

2013 HSC Mathematics Extension 1 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	С
2	D
3	С
4	D
5	А
6	В
7	А
8	D
9	В
10	С



Section II

Question 11 (a)

Criteria	Marks
Correct answer	1

For any cubic $\alpha\beta\gamma = -\frac{d}{a}$, so $\alpha\beta\gamma = -\frac{7}{2}$.

Question 11 (b)

	Criteria	Marks
•	Correct primitive	2
•	Attempts to use standard integral, or equivalent merit	1

$$\int \frac{1}{\sqrt{49 - 4x^2}} dx$$

= $\int \frac{1}{\sqrt{4\left(\frac{49}{4} - x^2\right)}} dx$
= $\frac{1}{2} \int \frac{1}{\sqrt{\left(\frac{7}{2}\right)^2 - x^2}} dx$
= $\frac{1}{2} \sin^{-1} \frac{x}{\frac{7}{2}} + c$
= $\frac{1}{2} \sin^{-1} \frac{2x}{7} + c$



Question 11 (c)

	Criteria	Marks
•	Correct answer	2
•	Identifies correct binomial coefficient, or equivalent merit	1

The probability is

$${}^{10}C_7 \left(\frac{1}{4}\right)^7 \left(\frac{3}{4}\right)^3 = \frac{10!}{7!3!} \frac{3^3}{4^{10}}$$
$$= 120 \cdot \frac{3^3}{4^{10}}$$
$$= 30 \cdot \frac{3^3}{4^9}.$$

For each question, the probability of choosing the correct option is $\frac{1}{4}$ and the probability of choosing an incorrect option is $\frac{3}{4}$. The probability of choosing the correct option for exactly 7 out of 10 questions is the term in $\left(\frac{1}{4} + \frac{3}{4}\right)^{10}$ containing $\left(\frac{1}{4}\right)^{7}$.

Question 11 (d) (i)

	Criteria	Marks
•	Correct solution	2
•	Finds $f'(x)$	1

Domain: all real $x \neq \pm 2$ Using the quotient rule

$$f'(x) = \frac{(4 - x^2) \cdot 1 - x(-2x)}{(4 - x^2)^2}$$
$$= \frac{4 - x^2 + 2x^2}{(4 - x^2)^2}$$
$$= \frac{4 + x^2}{(4 - x^2)^2}$$

f'(x) > 0 since $4 + x^2 > 0$ for all x, and $(4 - x^2)^2 > 0$ for all $x \neq \pm 2$ Hence f'(x) > 0 for all $x \neq \pm 2$.



Question 11 (d) (ii)

	Criteria	Marks
•	Correct graph	2
•	Finds both vertical asymptotes, or equivalent merit	1



Question 11 (e)

1



Question 11 (f)

	Criteria	Marks
•	Correct solution	3
•	Finds a primitive of the form of $a \tan^{-1} u$	2
•	Attempts to use given substitution	1

$$u = e^{3x}, \quad \frac{du}{dx} = 3e^{3x}, \quad u^2 = e^{6x}$$

If $x = 0$, then $u = e^0 = 1$
If $x = \frac{1}{3}$, then $u = e^{3 \cdot \frac{1}{3}} = e$
Hence

$$\int_{0}^{\frac{1}{3}} \frac{e^{3x}}{e^{6x} + 1} dx$$

= $\frac{1}{3} \int_{1}^{e} \frac{1}{u^{2} + 1} du$
= $\frac{1}{3} \left[\tan^{-1} u \right]_{1}^{e}$
= $\frac{1}{3} \left(\tan^{-1} e - \tan^{-1} 1 \right)$
= $\frac{1}{3} \left(\tan^{-1} e - \frac{\pi}{4} \right).$

Question 11 (g)

	Criteria	Marks
•	Correct solution	2
•	Uses correct derivative of $\sin^{-1}5x$, or equivalent merit	1

Using the product rule

$$f'(x) = x^2 \frac{5}{\sqrt{1 - 25x^2}} + 2x \sin^{-1} 5x$$
$$= \frac{5x^2}{\sqrt{1 - 25x^2}} + 2x \sin^{-1} 5x.$$



Question 12 (a) (i)

Criteria	Marks
Correct answer	1

We want

$$\sqrt{3}\cos x - \sin x = 2\cos(x + \alpha)$$

$$\frac{\sqrt{3}}{2}\cos x - \frac{1}{2}\sin x = \cos(x + \alpha)$$

$$= \cos x \cos \alpha - \sin x \sin \alpha,$$
So $\cos \alpha = \frac{\sqrt{3}}{2}$ and $\sin \alpha = \frac{1}{2}$. Hence $\tan \alpha = \frac{1}{\sqrt{3}}$.

$$\therefore \ \alpha = \frac{\pi}{6} \left(\text{and } 0 < \frac{\pi}{6} < \frac{\pi}{2} \right)$$

Question 12 (a) (ii)

Criteria	Marks
Correct solution	2
• Obtains an equation of the form $\cos(\underline{}) = k$, or equivalent merit	1

Rearrange the equation and use part (i):

$$\sqrt{3}\cos x - \sin x = 1, \text{ ie } 2\cos\left(x + \frac{\pi}{6}\right) = 1$$
$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$
$$\therefore x + \frac{\pi}{6} = \frac{\pi}{3} \text{ or } x + \frac{\pi}{6} = \frac{5\pi}{3}$$
Hence the solutions satisfying $0 < x < 2\pi$ are

$$x = \frac{\pi}{3} - \frac{\pi}{6} = \frac{\pi}{6}$$

and $x = \frac{5\pi}{3} - \frac{\pi}{6} = \frac{3\pi}{2}$.



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Question 12 (b)

	Criteria	Marks
•	Correct solution	3
•	Finds correct primitive, or equivalent merit	2
•	Finds correct integrand, or equivalent merit	1



Volume of solid is given by

$$V = \pi \int y^2 dx$$

$$V = \pi \int_0^{\frac{3\pi}{2}} \left(3\sin\frac{x}{2}\right)^2 dx$$

$$= 9\pi \int_0^{\frac{3\pi}{2}} \sin^2\frac{x}{2} dx$$

$$V = 9\pi \int_0^{\frac{3\pi}{2}} \frac{1}{2}(1 - \cos x) dx$$

$$\left(\text{Note: } \cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} - \sin$$



Question 12 (c)

	Criteria	Marks
•	Correct solution	3
•	Finds values for A, B and k, or equivalent merit	2
•	Finds value for A, or equivalent merit	1

As $t \to \infty$, T approaches the temperature of the room, and so A = 22.

When t = 0, T = 22 + B

= 80 $\therefore B = 58.$

Hence

$$T = 22 + 58e^{-kt}$$

To find k: After 10 minutes the temperature is 60° (t = 10, T = 60)

$$60 = 22 + 58e^{-k \times 10}$$
$$38 = 58e^{-10k}$$
$$\log_e \left(\frac{38}{58}\right) = -10k,$$
$$\text{so } k = \frac{-1}{10}\log_e \left(\frac{38}{58}\right)$$

To determine time to drop to 40°C:

$$40 = 22 + 58e^{\frac{1}{10}\log_e\left(\frac{38}{58}\right)t}$$
$$\frac{18}{58} = e^{\frac{1}{10}\log_e\left(\frac{38}{58}\right)t}$$
$$\log_e \frac{18}{58} = \frac{1}{10}\log_e\left(\frac{38}{58}\right)t$$
$$t = 10\frac{\log_e\left(\frac{38}{58}\right)}{\log_e\left(\frac{38}{58}\right)},$$
$$t = 27.67...$$

: temperature drops to 40° after 28 minutes (to the nearest minute).



Question 12 (d) (i)

	Criteria	Marks
•	Correct solution	2
•	• Uses an appropriate method to attempt to find $D(t)$	1

Using the perpendicular distance formula from $P(t, t^2 + 3)$ to the line y = 2x - 1:

$$D(t) = \frac{\left|2t - (t^2 + 3) - 1\right|}{\sqrt{4 + 1}}$$

= $\frac{\left|-t^2 + 2t - 4\right|}{\sqrt{5}}$
= $\frac{\left|-(t^2 - 2t + 4)\right|}{\sqrt{5}}$
= $\frac{\left|t^2 - 2t + 4\right|}{\sqrt{5}}$
Now $t^2 - 2t + 4 = (t - 1)^2 + 3 > 0$ for all t, so
 $D(t) = \frac{t^2 - 2t + 4}{\sqrt{5}}$ is the perpendicular distance.

Question 12 (d) (ii)

	Criteria	Marks
•	Correct answer	1

To find minimum distance, D'(t) = 0.

$$D'(t) = \frac{1}{\sqrt{5}}(2t - 2)$$
$$= \frac{2}{\sqrt{5}}(t - 1)$$
$$= 0$$
$$\therefore t = 1$$

Minimum because $D(t) \rightarrow \infty$ as $t \rightarrow \pm \infty$.

Alternative: D(t) is a quadratic with minimum at t = 1 since $D(t) = \frac{1}{\sqrt{5}} ((t-1)^2 + 3)$.



Question 12 (d) (iii)

	Criteria	Marks
•	Correct solution	1

t = 1 corresponds to x = 1, on the curve given by $y = x^2 + 3$.

y' = 2x, so the slope of the tangent at x = 1 is 2.

Also, the slope of the line y = 2x - 1 is 2.

 \therefore the line and the tangent are parallel, since their gradients are equal.

Question 12 (e)

	Criteria	Marks
•	Correct solution	2
•	Correctly differentiates v^2 with respect to x, or equivalent merit	1

$$v^{2} + 9x^{2} = k$$

$$v^{2} = k - 9x^{2}$$

$$\frac{1}{2}v^{2} = \frac{1}{2}k - \frac{9}{2}x^{2}$$

$$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^{2}\right) = -9x$$

For SHM, $\ddot{x} = -n^2(x-b)$.

Now, $\ddot{x} = -9x$, is of this form with

$$n = 3$$
 and $b = 0$
As $n = 3$, the period is $\frac{2\pi}{3}$.



Question 13 (a) (i)

Criteria	Marks
Correct solution	1
Using $V = \frac{4}{3}\pi r^3$ and $A = 4\pi r^2$,	
$-10^{-4}A = \frac{dV}{dV}$	

$$A = \frac{dt}{dt}$$
$$= \frac{dV}{dr}\frac{dr}{dt}$$
$$= 4\pi r^2 \frac{dr}{dt}$$
$$= A\frac{dr}{dt}$$

Hence $\frac{dr}{dt} = -10^{-4}$, which is constant.

Question 13 (a) (ii)

	Criteria	Marks
•	Correct answer	2
•	Finds initial radius, or equivalent merit	1

From part (i) $\frac{dr}{dt} = 10^{-4}$

To determine the initial radius of the raindrop:

$$\frac{4}{3}\pi r^{3} = 10^{-6},$$

so $r^{3} = \frac{3}{4\pi}10^{-6}$
 $r = \sqrt[3]{\frac{3}{4\pi}}10^{-2}$
 $= \frac{1}{10^{2}}\sqrt[3]{\frac{3}{4\pi}}$

Time to evaporate;

$$t = \frac{r}{10^{-4}} = 10^2 \sqrt[3]{\frac{3}{4\pi}}$$

 \therefore time = 62 seconds (to the nearest second).



Question 13 (b) (i)

	Criteria	Marks
•	Correct solution	2
•	Writes an unsimplified expression for the x - or y - coordinate of G , or equivalent merit	1

The point G divides NT in the ratio 2:1 externally means that T is the midpoint of NG.

Hence if (x_0, y_0) are the coordinates of G, then $\frac{0 + x_0}{2} = ap \qquad (x \text{-coordinate of } T)$ $\frac{2a + ap^2 + y_0}{2} = 0 \qquad (y \text{-coordinate of } T)$ $\therefore x_0 = 2ap \text{ and } y_0 = -2a - ap^2$ ie G is the point $(2ap, -2a - ap^2)$, as required



Question 13 (b) (ii)

	Criteria	Marks
•	Correct solution for G	2
•	Shows that G lies on a parabola, or equivalent merit	1

The original parabola has focal length *a* and directrix y = -a. To show

that G $(2ap, -2a - ap)^2$ lies on a parabola, eliminate p from x = 2ap, $y = -2a - ap^2$

Now,
$$p = \frac{x}{2a}$$
, so
 $y = -2a - a\left(\frac{x}{2a}\right)^2$
 $y = -2a - \frac{ax^2}{4a^2}$
 $= -2a - \frac{x^2}{4a}$
 $y + 2a = -\frac{x^2}{4a}$
 $x^2 = -4a(y + 2a)$

This is a parabola with focal length *a* and vertex (0, -2a). So the directrix is

$$y = -2a + a$$
$$= -a$$

Hence focal length and directrix are the same for both parabolas.

Alternative:

 $(2ap, -ap^2)$ is the parabola obtained if the original parabola is reflected about the x-axis. It has focal length *a* and directrix y = +a. Now $G(2ap, -2a - ap^2)$ is obtained by translating the reflected parabola down by 2a units. Translation preserves the focal length *a*, and the directrix becomes y = a - 2a = -a.



Question 13 (c) (i)

	Criteria	Marks
•	Correct solution	2
•	Obtains an expression for \dot{y} , or equivalent merit	1

We need to find the time when the quadratic $y = ut \sin \alpha - \frac{g}{2}t^2$ is a maximum.

 $\frac{dy}{dt} = u\sin\alpha - gt$ = 0 for max/min $t = \frac{u\sin\alpha}{g}$

This will be at the maximum height since $y = ut \sin \alpha - \frac{g}{2}t^2$ is a concave down parabola.

Question 13 (c) (ii)

	Criteria	Marks
•	Correct solution	1

Similarly from part (i), the projectile fired from B reaches its maximum height when

$$t = \frac{w \sin \beta}{g}.$$

The projectiles collide when they both reach their maximum height.

ie
$$\frac{u\sin\alpha}{g} = t = \frac{w\sin\beta}{g}$$
,
so $u\sin\alpha = w\sin\beta$.



Question 13 (c) (iii)

Criteria	Marks
Correct solution	2
Finds the distance one projectile has travelled	1

When the projectiles collide, the sum of the two distances travelled must be d. Both projectiles have travelled for time t.

$$t = \frac{u\sin\alpha}{g} = \frac{w\sin\beta}{g}, \text{ so}$$
$$d = u\frac{u\sin\alpha}{g}\cos\alpha + w\frac{w\sin\beta}{g}\cos\beta$$
$$= u\frac{w\sin\beta}{g}\cos\alpha + w\frac{u\sin\alpha}{g}\cos\beta$$
$$= \frac{uw}{g}(\sin\beta\cos\alpha + \sin\alpha\cos\beta)$$
$$= \frac{uw}{g}\sin(\alpha + \beta).$$



Question 13 (d)

	Marks	
•	Correct solution	3
•	Proves that $\angle QBT$ and $\angle PAT$ are equal AND uses the properties of the common tangent, or equivalent merit	2
•	Proves that $\angle QBT$ and $\angle PAT$ are equal or uses the properties of the common tangent, or equivalent merit	1



The circles C_1 and C_2 have a common tangent at T.

Choose points X and Y on that tangent as marked on the diagram above.

 $\angle YTP = \angle PAT = \alpha \quad (angle in alternate segment, chord PT and tangent XY)$ $\angle XTQ = \angle QBT = \beta \quad (angle in alternate segment, chord QT and tangent XY)$ $But \angle PAT = \angle QBT \qquad (alternate angles equal, BQ || PA)$

ie $\alpha = \beta$

$$\therefore \ \angle YTP = \angle XTQ$$

Hence QP is a straight line making equal vertically opposite angles with line XY at T. $\therefore P,T,Q$ are collinear.



Question 14 (a) (i)

Criteria	Marks
Correct solution	1

$$\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} = \frac{1}{(k+1)^2} - \frac{k+1}{k(k+1)} + \frac{k}{k(k+1)}$$
$$= \frac{1}{k+1} \left[\frac{1}{k+1} - \frac{1}{k} \right]$$
$$< 0 \text{ (since } k+1 > k \text{ and } k > 0 \text{)}.$$

Question 14 (a) (ii)

	Marks	
•	Correct solution	3
•	Uses correct assumption in an attempt to prove that $p(k) \Rightarrow p(k+1)$	2
•	Establishes initial case, or equivalent merit	1

Case
$$n = 2$$
: LHS $\frac{1}{1} + \frac{1}{2^2} = 1 + \frac{1}{4} = \frac{5}{4}$ RHS: $2 - \frac{1}{2} = \frac{3}{2}$
LHS < RHS

Hence true for n = 2

Assume true for some $k \ge 2$

ie
$$\sum_{p=1}^{k} \frac{1}{p^2} < 2 - \frac{1}{k}$$

Need to show true for n = k + 1

ie
$$\sum_{p=1}^{k+1} \frac{1}{p^2} < 2 - \frac{1}{k+1}$$

Now

$$LHS = \sum_{p=1}^{k+1} \frac{1}{p^2} = \sum_{p=1}^{k} \frac{1}{p^2} + \frac{1}{(k+1)^2}$$

$$LHS < 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \text{ (by induction assumption)}$$

$$< 2 - \frac{1}{k} + \frac{1}{k} - \frac{1}{k+1} \quad \left(\text{using part (i), ie } \frac{1}{(k+1)^2} < \frac{1}{k} - \frac{1}{k+1} \right)$$

$$\text{ie } \sum_{p=1}^{k+1} \frac{1}{p^2} < 2 - \frac{1}{k+1}$$

By the principle of Mathematical Induction, it is true for all $n \ge 2$.



Question 14 (b) (i)

	Criteria	Marks
•	Correct answer	1

From the binomial theorem

$$(1+x)^{4n} = \sum_{k=0}^{4n} {4n \choose k} x^{4n-k}$$

The coefficient of x^{2n} is ${4n \choose 2n}$, choosing $k = 2n$.

Question 14 (b) (ii)

	Criteria	Marks
•	Correct solution	2
•	Writes $(1 + x^2 + 2x)^{2n}$ as $(1 + x(x + 2))^{2n}$ or uses the binomial expansion, or equivalent merit	1

From the binomial theorem

$$(1 + x^{2} + 2x)^{2n} = (1 + x(x + 2))^{2n}$$
$$= \sum_{k=0}^{2n} {2n \choose k} (x(x + 2))^{2n-k}$$
$$= \sum_{k=0}^{2n} {2n \choose k} x^{2n-k} (x + 2)^{2n-k}.$$



Question 14 (b) (iii)

	Marks	
•	Correct solution	3
•	Substitutes the given expression into part (ii) and makes progress towards the solution	2
•	Connects $(1+x)^{4n}$ and $(1+x^2+2x)^{2n}$, or equivalent merit	1

$$(1 + x^{2} + 2x) = (1 + x)^{2}$$
, so
 $(1 + x^{2} + 2x)^{2n} = (1 + x)^{4n}$
By part (i), $\binom{4n}{2n}$ is the coefficient of x^{2n} in $(1 + x^{2} + 2x)^{2n}$.

From part (ii) these are

the terms in $\binom{2n}{k} x^{2n-k} (x+2)^{2n-k}$ for k = 0, ..., n. If k > n there is no term involving x^{2n} .

Using the given identity the coefficients of x^{2n} are:

$$k = 0 : {\binom{2n}{0}} {\binom{2n-0}{0}} 2^{2n-0}$$

$$k = 1 : {\binom{2n}{1}} {\binom{2n-1}{1}} 2^{2n-1-1}$$

$$k = 2 : {\binom{2n}{2}} {\binom{2n-2}{2}} 2^{2n-2-2}$$

$$\vdots$$

$$k = n : {\binom{2n}{n}} {\binom{2n-n}{n}} 2^{2n-n-n}$$

Hence

$$\begin{pmatrix} 4n \\ 2n \end{pmatrix} = {\binom{2n}{0}} {\binom{2n}{0}} 2^{2n} + {\binom{2n}{1}} {\binom{2n-1}{1}} 2^{2n-2} + \dots + {\binom{2n}{n}} {\binom{n}{n}} 2^0$$
$$= \sum_{k=0}^n {\binom{2n}{k}} {\binom{2n-k}{k}} 2^{2n-2k}.$$



Question 14 (c) (i)

	Criteria	Marks
•	Correct solution	2
•	Correctly applies Newton's method	1

Take
$$f(t) = e^t - \frac{1}{t}$$

Then $f'(t) = e^t + \frac{1}{t^2}$

Apply Newton's method

$$t_1 = t_0 - \frac{f(t_0)}{f'(t_0)}$$
$$= 0.5 - \frac{e^{0.5} - 2}{e^{0.5} + 4}$$
$$= 0.5621...$$

Hence $t_1 = 0.56$ is another approximate solution.



Question 14 (c) (ii)

	Marks	
•	Correct solution	3
•	Uses part (c) (i) to find an approximate solution to $re^{rx} = \frac{1}{x}$, or equivalent merit	2
•	Establishes $re^{rx} = \frac{1}{x}$, or equivalent merit	1

We need to determine r and x so that

$$e^{rx} = \log_e x$$
 and $re^{rx} = \frac{1}{x}$

(common point and equal slope of tangents).

From the equal slopes

$$e^{rx} = \frac{1}{rx}$$

Setting rx = t and using part (i), we have an approximate solution

$$rx = 0.56$$

Since the curves intersect at x,

$$\log_{e} x = e^{rx} = e^{0.56}$$

$$x = e^{e^{0.56}} = 5.758...$$
now $rx = 0.56$

$$\therefore r = \frac{0.56}{5.758...}$$

$$= 0.0972$$

$$r \neq 0.097$$

Mathematics Extension 1 2013 HSC Examination Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	16.2	PE3
2	1	15.1	HE4
3	1	2.8	PE3
4	1	16.1	PE3
5	1	11.5	HE6
6	1	15.2	HE4
7	1	18.1	H5, PE3
8	1	5.7	H5, PE2
9	1	15.3	HE4
10	1	1.4E	PE3

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	1	16.3	PE3
11 (b)	2	15.5	HE4
11 (c)	2	18.2	HE3
11 (d) (i)	2	8.8	P7, HE4
11 (d) (ii)	2	10.5	H6, HE4
11 (e)	1	13.4	H5, HE7
11 (f)	3	11.5, 15.5	HE4, HE6
11 (g)	2	8.8, 15.5	P7, HE4
12 (a) (i)	1	5.7, 5.9	PE2
12 (a) (ii)	2	5.7, 5.9	PE2
12 (b)	3	11.4, 13.6E	H5, H8, HE7
12 (c)	3	14.2E	HE3
12 (d) (i)	2	6.5	P4
12 (d) (ii)	1	8.5, 10.6	P4, H5
12 (d) (iii)	1	8.5	P6
12 (e)	2	14.4	HE3
13 (a) (i)	1	14.1	HE5
13 (a) (ii)	2	14.1	HE5

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