Mathematics Extension 1

General Instructions

• Reading time – 5 minutes
• Working time – 2 hours
• Write using black or blue pen
• Board-approved calculators may be used
• A table of standard integrals is provided at the back of this paper
• All necessary working should be shown in every question

Total marks – 84

• Attempt Questions 1–7
• All questions are of equal value
**Question 1** (12 marks) Use a SEPARATE writing booklet.

(a) Use the table of standard integrals to find the exact value of

\[
\int_{0}^{2} \frac{dx}{\sqrt{16-x^2}}.
\]

(b) Find \(\frac{d}{dx}(x\sin^2 x)\).

(c) Evaluate \(\sum_{n=4}^{7} (2n+3)\).

(d) Let \(A\) be the point \((-2, 7)\) and let \(B\) be the point \((1, 5)\). Find the coordinates of the point \(P\) which divides the interval \(AB\) externally in the ratio 1 : 2.

(e) Is \(x + 3\) a factor of \(x^3 - 5x + 12\)? Give reasons for your answer.

(f) Use the substitution \(u = 1 + x\) to evaluate

\[
15 \int_{-1}^{0} x\sqrt{1+x} \, dx.
\]
Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Let \( f(x) = 3x^2 + x \). Use the definition

\[
f'(a) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}
\]

to find the derivative of \( f(x) \) at the point \( x = a \).

(b) Find

(i) \[ \int \frac{e^x}{1 + e^x} \, dx \]  

(ii) \[ \int_0^\pi \cos^2 3x \, dx. \]

(c) The letters \( A, E, I, O, \) and \( U \) are vowels.

(i) How many arrangements of the letters in the word ALGEBRAIC are possible?  

(ii) How many arrangements of the letters in the word ALGEBRAIC are possible if the vowels must occupy the 2nd, 3rd, 5th and 8th positions?

(d) Find the term independent of \( x \) in the binomial expansion of

\[ \left(x^2 - \frac{1}{x}\right)^9. \]
Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) The function \( f(x) = \sin x + \cos x - x \) has a zero near \( x = 1.2 \).

Use one application of Newton’s method to find a second approximation to the zero. Write your answer correct to three significant figures.

(b) Two circles, \( C_1 \) and \( C_2 \), intersect at points \( A \) and \( B \). Circle \( C_1 \) passes through the centre \( O \) of circle \( C_2 \). The point \( P \) lies on circle \( C_2 \) so that the line \( PAT \) is tangent to circle \( C_1 \) at point \( A \). Let \( \angle APB = \theta \).

Copy or trace the diagram into your writing booklet.

(i) Find \( \angle AOB \) in terms of \( \theta \). Give a reason for your answer.

(ii) Explain why \( \angle TAB = 2\theta \).

(iii) Deduce that \( PA = BA \).

(c) (i) Starting from the identity \( \sin(\theta + 2\theta) = \sin \theta \cos 2\theta + \cos \theta \sin 2\theta \), and using the double angle formulae, prove the identity

\[
\sin 3\theta = 3\sin \theta - 4\sin^3 \theta.
\]

(ii) Hence solve the equation

\[
\sin 3\theta = 2\sin \theta \quad \text{for} \quad 0 \leq \theta \leq 2\pi.
\]
Question 4 (12 marks) Use a SEPARATE writing booklet.

(a) Solve \( \frac{3x}{x-2} \leq 1 \).  

(b) An aircraft flying horizontally at \( V \text{ m s}^{-1} \) releases a bomb that hits the ground 4000 m away, measured horizontally. The bomb hits the ground at an angle of 45° to the vertical.

Assume that, \( t \) seconds after release, the position of the bomb is given by

\[
x = Vt, \quad y = -5t^2.
\]

Find the speed \( V \) of the aircraft.

(c) A particle, whose displacement is \( x \), moves in simple harmonic motion.

Find \( x \) as a function of \( t \) if

\[
\ddot{x} = -4x
\]

and if \( x = 3 \) and \( \dot{x} = -6\sqrt{3} \) when \( t = 0 \).
Question 5 (12 marks) Use a SEPARATE writing booklet.

(a)

The sketch shows the graph of the curve \( y = f(x) \) where \( f(x) = 2\cos^{-1}\frac{x}{3} \). The area under the curve for \( 0 \leq x \leq 3 \) is shaded.

(i) Find the \( y \) intercept.

(ii) Determine the inverse function \( y = f^{-1}(x) \), and write down the domain \( D \) of this inverse function.

(iii) Calculate the area of the shaded region.

(b) By using the binomial expansion, show that

\[
(q + p)^n - (q - p)^n = 2 \binom{n}{1} q^{n-1} p + 2 \binom{n}{3} q^{n-3} p^3 + \cdots
\]

What is the last term in the expansion when \( n \) is odd? What is the last term in the expansion when \( n \) is even?

(c) A fair six-sided die is randomly tossed \( n \) times.

(i) Suppose \( 0 \leq r \leq n \). What is the probability that exactly \( r \) ‘sixes’ appear in the uppermost position?

(ii) By using the result of part (b), or otherwise, show that the probability that an odd number of ‘sixes’ appears is

\[
\frac{1}{2} \left[ 1 - \left( \frac{2}{3} \right)^n \right].
\]
Question 6 (12 marks) Use a SEPARATE writing booklet.

(a) Prove by induction that

\[ n^3 + (n + 1)^3 + (n + 2)^3 \]

is divisible by 9 for \( n = 1, 2, 3, \ldots \)

(b) Consider the variable point \( P(2at, at^2) \) on the parabola \( x^2 = 4ay \).

   (i) Prove that the equation of the normal at \( P \) is \( x + ty = at^3 + 2at \).

   (ii) Find the coordinates of the point \( Q \) on the parabola such that the normal at \( Q \) is perpendicular to the normal at \( P \).

   (iii) Show that the two normals of part (ii) intersect at the point \( R \), whose coordinates are

   \[ x = a \left( t - \frac{1}{t} \right), \quad y = a \left( t^2 + 1 + \frac{1}{t^2} \right). \]

   (iv) Find the equation in Cartesian form of the locus of the point \( R \) given in part (iii).
Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) A particle moves in a straight line so that its acceleration is given by

\[ \frac{dv}{dt} = x - 1 \]

where \( v \) is its velocity and \( x \) is its displacement from the origin.

Initially, the particle is at the origin and has velocity \( v = 1 \).

(i) Show that \( v^2 = (x - 1)^2 \).  

(ii) By finding an expression for \( \frac{dt}{dx} \), or otherwise, find \( x \) as a function of \( t \).

Question 7 continues on page 9
Consider the diagram, which shows a vertical tower $OT$ of height $h$ metres, a fixed point $A$, and a variable point $P$ that is constrained to move so that angle $AOP$ is $\frac{\pi}{3}$ radians. The angle of elevation of $T$ from $A$ is $\frac{\pi}{4}$ radians.

Let the angle of elevation of $T$ from $P$ be $\alpha$ radians and let angle $ATP$ be $\theta$ radians.

(i) By considering triangle $AOP$, show that
\[ AP^2 = h^2 + h^2 \cot^2 \alpha - h^2 \cot \alpha. \]

(ii) By finding a second expression for $AP^2$, deduce that
\[ \cos \theta = \frac{1}{\sqrt{2}} \sin \alpha + \frac{1}{2\sqrt{2}} \cos \alpha. \]

(iii) Sketch a graph of $\theta$ for $0 < \alpha < \frac{\pi}{2}$, identifying and classifying any turning points. Discuss the behaviour of $\theta$ as $\alpha \to 0$ and as $\alpha \to \frac{\pi}{2}$.

End of paper
STANDARD INTEGRALS

\[ \int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0 \]

\[ \int \frac{1}{x} \, dx = \ln x, \quad x > 0 \]

\[ \int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0 \]

\[ \int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0 \]

\[ \int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0 \]

\[ \int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0 \]

\[ \int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0 \]

\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0 \]

\[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a \]

\[ \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0 \]

\[ \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left( x + \sqrt{x^2 + a^2} \right) \]

NOTE: \( \ln x = \log_e x, \quad x > 0 \)