BOARD OF STUDIES
NEW SOUTH WALES

2013
HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 2–7

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 8–15

60 marks

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 The polynomial $P(x) = x^3 - 4x^2 - 6x + k$ has a factor $x - 2$.

What is the value of k?

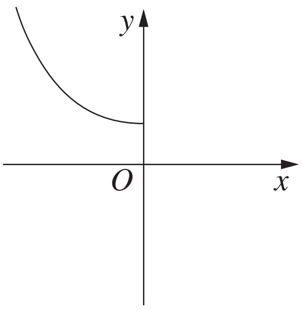
(A) 2

(B) 12

(C) 20

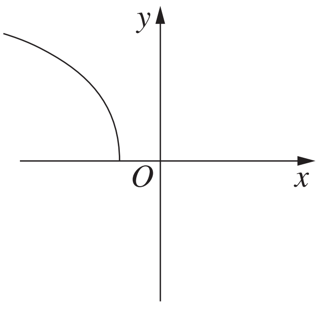
(D) 36

2 The diagram shows the graph $y = f(x)$.



Which diagram shows the graph $y = f^{-1}(x)$?

(A)



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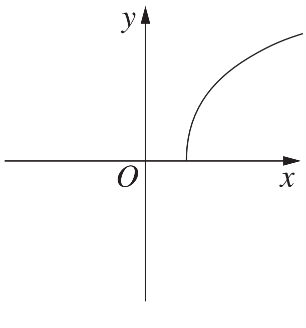

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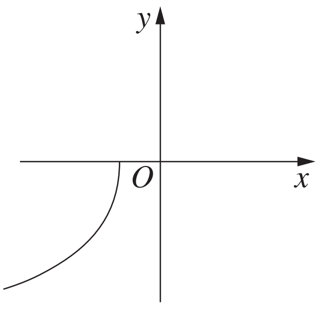
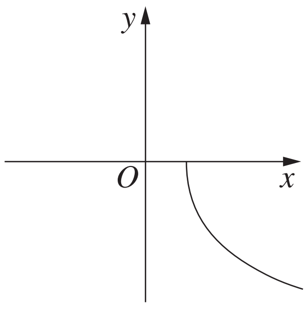
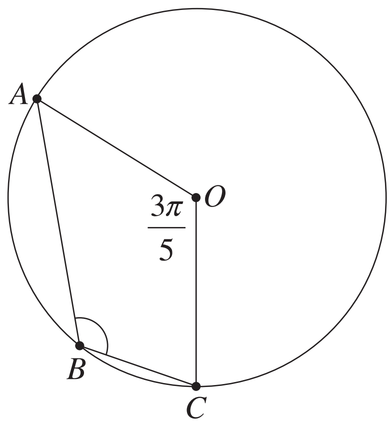


(D)



3 The points A, B and C lie on a circle with centre O, as shown in the diagram.

The size of $\angle AOC$ is $\dfrac{3\pi}{5}$ radians.



NOT TO SCALE

What is the size of $\angle ABC$ in radians?

(A) $\dfrac{3\pi}{10}$

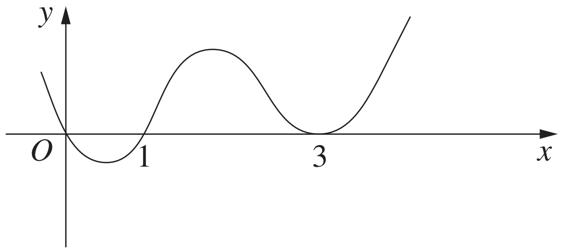
(B) $\dfrac{2\pi}{5}$

(C) $\dfrac{7\pi}{10}$

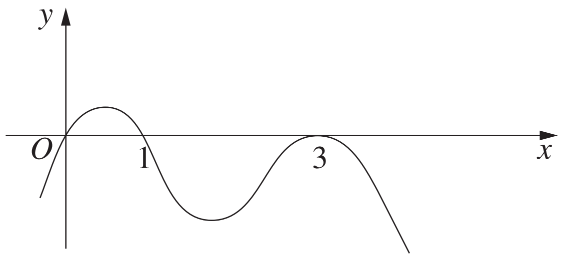
(D) $\dfrac{4\pi}{5}$

4 Which diagram best represents the graph $y = x(1-x)^3(3-x)^2$?

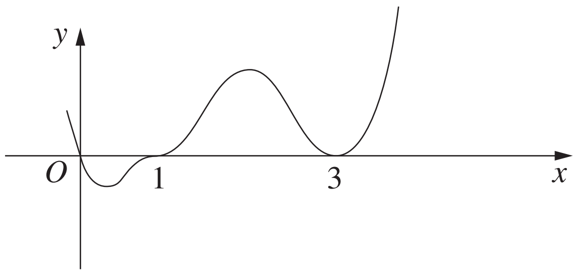
(A)



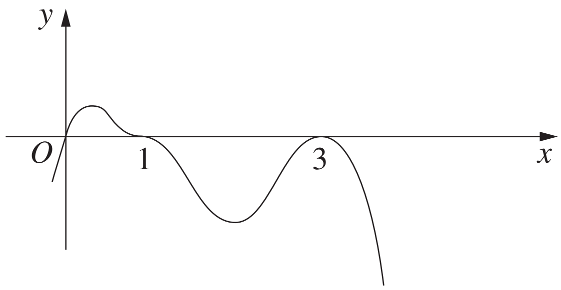
(B)



(C)



(D)



5 Which integral is obtained when the substitution $u = 1 + 2x$ is applied to $\int x\sqrt{1+2x}\,dx$?

(A) $\frac{1}{4}\int (u-1)\sqrt{u}\,du$

(B) $\frac{1}{2}\int (u-1)\sqrt{u}\,du$

(C) $\int (u-1)\sqrt{u}\,du$

(D) $2\int (u-1)\sqrt{u}\,du$

6 Let $|a| \le 1$. What is the general solution of $\sin 2x = a$?

(A) $x = n\pi + (-1)^n \frac{\sin^{-1} a}{2}$, n is an integer

(B) $x = \frac{n\pi + (-1)^n \sin^{-1} a}{2}$, n is an integer

(C) $x = 2n\pi \pm \frac{\sin^{-1} a}{2}$, n is an integer

(D) $x = \frac{2n\pi \pm \sin^{-1} a}{2}$, n is an integer

7 A family of eight is seated randomly around a circular table.

What is the probability that the two youngest members of the family sit together?

(A) $\dfrac{6!2!}{7!}$

(B) $\dfrac{6!}{7!2!}$

(C) $\dfrac{6!2!}{8!}$

(D) $\dfrac{6!}{8!2!}$

8 The angle θ satisfies $\sin\theta = \dfrac{5}{13}$ and $\dfrac{\pi}{2} < \theta < \pi$.

What is the value of $\sin 2\theta$?

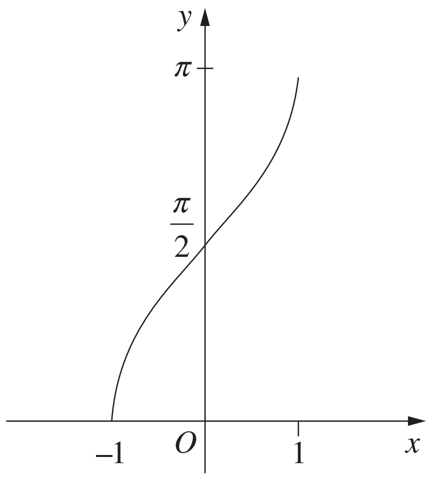
(A) $\dfrac{10}{13}$

(B) $-\dfrac{10}{13}$

(C) $\dfrac{120}{169}$

(D) $-\dfrac{120}{169}$

9 The diagram shows the graph of a function.



Which function does the graph represent?

(A) $y = \cos^{-1} x$

(B) $y = \dfrac{\pi}{2} + \sin^{-1} x$

(C) $y = -\cos^{-1} x$

(D) $y = -\dfrac{\pi}{2} - \sin^{-1} x$

10 Which inequality has the same solution as $|x + 2| + |x - 3| = 5$?

(A) $\dfrac{5}{3 - x} \geq 1$

(B) $\dfrac{1}{x - 3} - \dfrac{1}{x + 2} \leq 0$

(C) $x^2 - x - 6 \leq 0$

(D) $|2x - 1| \geq 5$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) The polynomial equation $2x^3 - 3x^2 - 11x + 7 = 0$ has roots α, β and γ. **1**

Find $\alpha\beta\gamma$.

(b) Find $\int \frac{1}{\sqrt{49 - 4x^2}}\,dx$. **2**

(c) An examination has 10 multiple-choice questions, each with 4 options. In each question, only one option is correct. For each question a student chooses one option at random. **2**

Write an expression for the probability that the student chooses the correct option for exactly 7 questions.

(d) Consider the function $f(x) = \frac{x}{4 - x^2}$.

(i) Show that $f'(x) > 0$ for all x in the domain of $f(x)$. **2**

(ii) Sketch the graph $y = f(x)$, showing all asymptotes. **2**

Question 11 continues on page 9

Question 11 (continued)

(e) Find $\lim_{x \to 0} \frac{\sin \frac{x}{2}}{3x}$. **1**

(f) Use the substitution $u = e^{3x}$ to evaluate $\int_0^{\frac{1}{3}} \frac{e^{3x}}{e^{6x}+1}\, dx$. **3**

(g) Differentiate $x^2 \sin^{-1} 5x$. **2**

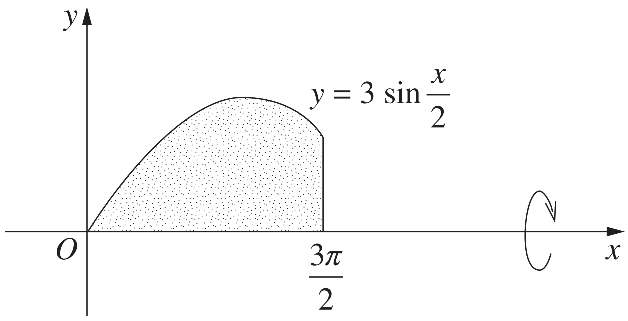
End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Write $\sqrt{3}\cos x - \sin x$ in the form $2\cos(x + \alpha)$, where $0 < \alpha < \frac{\pi}{2}$. **1**

(ii) Hence, or otherwise, solve $\sqrt{3}\cos x = 1 + \sin x$, where $0 < x < 2\pi$. **2**

(b) The region bounded by the graph $y = 3\sin\frac{x}{2}$ and the x-axis between $x = 0$ and $x = \frac{3\pi}{2}$ is rotated about the x-axis to form a solid. **3**



Find the exact volume of the solid.

(c) A cup of coffee with an initial temperature of 80°C is placed in a room with a constant temperature of 22°C. **3**

The temperature, T°C, of the coffee after t minutes is given by

$$T = A + Be^{-kt},$$

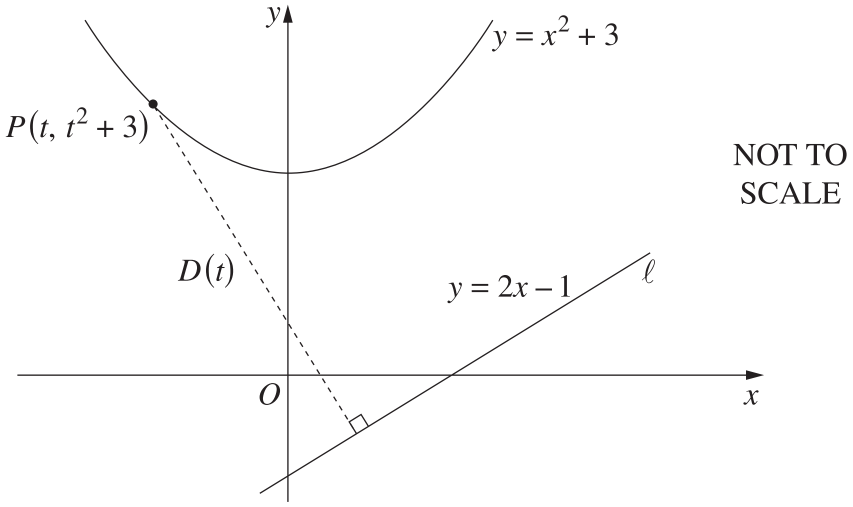
where A, B and k are positive constants. The temperature of the coffee drops to 60°C after 10 minutes.

How long does it take for the temperature of the coffee to drop to 40°C? Give your answer to the nearest minute.

Question 12 continues on page 11

Question 12 (continued)

(d) The point $P(t, t^2 + 3)$ lies on the curve $y = x^2 + 3$. The line ℓ has equation $y = 2x - 1$. The perpendicular distance from P to the line ℓ is $D(t)$.



(i) Show that $D(t) = \dfrac{t^2 - 2t + 4}{\sqrt{5}}$. **2**

(ii) Find the value of t when P is closest to ℓ. **1**

(iii) Show that, when P is closest to ℓ, the tangent to the curve at P is parallel to ℓ. **1**

(e) A particle moves along a straight line. The displacement of the particle from the origin is x, and its velocity is v. The particle is moving so that $v^2 + 9x^2 = k$, where k is a constant. **2**

Show that the particle moves in simple harmonic motion with period $\dfrac{2\pi}{3}$.

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A spherical raindrop of radius r metres loses water through evaporation at a rate that depends on its surface area. The rate of change of the volume V of the raindrop is given by

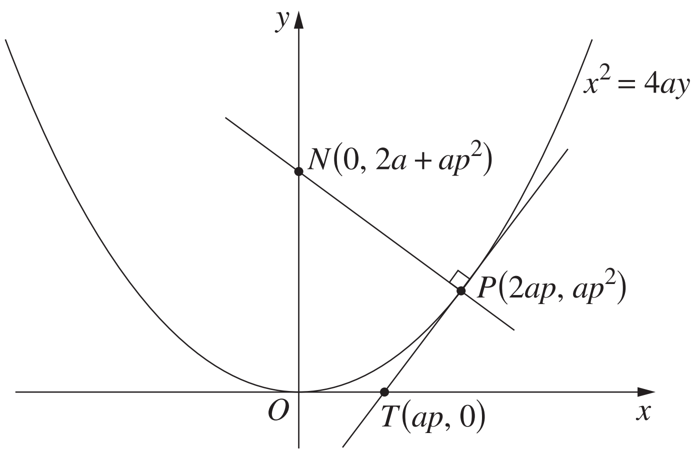
$$\frac{dV}{dt} = -10^{-4} A,$$

where t is time in seconds and A is the surface area of the raindrop. The surface area and the volume of the raindrop are given by $A = 4\pi r^2$ and $V = \frac{4}{3}\pi r^3$ respectively.

(i) Show that $\frac{dr}{dt}$ is constant. **1**

(ii) How long does it take for a raindrop of volume 10^{-6} m^3 to completely evaporate? **2**

(b) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$. The tangent to the parabola at P meets the x-axis at $T(ap, 0)$. The normal to the tangent at P meets the y-axis at $N(0,\ 2a + ap^2)$.



The point G divides NT externally in the ratio $2:1$.

(i) Show that the coordinates of G are $(2ap, -2a - ap^2)$. **2**

(ii) Show that G lies on a parabola with the same directrix and focal length as the original parabola. **2**

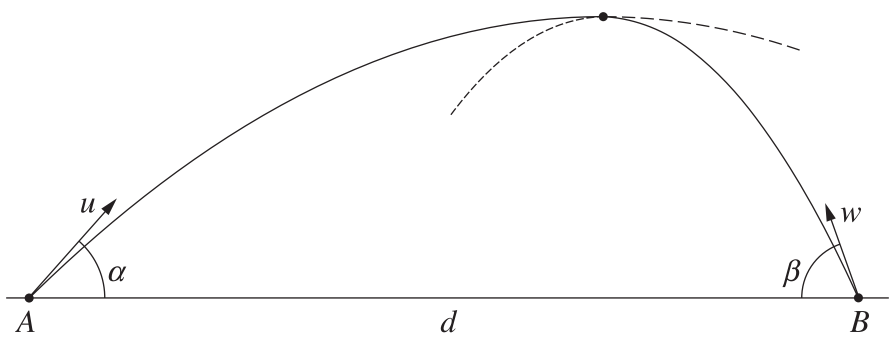
Question 13 continues on page 13

Question 13 (continued)

(c) Points A and B are located d metres apart on a horizontal plane. A projectile is fired from A towards B with initial velocity u m s^{-1} at angle α to the horizontal.

At the same time, another projectile is fired from B towards A with initial velocity w m s^{-1} at angle β to the horizontal, as shown on the diagram.

The projectiles collide when they both reach their maximum height.



The equations of motion of a projectile fired from the origin with initial velocity V m s^{-1} at angle θ to the horizontal are

$$x = Vt\cos\theta \quad \text{and} \quad y = Vt\sin\theta - \frac{g}{2}t^2. \qquad \text{(Do NOT prove this.)}$$

(i) How long does the projectile fired from A take to reach its maximum height? **2**

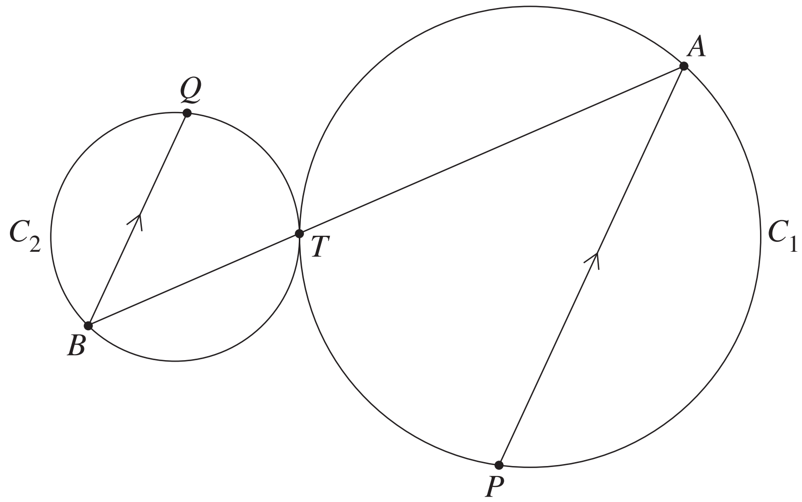
(ii) Show that $u\sin\alpha = w\sin\beta$. **1**

(iii) Show that $d = \dfrac{uw}{g}\sin(\alpha + \beta)$. **2**

Question 13 continues on page 14

Question 13 (continued)

(d) The circles C_1 and C_2 touch at the point T. The points A and P are on C_1. The line AT intersects C_2 at B. The point Q on C_2 is chosen so that BQ is parallel to PA. **3**



Copy or trace the diagram into your writing booklet.

Prove that the points Q, T and P are collinear.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that for $k > 0$, $\dfrac{1}{(k+1)^2} - \dfrac{1}{k} + \dfrac{1}{k+1} < 0$. **1**

(ii) Use mathematical induction to prove that for all integers $n \geq 2$, **3**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}.$$

(b) (i) Write down the coefficient of x^{2n} in the binomial expansion of $(1+x)^{4n}$. **1**

(ii) Show that $\left(1 + x^2 + 2x\right)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^{2n-k} (x+2)^{2n-k}$. **2**

(iii) It is known that **3**

$$x^{2n-k}(x+2)^{2n-k} = \binom{2n-k}{0} 2^{2n-k} x^{2n-k} + \binom{2n-k}{1} 2^{2n-k-1} x^{2n-k+1}$$

$$+ \cdots + \binom{2n-k}{2n-k} 2^0 x^{4n-2k}.$$ (Do NOT prove this.)

Show that

$$\binom{4n}{2n} = \sum_{k=0}^{n} 2^{2n-2k} \binom{2n}{k} \binom{2n-k}{k}.$$

(c) The equation $e^t = \dfrac{1}{t}$ has an approximate solution $t_0 = 0.5$.

(i) Use one application of Newton's method to show that $t_1 = 0.56$ is another approximate solution of $e^t = \dfrac{1}{t}$. **2**

(ii) Hence, or otherwise, find an approximation to the value of r for which the graphs $y = e^{rx}$ and $y = \log_e x$ have a common tangent at their point of intersection. **3**

End of paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$