude $V$ and begins to move along the arc of a circle in a counterclockwise direction.

Let $O$ be the centre of this circle and $A$ the initial position of $P$. Let $s$ denote the arc length $AP$, $v = \frac{ds}{dt}$, $\theta = \angle AOP$ and let the tension in the string be $T$. The acceleration due to gravity is $g$ and there are no frictional forces acting on $P$.

For parts (i) to (iv), assume that the mass is moving along the circle.

(i) Show that the tangential acceleration of $P$ is given by \[
\frac{d^2s}{dt^2} = \frac{1}{\ell} \frac{d}{d\theta} \left( \frac{1}{2} v^2 \right).
\]

(ii) Show that the equation of motion of $P$ is \[
\frac{1}{\ell} \frac{d}{d\theta} \left( \frac{1}{2} v^2 \right) = -g \sin \theta.
\]

(iii) Deduce that $V^2 = v^2 + 2g\ell(1 - \cos \theta)$.

(iv) Explain why $T - mg \cos \theta = \frac{1}{\ell} mv^2$.

(v) Suppose that $V^2 = 3g\ell$. Find the value of $\theta$ at which $T = 0$.

(vi) Consider the situation in part (v). Briefly describe, in words, the path of $P$ after the tension $T$ becomes zero.
QUESTION 6 Use a SEPARATE Writing Booklet. 

(a) In the diagram, $C_1$ and $C_2$ are semicircles of radii $r_1$ and $r_2$, with centres $O_1$ and $O_2$ on $AB$. The two semicircles touch at the point $S$ on $AB$. The semicircle $C_3$ has diameter $AB$, and $R$ is the point on $C_3$ such that $RS$ is tangential to both $C_1$ and $C_2$ (so $RS$ is perpendicular to $AB$). The other common tangent to $C_1$ and $C_2$ touches $C_1$ at $P$ and $C_2$ at $Q$. The tangents $PQ$ and $RS$ intersect at $M$.

(i) State why $MP = MS = MQ$.

(ii) By using the ‘intersecting chords theorem’ (applied to $C_3$), or otherwise, prove that $RS^2 = 4r_1r_2$.

(The intersecting chords theorem states that the products of the intercepts of two intersecting chords are equal.)

(iii) Show that $\angle O_1MO_2$ is a right angle, and deduce that $MS^2 = r_1r_2$.

(iv) Deduce that $PSQR$ is a rectangle.

(b) (i) Evaluate $\int_{0}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$.

(ii) Explain carefully why, for $n \geq 2$,

$$\frac{1}{2} \leq \int_{0}^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}} \leq \frac{\pi}{6}.$$
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(a) (i) Show that, for $x > 0$,

$$\ln x \leq x - 1, \text{ with equality only at } x = 1.$$  

(ii) From (i) deduce that

$$\sum_{i=1}^{n} x_i \ln \frac{y_i}{x_i} \leq 0$$

whenever $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1$, where $x_i > 0$, $y_i > 0$ for $i = 1, 2, \ldots, n$.

Show also that equality occurs only if $x_i = y_i$ for $i = 1, 2, \ldots, n$.

(iii) By considering part (ii) with equal values of $y_i$ for $i = 1, 2, \ldots, n$, prove that the maximum value of

$$\sum_{i=1}^{n} x_i \ln \frac{1}{x_i}$$

is $\ln n$, where $\sum_{i=1}^{n} x_i = 1$ and $x_i > 0$ for $i = 1, 2, \ldots, n$.

(iv) Does the result of part (iii) hold if $\ln$ is replaced by $\log_2$? Give reasons for your answer.
In the diagram, $P$ is an arbitrary point on the ellipse, and $QPT$ is the tangent to the ellipse at $P$. The points $S'$ and $S$ are the foci of the ellipse, and $S^*$ is the reflection of $S$ across the tangent, as shown. Let the line $S'Q$ intersect the ellipse at $R$.

(i) Assuming $Q \neq P$, prove that

$$S'Q + QS > S'R + RS.$$  

(ii) Deduce that the shortest path from $S'$ to $S$ passing through a point on the tangent is that through $P$, having length $S'P + PS$.

(iii) By considering the point $S^*$, deduce that $\angle QPS' = \angle TPS$.

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QUESTION 8 Use a SEPARATE Writing Booklet.

(a) 
(i) Use the formula for the sum of a geometric series to show that

\[ \sum_{k=1}^{n} (z + z^2 + \ldots + z^k) = \frac{nz}{1-z} - \frac{z^2}{(1-z)^2} (1-z^n), \quad z \neq 1. \]

(ii) Let \( z = \cos \theta + i \sin \theta \), where \( 0 < \theta < 2\pi \). By considering the imaginary part of the left-hand side of the equation of part (i), deduce that

\[ \sum_{k=1}^{n} (\sin \theta + \sin 2\theta + \ldots + \sin k\theta) = \frac{(n+1)\sin \theta - \sin(n+1)\theta}{4\sin^2 \frac{\theta}{2}}. \]

(You may assume that \( \frac{z}{1-z} = \frac{i}{2\sin \frac{\theta}{2}} (\cos \frac{\theta}{2} + i \sin \frac{\theta}{2}) \).

(b) A fair coin is to be tossed repeatedly. For integers \( r \) and \( s \), not both zero, let \( P(r, s) \) be the probability that a total of \( r \) heads are tossed before a total of \( s \) tails are tossed so that \( P(0, 1) = 1 \) and \( P(1, 0) = 0 \).

(i) Explain why, for \( r, s \geq 1 \),

\[ P(r, s) = \frac{1}{2} P(r-1, s) + \frac{1}{2} P(r, s-1) \]

(ii) Find \( P(2, 3) \) by using part (i).

(iii) By using induction on \( n = r + s - 1 \), or otherwise, prove that

\[ P(r, s) = \frac{1}{2^n} \left( \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{s-1} \right) \quad \text{for} \quad s \geq 1. \]

End of paper
STANDARD INTEGRALS

\[ \int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0 \]

\[ \int \frac{1}{x} \, dx = \ln x, \quad x > 0 \]

\[ \int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0 \]

\[ \int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0 \]

\[ \int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0 \]

\[ \int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0 \]

\[ \int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0 \]

\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0 \]

\[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a \]

\[ \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0 \]

\[ \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left( x + \sqrt{x^2 + a^2} \right) \]

NOTE : \( \ln x = \log_e x, \quad x > 0 \)