General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question

Total marks – 84

- Attempt Questions 1–7
- All questions are of equal value
Question 1 (12 marks) Use a SEPARATE writing booklet.

(a) Evaluate \( \lim_{x \to 0} \frac{\sin 3x}{x} \). 

(b) Find \( \frac{d}{dx}(3x^2 \ln x) \) for \( x > 0 \).

(c) Use the table of standard integrals to evaluate \( \int_{0}^{\frac{\pi}{4}} \sec 2x \tan 2x \, dx \).

(d) State the domain and range of the function \( f(x) = 3 \sin^{-1} \left( \frac{x}{2} \right) \).

(e) The variable point \( (3t, 2t^2) \) lies on a parabola. Find the Cartesian equation for this parabola.

(f) Use the substitution \( u = 1 - x^2 \) to evaluate \( \int_{0}^{3} \frac{2x}{2 \left(1 - x^2 \right)^2} \, dx \).
Question 2 (12 marks) Use a SEPARATE writing booklet.

(a) Solve $2^x = 3$.
Express your answer correct to two decimal places.

(b) Find the general solution to $2 \cos x = \sqrt{3}$.
Express your answer in terms of $\pi$.

(c) Suppose $x^3 - 2x^2 + a \equiv (x + 2)Q(x) + 3$ where $Q(x)$ is a polynomial.
Find the value of $a$.

(d) Evaluate $2 \int_0^{\frac{\pi}{4}} \sin^2 4x \, dx$.

(e) In the diagram the points $A$, $B$ and $C$ lie on the circle and $CB$ produced meets
the tangent from $A$ at the point $T$. The bisector of the angle $ATC$ intersects $AB$
and $AC$ at $X$ and $Y$ respectively. Let $\angle TAB = \beta$.
Copy or trace the diagram into your writing booklet.

(i) Explain why $\angle ACB = \beta$.

(ii) Hence prove that triangle $AXY$ is isosceles.
Question 3 (12 marks) Use a SEPARATE writing booklet.

(a) Seven people are to be seated at a round table.

(i) How many seating arrangements are possible?  1

(ii) Two people, Kevin and Jill, refuse to sit next to each other. How many seating arrangements are then possible?  2

(b) (i) Show that \( f(x) = e^x - 3x^2 \) has a root between \( x = 3.7 \) and \( x = 3.8 \).  1

(ii) Starting with \( x = 3.8 \), use one application of Newton’s method to find a better approximation for this root. Write your answer correct to three significant figures.  3

(c) A household iron is cooling in a room of constant temperature 22°C. At time \( t \) minutes its temperature \( T \) decreases according to the equation

\[
\frac{dT}{dt} = -k(T - 22) \quad \text{where} \quad k \text{ is a positive constant.}
\]

The initial temperature of the iron is 80°C and it cools to 60°C after 10 minutes.

(i) Verify that \( T = 22 + Ae^{-kt} \) is a solution of this equation, where \( A \) is a constant.  1

(ii) Find the values of \( A \) and \( k \).  2

(iii) How long will it take for the temperature of the iron to cool to 30°C? Give your answer to the nearest minute.  2
(a) Lyndal hits the target on average 2 out of every 3 shots in archery competitions. During a competition she has 10 shots at the target.

(i) What is the probability that Lyndal hits the target exactly 9 times? Leave your answer in unsimplified form.

(ii) What is the probability that Lyndal hits the target fewer than 9 times? Leave your answer in unsimplified form.

(b) The polynomial $P(x) = x^3 - 2x^2 + kx + 24$ has roots $\alpha, \beta, \gamma$.

(i) Find the value of $\alpha + \beta + \gamma$.

(ii) Find the value of $\alpha \beta \gamma$.

(iii) It is known that two of the roots are equal in magnitude but opposite in sign. Find the third root and hence find the value of $k$.

(c) A particle, whose displacement is $x$, moves in simple harmonic motion such that $\ddot{x} = -16x$. At time $t = 0$, $x = 1$ and $\dot{x} = 4$.

(i) Show that, for all positions of the particle,

$$|\dot{x}| = 4\sqrt{2 - x^2}.$$ 

(ii) What is the particle’s greatest displacement?

(iii) Find $x$ as a function of $t$. You may assume the general form for $x$. 
Question 5 (12 marks) Use a SEPARATE writing booklet.

(a) Use the principle of mathematical induction to show that

\[2 \times 1! + 5 \times 2! + 10 \times 3! + \ldots + (n^2 + 1)n! = n(n + 1)!\]

for all positive integers \(n\).

(b) The diagram shows a conical drinking cup of height 12 cm and radius 4 cm. The cup is being filled with water at the rate of 3 cm\(^3\) per second. The height of water at time \(t\) seconds is \(h\) cm and the radius of the water’s surface is \(r\) cm.

(i) Show that \(r = \frac{1}{3}h\).

(ii) Find the rate at which the height is increasing when the height of water is 9 cm. (Volume of cone = \(\frac{1}{3} \pi r^2 h\).)

(c) Consider the function

\[f(x) = 2 \sin^{-1} \sqrt{x} - \sin^{-1}(2x - 1)\] for \(0 \leq x \leq 1\).

(i) Show that \(f'(x) = 0\) for \(0 < x < 1\).

(ii) Sketch the graph of \(y = f(x)\).
Question 6 (12 marks) Use a SEPARATE writing booklet.

(a)

An angler casts a fishing line so that the sinker is projected with a speed $V$ m s$^{-1}$ from a point 5 metres above a flat sea. The angle of projection to the horizontal is $\theta$, as shown.

Assume that the equations of motion of the sinker are

\[ \ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -10, \]

referred to the coordinate axes shown.

(i) Let $(x, y)$ be the position of the sinker at time $t$ seconds after the cast, and before the sinker hits the water.

It is known that $x = Vt \cos \theta$.

Show that \[ y = Vt \sin \theta - 5t^2 + 5. \]

(ii) Suppose the sinker hits the sea 60 metres away as shown in the diagram.

Find the value of $V$ if $\theta = \tan^{-1} \frac{3}{4}$.

(iii) For the cast described in part (ii), find the maximum height above sea level that the sinker achieved.

Question 6 continues on page 9
Question 6 (continued)

(b) Let \( n \) be a positive integer.

(i) By considering the graph of \( y = \frac{1}{x} \) show that

\[
\frac{1}{n+1} < \int_{n}^{n+1} \frac{dx}{x} < \frac{1}{n}. 
\]

(ii) Hence deduce that

\[
\left(1 + \frac{1}{n}\right)^n < e < \left(1 + \frac{1}{n}\right)^{n+1}. 
\]

End of Question 6

Please turn over
Question 7 (12 marks) Use a SEPARATE writing booklet.

(a) Let \( g(x) = e^x + \frac{1}{e^x} \) for all real values of \( x \) and let \( f(x) = e^x + \frac{1}{e^x} \) for \( x \leq 0 \).

(i) Sketch the graph \( y = g(x) \) and explain why \( g(x) \) does not have an inverse function.  

(ii) On a separate diagram, sketch the graph of the inverse function \( y = f^{-1}(x) \).  

(iii) Find an expression for \( y = f^{-1}(x) \) in terms of \( x \).  

(b) The coefficient of \( x^k \) in \((1 + x)^n\), where \( n \) is a positive integer, is denoted by \( c_k \) (so \( c_k = \binom{n}{k} \)).

(i) Show that \( c_0 + 2c_1 + 3c_2 + \ldots + (n+1)c_n = (n+2)2^{n-1} \).  

(ii) Find the sum \( \frac{c_0}{1.2} - \frac{c_1}{2.3} + \frac{c_2}{3.4} - \ldots + (-1)^n \frac{c_n}{(n+1)(n+2)} \).  

Write your answer as a simple expression in terms of \( n \).  

End of paper
STANDARD INTEGRALS

\[
\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0
\]

\[
\int \frac{1}{x} \, dx = \ln x, \quad x > 0
\]

\[
\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0
\]

\[
\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0
\]

\[
\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0
\]

\[
\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0
\]

\[
\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0
\]

\[
\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0
\]

\[
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a
\]

\[
\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0
\]

\[
\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left( x + \sqrt{x^2 + a^2} \right)
\]

NOTE: \( \ln x = \log_e x, \quad x > 0 \)