Mathematics Extension 2

General Instructions
• Reading time – 5 minutes
• Working time – 3 hours
• Write using black or blue pen
• Board-approved calculators may be used
• A table of standard integrals is provided at the back of this paper
• All necessary working should be shown in every question

Total marks – 120
• Attempt Questions 1–8
• All questions are of equal value
Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Evaluate \( \int_{0}^{1} \frac{e^x}{(1 + e^x)^2} \, dx \).  

(b) Use integration by parts to find \( \int x^3 \log_e x \, dx \).

(c) By completing the square and using the table of standard integrals, find \( \int \frac{dx}{\sqrt{x^2 - 2x + 5}} \).

(d) (i) Find the real numbers \( a \) and \( b \) such that
\[
\frac{5x^2 - 3x + 13}{(x - 1)(x^2 + 4)} \equiv \frac{a}{x - 1} + \frac{bx - 1}{x^2 + 4}.
\]

(ii) Find \( \int \frac{5x^2 - 3x + 13}{(x - 1)(x^2 + 4)} \, dx \).

(e) Use the substitution \( x = 3 \sin \theta \) to evaluate
\[
\int_{0}^{\sqrt{2}/3} \frac{dx}{(9 - x^2)^{3/2}}.
\]
Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let \( z = 2 + i \) and \( w = 1 - i \).

Find, in the form \( x + iy \),

(i) \( z\bar{w} \)  

(ii) \( \frac{4}{z} \).  

(b) Let \( \alpha = -1 + i \).

(i) Express \( \alpha \) in modulus-argument form. 

(ii) Show that \( \alpha \) is a root of the equation \( z^4 + 4 = 0 \).

(iii) Hence, or otherwise, find a real quadratic factor of the polynomial \( z^4 + 4 \).

(c) Sketch the region in the complex plane where the inequalities

\[ |z - 1 - i| < 2 \quad \text{and} \quad 0 < \arg(z - 1 - i) < \frac{\pi}{4} \]

hold simultaneously.

(d) By applying de Moivre’s theorem and by also expanding \((\cos \theta + i \sin \theta)^5\), express \( \cos 5\theta \) as a polynomial in \( \cos \theta \).

(e) Suppose that the complex number \( z \) lies on the unit circle, and \( 0 \leq \arg(z) \leq \frac{\pi}{2} \).

Prove that \( 2 \arg(z + 1) = \arg(z) \).
Question 3 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the graph of \( y = f(x) \).

\[
\begin{align*}
\text{Draw separate one-third page sketches of the graphs of the following:} \\
(i) \quad y = \frac{1}{f(x)} & \quad 2 \\
(ii) \quad y = f(x) + |f(x)| & \quad 2 \\
(iii) \quad y = (f(x))^2 & \quad 1 \\
(iv) \quad y = e^{f(x)} & \quad 2
\end{align*}
\]

(b) Find the eccentricity, foci and the equations of the directrices of the ellipse \( \frac{x^2}{9} + \frac{y^2}{4} = 1 \).

Question 3 continues on page 5
Question 3 (continued)

(c) The region bounded by the curve \( y = (x - 1)(3 - x) \) and the \( x \)-axis is rotated about the line \( x = 3 \) to form a solid. When the region is rotated, the horizontal line segment \( \ell \) at height \( y \) sweeps out an annulus.

(i) Show that the area of the annulus at height \( y \) is given by \( 4\pi \sqrt{1 - y} \).

(ii) Find the volume of the solid.

End of Question 3

Please turn over
Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) A particle $P$ of mass $m$ moves with constant angular velocity $\omega$ on a circle of radius $r$. Its position at time $t$ is given by:

\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta,
\end{align*}
\]

where $\theta = \omega t$.

(i) Show that there is an inward radial force of magnitude $mr\omega^2$ acting on $P$.  

(ii) A telecommunications satellite, of mass $m$, orbits Earth with constant angular velocity $\omega$ at a distance $r$ from the centre of Earth. The gravitational force exerted by Earth on the satellite is $\frac{Am}{r^2}$, where $A$ is a constant. By considering all other forces on the satellite to be negligible, show that

\[
r = \sqrt[3]{\frac{A}{\omega^2}}.
\]

(b) (i) Derive the equation of the tangent to the hyperbola

\[
\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1
\]

at the point $P (a \sec \theta, b \tan \theta)$.

(ii) Show that the tangent intersects the asymptotes of the hyperbola at the points

\[
A\left(\frac{a \cos \theta}{1 - \sin \theta}, \frac{b \cos \theta}{1 - \sin \theta}\right) \quad \text{and} \quad B\left(\frac{a \cos \theta}{1 + \sin \theta}, \frac{-b \cos \theta}{1 + \sin \theta}\right)
\]

(iii) Prove that the area of the triangle $OAB$ is $ab$.  

Question 4 continues on page 7
(c) A hall has $n$ doors. Suppose that $n$ people each choose any door at random to enter the hall.

(i) In how many ways can this be done?  

(ii) What is the probability that at least one door will not be chosen by any of the people?

End of Question 4

Please turn over
(a) Let $\alpha$, $\beta$ and $\gamma$ be the three roots of $x^3 + px + q = 0$, and define $s_n$ by

$$s_n = \alpha^n + \beta^n + \gamma^n \quad \text{for} \quad n = 1, 2, 3, \ldots$$

(i) Explain why $s_1 = 0$, and show that $s_2 = -2p$ and $s_3 = -3q$.

(ii) Prove that for $n > 3$

$$s_n = -ps_{n-2} - qs_{n-3}.$$ 

(iii) Deduce that

$$\frac{\alpha^5 + \beta^5 + \gamma^5}{5} = \left(\frac{\alpha^2 + \beta^2 + \gamma^2}{2}\right) \left(\frac{\alpha^3 + \beta^3 + \gamma^3}{3}\right).$$

Question 5 continues on page 9
Question 5 (continued)

(b) A particle of mass $m$ is thrown from the top, $O$, of a very tall building with an initial velocity $u$ at an angle $\alpha$ to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both the horizontal and vertical directions. The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -k\dot{x} \quad \text{and} \quad \ddot{y} = -k\dot{y} - g,$$

where $k$ is a constant and the acceleration due to gravity is $g$. (You are NOT required to show these.)

(i) Derive the result $\dot{x} = ue^{-kt} \cos \alpha$ from the relevant equation of motion.

(ii) Verify that $\dot{y} = \frac{1}{k} \left( (ku \sin \alpha + g)e^{-kt} - g \right)$ satisfies the appropriate equation of motion and initial condition.

(iii) Find the value of $t$ when the particle reaches its maximum height.

(iv) What is the limiting value of the horizontal displacement of the particle?

End of Question 5

Please turn over
Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove the identity \( \cos(a + b)x + \cos(a - b)x = 2\cos ax \cos bx \).  
\[ \int \cos 3x \cos 2x \, dx. \]

(ii) Hence find \( \int \cos 3x \cos 2x \, dx. \)

(b) A sequence \( s_n \) is defined by \( s_1 = 1, s_2 = 2 \) and, for \( n > 2 \),
\[ s_n = s_{n-1} + (n-1)s_{n-2}. \]

(i) Find \( s_3 \) and \( s_4. \)

(ii) Prove that \( \sqrt{x} + x \geq \sqrt{x(x + 1)} \) for all real numbers \( x \geq 0 \).

(iii) Prove by induction that \( s_n \geq \sqrt{n!} \) for all integers \( n \geq 1. \)

(c) (i) Let \( x \) and \( y \) be real numbers such that \( x \geq 0 \) and \( y \geq 0 \).

\[ \text{Prove that } \frac{x + y}{2} \geq \sqrt{xy}. \]

(ii) Suppose that \( a, b, c \) are real numbers.

\[ \text{Prove that } a^4 + b^4 + c^4 \geq a^2b^2 + a^2c^2 + b^2c^2. \]

(iii) Show that \( a^2b^2 + a^2c^2 + b^2c^2 \geq a^2bc + b^2ac + c^2ab. \)

(iv) Deduce that if \( a + b + c = d \), then \( a^4 + b^4 + c^4 \geq abc. \)
(a) The region bounded by $0 \leq x \leq \sqrt{3}, \ 0 \leq y \leq x\left(3 - x^2\right)$ is rotated about the $y$-axis to form a solid.

Use the method of cylindrical shells to find the volume of the solid.

(b) Two circles $C_1$ and $C_2$ intersect at the points $A$ and $B$. Let $P$ be a point on $AB$ produced and let $PS$ and $PT$ be tangents to $C_1$ and $C_2$ respectively, as shown in the diagram.

Copy or trace the diagram into your writing booklet.

(i) Prove that $\triangle ASP \parallel \triangle SBP$.  

(ii) Hence, prove that $SP^2 = AP \times BP$ and deduce that $PT = PS$.  

(iii) The perpendicular to $SP$ drawn from $S$ meets the bisector of $\angle SPT$ at $D$. Prove that $DT$ passes through the centre of $C_2$.  

Question 7 continues on page 13
Question 7 (continued)

(c) Suppose that $\alpha$ is a real number with $0 < \alpha < \pi$.

Let $P_n = \cos\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{4}\right) \cos\left(\frac{\alpha}{8}\right) \ldots \cos\left(\frac{\alpha}{2^n}\right)$.

(i) Show that $P_n \sin\left(\frac{\alpha}{2^n}\right) = \frac{1}{2} P_{n-1} \sin\left(\frac{\alpha}{2^{n-1}}\right)$.

(ii) Deduce that $P_n = \frac{\sin\alpha}{2^n \sin\left(\frac{\alpha}{2^n}\right)}$.

(iii) Given that $\sin x < x$ for $x > 0$, show that

$$\frac{\sin\alpha}{\cos\left(\frac{\alpha}{2}\right) \cos\left(\frac{\alpha}{4}\right) \cos\left(\frac{\alpha}{8}\right) \ldots \cos\left(\frac{\alpha}{2^n}\right)} < \alpha.$$ 

End of Question 7

Please turn over
Question 8 (15 marks) Use a SEPARATE writing booklet.

(a) Suppose that $\omega^3 = 1$, $\omega \neq 1$, and $k$ is a positive integer.

(i) Find the two possible values of $1 + \omega^k + \omega^{2k}$.  

(ii) Use the binomial theorem to expand $(1 + \omega)^n$ and $(1 + \omega^2)^n$, where $n$ is a positive integer.  

(iii) Let $\ell$ be the largest integer such that $3\ell \leq n$.

Deduce that
\[
\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots + \binom{n}{3\ell} = \frac{1}{3} \left( 2^n + (1 + \omega)^n + (1 + \omega^2)^n \right).
\]

(iv) If $n$ is a multiple of 6, prove that
\[
\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \ldots + \binom{n}{n} = \frac{1}{3} \left( 2^n + 2 \right).
\]
Question 8 (continued)

(b) Suppose that $\pi$ could be written in the form $\frac{p}{q}$, where $p$ and $q$ are positive integers.

Define the family of integrals $I_n$ for $n = 0, 1, 2, \ldots$ by

$$I_n = \frac{q^{2n}}{n!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\pi^2}{4} - x^2 \right)^n \cos x \, dx.$$ 

You are given that $I_0 = 2$ and $I_1 = 4q^2$. (Do NOT prove this.)

(i) Use integration by parts twice to show that, for $n \geq 2$,

$$I_n = \frac{2q^{2n}}{(n-1)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( \frac{\pi^2}{4} - x^2 \right)^{n-1} \cos x \, dx - \frac{4q^{2n}}{(n-2)!} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x^2 \left( \frac{\pi^2}{4} - x^2 \right)^{n-2} \cos x \, dx.$$ 

(ii) By writing $x^2$ as $\frac{\pi^2}{4} - \left( \frac{\pi^2}{4} - x^2 \right)$ where appropriate, deduce that $I_n = (4n-2)q^2 I_{n-1} - p^2 q^2 I_{n-2}$, for $n \geq 2$.

(iii) Explain briefly why $I_n$ is an integer for $n = 0, 1, 2, \ldots$ 

(iv) Prove that

$$0 < I_n < \frac{p}{q} \left( \frac{p}{2} \right)^{2n} \frac{1}{n!} \quad \text{for} \quad n = 0, 1, 2, \ldots$$

(v) Given that $\frac{p}{q} \left( \frac{p}{2} \right)^{2n} \frac{1}{n!} < 1$, if $n$ is sufficiently large, deduce that $\pi$ is irrational.

End of paper
STANDARD INTEGRALS

\[ \int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0 \]

\[ \int \frac{1}{x} \, dx = \ln x, \quad x > 0 \]

\[ \int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0 \]

\[ \int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0 \]

\[ \int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0 \]

\[ \int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0 \]

\[ \int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0 \]

\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0 \]

\[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a \]

\[ \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0 \]

\[ \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left( x + \sqrt{x^2 + a^2} \right) \]

NOTE :  \( \ln x = \log_e x, \quad x > 0 \)