General Instructions
• Reading time – 5 minutes
• Working time – 3 hours
• Write using black or blue pen
• Board-approved calculators may be used
• A table of standard integrals is provided at the back of this paper
• All necessary working should be shown in every question

Total marks – 120
• Attempt Questions 1–8
• All questions are of equal value
Question 1 (15 marks) Use a SEPARATE writing booklet.

(a) Find \( \int \frac{x}{\sqrt{1 + 3x^2}} \, dx \). \hspace{1cm} 2

(b) Evaluate \( \int_{0}^{\frac{\pi}{4}} \tan x \, dx \). \hspace{1cm} 3

(c) Find \( \int \frac{1}{x(x^2 + 1)} \, dx \). \hspace{1cm} 3

(d) Using the substitution \( t = \tan \frac{x}{2} \), or otherwise, evaluate \( \int_{0}^{\frac{\pi}{2}} \frac{dx}{1 + \sin x} \). \hspace{1cm} 4

(e) Find \( \int \frac{dx}{1 + \sqrt{x}} \). \hspace{1cm} 3
Question 2 (15 marks) Use a SEPARATE writing booklet.

(a) Let \( z = 5 - i \).

(i) Find \( z^2 \) in the form \( x + iy \).  

(ii) Find \( z + 2 \overline{z} \) in the form \( x + iy \).  

(iii) Find \( \frac{i}{z} \) in the form \( x + iy \).  

(b) (i) Express \( -\sqrt{3} - i \) in modulus–argument form.  

(ii) Show that \( \left(-\sqrt{3} - i\right)^6 \) is a real number.  

(c) Sketch the region in the complex plane where the inequalities \( 1 \leq |z| \leq 2 \) and \( 0 \leq z + \overline{z} \leq 3 \) hold simultaneously.

Question 2 continues on page 5
Question 2 (continued)

(d) Let \( z = \cos \theta + i \sin \theta \) where \( 0 < \theta < \frac{\pi}{2} \).

On the Argand diagram the point \( A \) represents \( z \), the point \( B \) represents \( z^2 \) and the point \( C \) represents \( z + z^2 \).

Copy or trace the diagram into your writing booklet.

(i) Explain why the parallelogram \( OACB \) is a rhombus. 1

(ii) Show that \( \arg(z + z^2) = \frac{3\theta}{2} \). 1

(iii) Show that \( |z + z^2| = 2 \cos \frac{\theta}{2} \). 2

(iv) By considering the real part of \( z + z^2 \), or otherwise, deduce that \( \cos \theta + \cos 2\theta = 2 \cos \frac{\theta}{2} \cos \frac{3\theta}{2} \). 1

End of Question 2
Question 3 (15 marks) Use a SEPARATE writing booklet.

(a)  
(i) Sketch the graph \( y = x^2 + 4x \).  
(ii) Sketch the graph \( y = \frac{1}{x^2 + 4x} \).

(b) The region shaded in the diagram is bounded by the \( x \)-axis and the curve \( y = 2x - x^2 \).

The shaded region is rotated about the line \( x = 4 \).

Find the volume generated.

(c) Two identical biased coins are each more likely to land showing heads than showing tails.

The two coins are tossed together, and the outcome is recorded. After a large number of trials it is observed that the probability that the two coins land showing a head and a tail is 0.48.

What is the probability that both coins land showing heads?

Question 3 continues on page 7
(d) The diagram shows the rectangular hyperbola $xy = c^2$, with $c > 0$.

![Diagram of rectangular hyperbola with points A(c, c), R(ct, c/t), and Q(-ct, -c/t)].

The points $A(c, c)$, $R\left( ct, \frac{c}{t} \right)$, and $Q\left( -ct, -\frac{c}{t} \right)$ are points on the hyperbola, with $t \neq \pm 1$.

(i) The line $\ell_1$ is the line through $R$ perpendicular to $QA$. Show that the equation of $\ell_1$ is

$$y = -tx + c\left( t^2 + \frac{1}{t} \right).$$

(ii) The line $\ell_2$ is the line through $Q$ perpendicular to $RA$. Write down the equation of $\ell_2$.

(iii) Let $P$ be the point of intersection of the lines $\ell_1$ and $\ell_2$. Show that $P$ is the point $\left( \frac{c}{t^2}, ct^2 \right)$.

(iv) Give a geometric description of the locus of $P$.

End of Question 3
Question 4 (15 marks) Use a SEPARATE writing booklet.

(a) (i) A curve is defined implicitly by $\sqrt{x} + \sqrt{y} = 1$.

Use implicit differentiation to find $\frac{dy}{dx}$.

(ii) Sketch the curve $\sqrt{x} + \sqrt{y} = 1$.

(iii) Sketch the curve $\sqrt{|x|} + \sqrt{|y|} = 1$.

(b) A bend in a highway is part of a circle of radius $r$, centre $O$. Around the bend the highway is banked at an angle $\alpha$ to the horizontal.

A car is travelling around the bend at a constant speed $v$. Assume that the car is represented by a point $P$ of mass $m$. The forces acting on the car are a lateral force $F$, the gravitational force $mg$ and a normal reaction $N$ to the road, as shown in the diagram.

(i) By resolving forces, show that $F = mg \sin \alpha - \frac{mv^2}{r} \cos \alpha$.

(ii) Find an expression for $v$ such that the lateral force $F$ is zero.
Question 4 (continued)

(c) Let $k$ be a real number, $k \geq 4$.

Show that, for every positive real number $b$, there is a positive real number $a$ such that

$$\frac{1}{a} + \frac{1}{b} = \frac{k}{a + b}.$$

(d) A group of 12 people is to be divided into discussion groups.

(i) In how many ways can the discussion groups be formed if there are 8 people in one group, and 4 people in another?

(ii) In how many ways can the discussion groups be formed if there are 3 groups containing 4 people each?

End of Question 4
Question 5 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows two circles, \( C_1 \) and \( C_2 \), centred at the origin with radii \( a \) and \( b \), where \( a > b \).

The point \( A \) lies on \( C_1 \) and has coordinates \((a \cos \theta, a \sin \theta)\).

The point \( B \) is the intersection of \( OA \) and \( C_2 \).

The point \( P \) is the intersection of the horizontal line through \( B \) and the vertical line through \( A \).

(i) Write down the coordinates of \( B \).

(ii) Show that \( P \) lies on the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \).

(iii) Find the equation of the tangent to the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) at \( P \).

(iv) Assume that \( A \) is not on the \( y \)-axis.

Show that the tangent to the circle \( C_1 \) at \( A \), and the tangent to the ellipse \( \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \) at \( P \), intersect at a point on the \( x \)-axis.

Question 5 continues on page 11
Question 5 (continued)

(b) Show that

\[ \int \frac{dy}{y(1-y)} = \ln\left( \frac{y}{1-y} \right) + c \]

for some constant \( c \), where \( 0 < y < 1 \).

(c) A TV channel has estimated that if it spends \( x \) on advertising a particular program it will attract a proportion \( y(x) \) of the potential audience for the program, where

\[ \frac{dy}{dx} = ay(1-y) \]

and \( a > 0 \) is a given constant.

(i) Explain why \( \frac{dy}{dx} \) has its maximum value when \( y = \frac{1}{2} \).

(ii) Using part (b), or otherwise, deduce that

\[ y(x) = \frac{1}{ke^{-ax} + 1} \]

for some constant \( k > 0 \).

(iii) The TV channel knows that if it spends no money on advertising the program then the audience will be one-tenth of the potential audience.

Find the value of the constant \( k \) referred to in part (c) (ii).

(iv) What feature of the graph \( y = \frac{1}{ke^{-ax} + 1} \) is determined by the result in part (c) (i)?

(v) Sketch the graph \( y = \frac{1}{ke^{-ax} + 1} \).

End of Question 5
Question 6 (15 marks) Use a SEPARATE writing booklet.

(a) The diagram shows the frustum of a right square pyramid. (A frustum of a pyramid is a pyramid with its top cut off.)

The height of the frustum is \( h \) m. Its base is a square of side \( a \) m, and its top is a square of side \( b \) m (with \( a > b > 0 \)).

A horizontal cross-section of the frustum, taken at height \( x \) m, is a square of side \( s \) m, shown shaded in the diagram.

(i) Show that \( s = a - \frac{(a - b)}{h} x \).  

(ii) Find the volume of the frustum.  

(b) A sequence \( a_n \) is defined by

\[
a_n = 2a_{n-1} + a_{n-2},
\]

for \( n \geq 2 \), with \( a_0 = a_1 = 2 \).

Use mathematical induction to prove that

\[
a_n = \left(1 + \sqrt{2}\right)^n + \left(1 - \sqrt{2}\right)^n \quad \text{for all} \quad n \geq 0.
\]

Question 6 continues on page 13
Question 6 (continued)

(c)  
(i) Expand \((\cos \theta + i \sin \theta)^5\) using the binomial theorem.  

(ii) Expand \((\cos \theta + i \sin \theta)^5\) using de Moivre’s theorem, and hence show that  
\[\binom{5}{1} \cos^4 \theta \sin \theta + \binom{5}{2} \cos^2 \theta \sin^2 \theta + \binom{5}{3} \cos \theta \sin^3 \theta + \binom{5}{4} (\sin \theta)^5 + \binom{5}{5} i \sin^5 \theta = 16 \sin^5 \theta - 20 \sin^3 \theta + 5 \sin \theta.\]

(iii) Deduce that  
\[x = \sin \left(\frac{\pi}{10}\right)\]  

is one of the solutions to  
\[16x^5 - 20x^3 + 5x - 1 = 0.\]

(iv) Find the polynomial \(p(x)\) such that \((x - 1) p(x) = 16x^5 - 20x^3 + 5x - 1\).  

(v) Find the value of \(a\) such that \(p(x) = (4x^2 + ax - 1)^2\).  

(vi) Hence find an exact value for  
\[\sin \left(\frac{\pi}{10}\right).\]

End of Question 6
Question 7 (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram $ABCD$ is a cyclic quadrilateral. The point $K$ is on $AC$ such that $\angle ADK = \angle CDB$, and hence $\triangle ADK$ is similar to $\triangle BDC$.

Copy or trace the diagram into your writing booklet.

(i) Show that $\triangle ADB$ is similar to $\triangle KDC$.  

(ii) Using the fact that $AC = AK + KC$, show that $BD \times AC = AD \times BC + AB \times DC$.  

(iii) A regular pentagon of side length 1 is inscribed in a circle, as shown in the diagram.

Let $x$ be the length of a chord in the pentagon.

Use the result in part (ii) to show that $x = \frac{1 + \sqrt{5}}{2}$.

Question 7 continues on page 15
Question 7 (continued)

(b) The graphs of $y = 3x - 1$ and $y = 2^x$ intersect at $(1, 2)$ and at $(3, 8)$.

Using these graphs, or otherwise, show that $2^x \geq 3x - 1$ for $x \geq 3$.

(c) Let $P(x) = (n - 1)x^n - nx^{n-1} + 1$, where $n$ is an odd integer, $n \geq 3$.

(i) Show that $P(x)$ has exactly two stationary points.

(ii) Show that $P(x)$ has a double zero at $x = 1$.

(iii) Use the graph $y = P(x)$ to explain why $P(x)$ has exactly one real zero other than 1.

(iv) Let $\alpha$ be the real zero of $P(x)$ other than 1.

Using part (b), or otherwise, show that $-1 < \alpha \leq -\frac{1}{2}$.

(v) Deduce that each of the zeros of $4x^5 - 5x^4 + 1$ has modulus less than or equal to 1.

End of Question 7
**Question 8** (15 marks) Use a SEPARATE writing booklet.

Let

\[ A_n = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx \quad \text{and} \quad B_n = \int_0^{\frac{\pi}{2}} x^2 \cos^{2n} x \, dx, \]

where \( n \) is an integer, \( n \geq 0 \). (Note that \( A_n > 0, \ B_n > 0 \).)

(a) Show that \( nA_n = \frac{2n-1}{2} A_{n-1} \) for \( n \geq 1 \).

(b) Using integration by parts on \( A_n \), or otherwise, show that

\[ A_n = 2n \int_0^{\frac{\pi}{2}} x \sin x \cos^{2n-1} x \, dx \quad \text{for} \quad n \geq 1. \]

(c) Use integration by parts on the integral in part (b) to show that

\[ \frac{A_n}{n^2} = \frac{(2n-1)}{n} B_{n-1} - 2B_n \quad \text{for} \quad n \geq 1. \]

(d) Use parts (a) and (c) to show that

\[ \frac{1}{n^2} = 2 \left( \frac{B_{n-1}}{A_{n-1}} - \frac{B_n}{A_n} \right) \quad \text{for} \quad n \geq 1. \]

(e) Show that \( \sum_{k=1}^{n} \frac{1}{k^2} = \frac{\pi^2}{6} - \frac{2B_n}{A_n} \).

(f) Use the fact that \( \sin x \geq \frac{2}{\pi} x \) for \( 0 \leq x \leq \frac{\pi}{2} \) to show that

\[ B_n \leq \int_0^{\frac{\pi}{2}} x^2 \left( 1 - \frac{4x^2}{\pi^2} \right)^n \, dx. \]

**Question 8 continues on page 17**
Question 8 (continued)

(g) Show that $\int_{0}^{\frac{\pi}{2}} x^2 \left(1 - \frac{4x^2}{\pi^2}\right)^n \, dx = \frac{\pi^2}{8(n+1)} \int_{0}^{\frac{\pi}{2}} \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} \, dx$.

(h) From parts (f) and (g) it follows that

$$B_n \leq \frac{\pi^2}{8(n+1)} \int_{0}^{\frac{\pi}{2}} \left(1 - \frac{4x^2}{\pi^2}\right)^{n+1} \, dx.$$ 

Use the substitution $x = \frac{\pi}{2} \sin t$ in this inequality to show that

$$B_n \leq \frac{\pi^3}{16(n+1)} \int_{0}^{\frac{\pi}{2}} \cos^{2n+3} t \, dt \leq \frac{\pi^3}{16(n+1)} A_n.$$ 

(i) Use part (e) to deduce that

$$\frac{\pi^2}{6} - \frac{\pi^3}{8(n+1)} \leq \sum_{k=1}^{n} \frac{1}{k^2} < \frac{\pi^2}{6}.$$ 

(j) What is $\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k^2}$? 

End of paper
STANDARD INTEGRALS

\[
\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0
\]

\[
\int \frac{1}{x} \, dx = \ln x, \quad x > 0
\]

\[
\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0
\]

\[
\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0
\]

\[
\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0
\]

\[
\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0
\]

\[
\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0
\]

\[
\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0
\]

\[
\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a
\]

\[
\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0
\]

\[
\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left( x + \sqrt{x^2 + a^2} \right)
\]

NOTE: \( \ln x = \log_e x, \quad x > 0 \)