BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2014 HSC Mathematics Extension 2
Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	D
2	A
3	B
4	C
5	C
6	D
7	B
8	B
9	A
10	D

Section II

Question 11 (a) (i)

Criteria	Marks
• Provides a correct solution	2
• Finds modulus or argument, or equivalent merit	1

Sample answer:

$$z + w = (-2 - 2i) + 3 + i = 1 - i$$

$$|z + w| = \sqrt{2}, \quad \arg(z + w) = \frac{-\pi}{4}$$

$$\text{so } z + w = \sqrt{2}\left[\cos\left(\frac{-\pi}{4}\right) + i\sin\left(\frac{-\pi}{4}\right)\right]$$

Question 11 (a) (ii)

Criteria	Marks
• Provides a correct solution	2
• Identifies conjugate, or equivalent merit	1

Sample answer:

$$\begin{aligned}\frac{z}{w} &= \frac{-2 - 2i}{3 + i} \times \frac{3 - i}{3 - i} \\ &= \frac{-8 - 4i}{10} \\ &= \frac{-4 - 2i}{5} \\ &= \frac{-4}{5} - \frac{2}{5}i\end{aligned}$$

Question 11 (b)

Criteria	Marks
• Provides a correct solution	3
• Correctly uses limits in one of two terms, or equivalent merit	2
• Attempts integration by parts, or equivalent merit	1

Sample answer:

$$\int_0^{\frac{1}{2}} (3x-1)\cos \pi x \, dx = \left[(3x-1)\frac{\sin \pi x}{\pi}\right]_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{3}{\pi}\sin \pi x \, dx$$

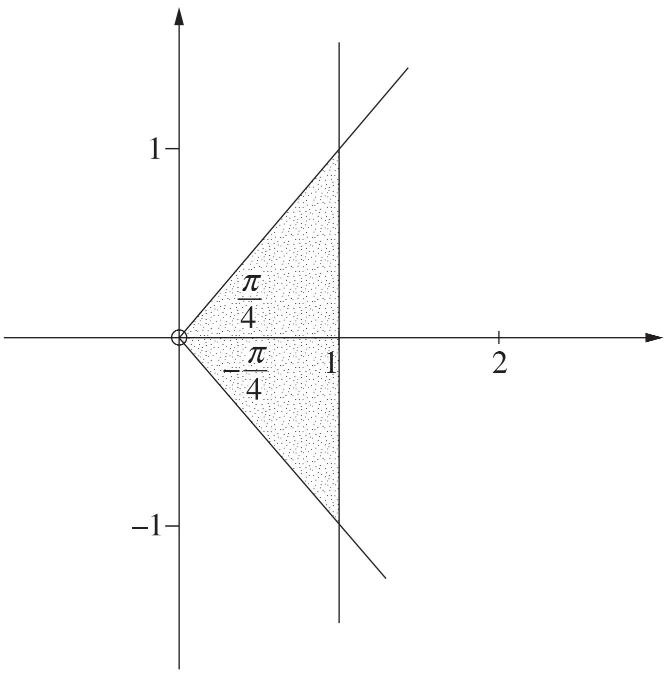
$$= \frac{1}{2\pi} - \left[\frac{-3}{\pi^2}\cos \pi x\right]_0^{\frac{1}{2}}$$

$$= \frac{1}{2\pi} - \frac{3}{\pi^2}$$

Question 11 (c)

Criteria	Marks
• Draws a correct sketch	3
• Correctly shows region satisfying $\lvert z \rvert \le \lvert z-2 \rvert$, or equivalent merit	2
• Correctly shows region satisfying $-\frac{\pi}{4} \le \arg(z) \le \frac{\pi}{4}$, or equivalent merit	1

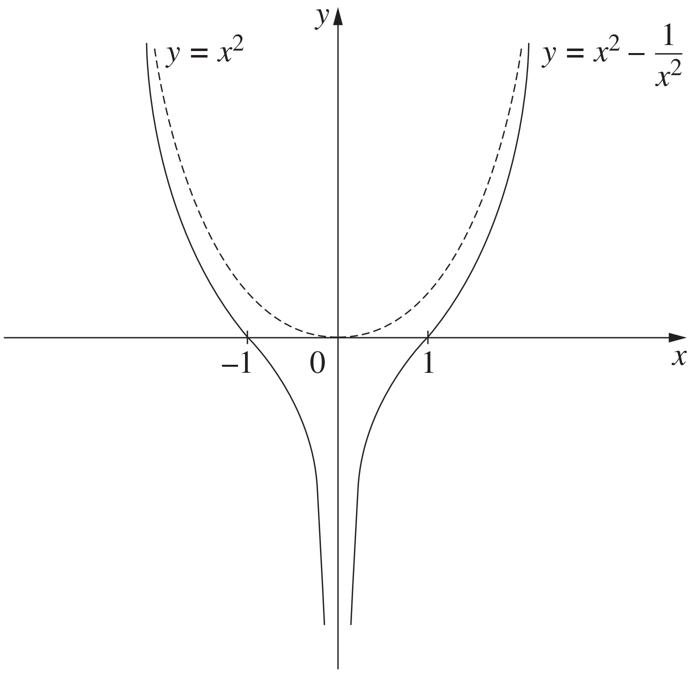
Sample answer:



Question 11 (d)

Criteria	Marks
• Draws correct sketch	2
• Sketches a graph with correct x-intercepts, or equivalent merit	1

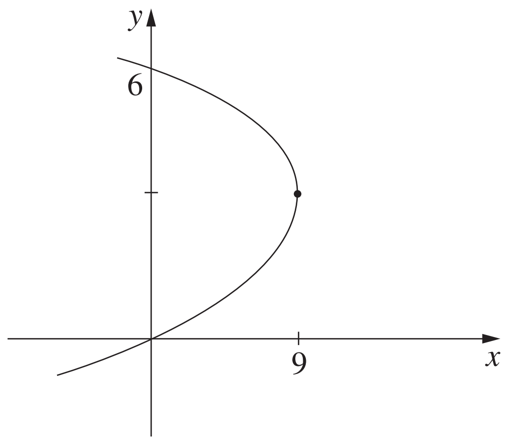
Sample answer:



Question 11 (e)

Criteria	Marks
• Provides a correct solution	3
• Attempts to evaluate the integral for the volume, or equivalent merit	2
• Applies a formula for the volume by cylindrical shells, or equivalent merit	1

Sample answer:



$$V = 2\pi \int_0^6 xy\,dy$$

$$= 2\pi \int_0^6 y(6-y)\,y\,dy$$

$$= 2\pi \int_0^6 \left(6y^2 - y^3\right) dy$$

$$= 2\pi \left[2y^3 - \frac{y^4}{4} \right]_0^6$$

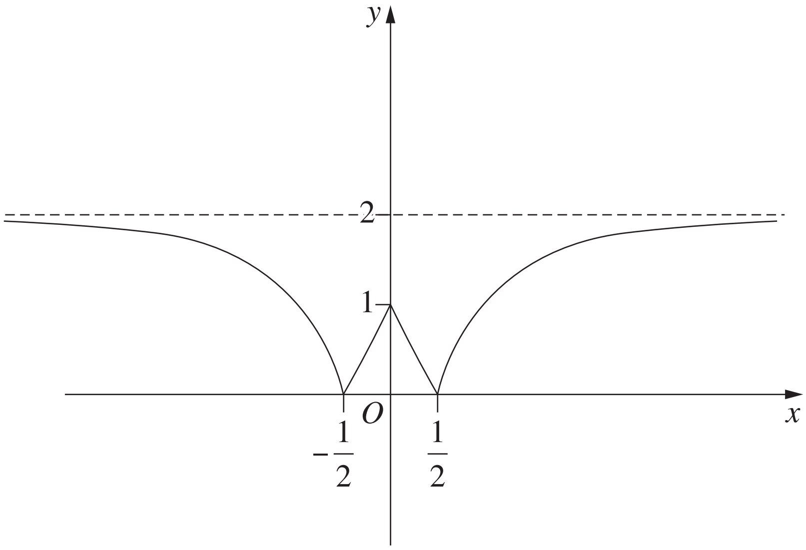
$$= 2\pi (2 \times 216 - 36 \times 9)$$

$$\text{Volume} = 216\pi \text{ units}^3$$

Question 12 (a) (i)

Criteria	Marks
• Draws a correct sketch	2
• Sketches graph symmetric with respect to y-axis or equivalent merit	1

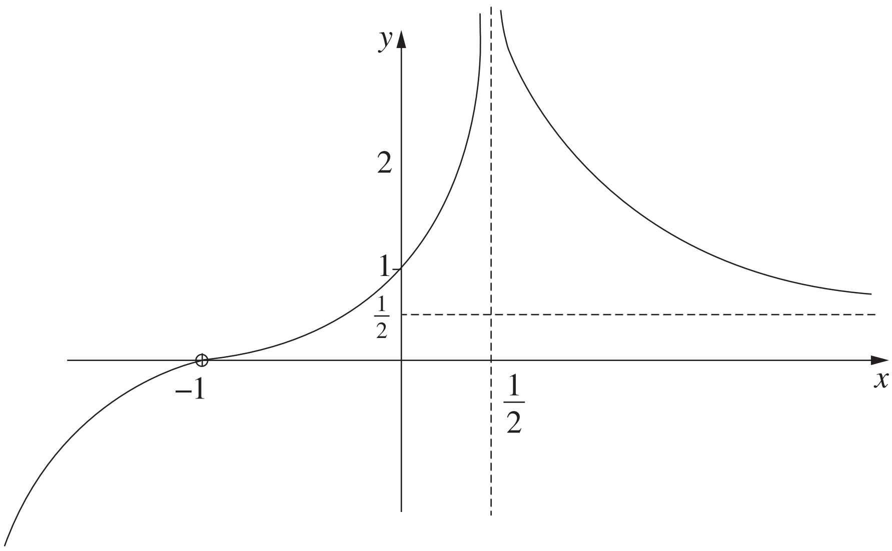
Sample answer:



Question 12 (a) (ii)

Criteria	Marks
• Draws a correct sketch	2
• Shows asymptote at $x = \frac{1}{2}$, or $y = \frac{1}{2}$ or equivalent merit	1

Sample answer:



Question 12 (b) (i)

Criteria	Marks
• Provides a correct solution	1

Sample answer:

Substituting $x = 2\cos\theta$

$$8\cos^3\theta - 6\cos\theta = \sqrt{3}$$

$$\text{so } 4\cos^3\theta - 3\cos\theta = \frac{\sqrt{3}}{2}$$

$$\therefore \cos 3\theta = \frac{\sqrt{3}}{2}$$

Question 12 (b) (ii)

Criteria	Marks
• Provides a correct solution	2
• Finds one real solution, or equivalent merit	1

Sample answer:

$$\cos 3\theta = \frac{\sqrt{3}}{2}$$

$$\text{so } 3\theta = \frac{\pi}{6}, \frac{11\pi}{6}, \frac{12\pi}{6}$$

$$\therefore \theta = \frac{\pi}{18}, \frac{11\pi}{18}, \frac{13\pi}{18}$$

Solutions are

$$x = 2\cos\frac{\pi}{18}, 2\cos\frac{11\pi}{18}, 2\cos\frac{13\pi}{18}$$

Question 12 (c)

Criteria	Marks
• Provides a correct solution	3
• Finds the slope of one curve, and attempts to find slope of other curve, or equivalent merit	2
• Attempts to find slope at (x_0, y_0) of one curve using implicit differentiation, or equivalent merit	1

Sample answer:

$$x^2 - y^2 = 5$$

Differentiating with respect to x:

$$2x - 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{x}{y}$$

at point (x_0, y_0) slope is $m_1 = \frac{x_0}{y_0}$

$$xy = 6 \qquad (x, y \neq 0)$$

Differentiating with respect to x:

$$y + x\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{y}{x}$$

at point (x_0, y_0) slope is $m_2 = -\frac{y_0}{x_0}$

If curves meet then $x_0, y_0 \neq 0$

and $m_1 \times m_2 = \frac{x_0}{y_0} \times \frac{-y_0}{x_0} = -1$

Then tangents are perpendicular.

Question 12 (d) (i)

Criteria	Marks
• Provides a correct solution	1

Sample answer:

$$I_0 = \int_0^1 \frac{1}{x^2+1}\,dx$$

$$= \left[\tan^{-1} x\right]_0^1$$

$$= \frac{\pi}{4}$$

Question 12 (d) (ii)

Criteria	Marks
• Provides a correct solution	2
• Writes the sum as a single integral, or equivalent merit	1

Sample answer:

$$I_n + I_{n-1} = \int_0^1 \left(\frac{x^{2n}}{x^2+1} + \frac{x^{2n-2}}{x^2+1}\right)dx$$

$$= \int_0^1 \frac{x^{2n-2}\left(x^2+1\right)}{\left(x^2+1\right)}\,dx$$

$$= \int_0^1 x^{2n-2}\,dx$$

$$= \left[\frac{1}{2n-1}x^{2n-1}\right]_0^1$$

$$= \frac{1}{2n-1}\left(1-0\right)$$

$$= \frac{1}{2n-1}$$

Question 12 (d) (iii)

Criteria	Marks
• Provides a correct answer	2
• Attempts to apply the recursion relation from part (ii), or equivalent merit	1

Sample answer:

$$\int_0^1 \frac{x^4}{x^2+1}\,dx = I_2$$

Now, $I_2 + I_1 = \frac{1}{3}$

$$I_1 + I_0 = 1$$

Subtracting:

$$I_2 - I_0 = \frac{1}{3} - 1$$

$$= -\frac{2}{3}$$

$$\therefore\ I_2 = I_0 - \frac{2}{3}$$

$$= \frac{\pi}{4} - \frac{2}{3}$$

Question 13 (a)

Criteria	Marks
• Provides a correct solution	3
• Finds a correct primitive, or equivalent merit	2
• Attempts to obtain an integral in terms of *t*, or equivalent merit	1

Sample answer:

$$t = \tan\frac{x}{2}, \quad \sin x = \frac{2t}{1+t^2}, \quad \cos x = \frac{1-t^2}{1+t^2}$$

$$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2} = \frac{1}{2}\left(1+\tan^2\frac{x}{2}\right)$$

$$= \frac{1}{2}\left(1+t^2\right)$$

$$\therefore\ dx = \frac{2dt}{1+t^2}$$

when $x = \frac{\pi}{2}, \quad t = 1$ and when $x = \frac{\pi}{3}, \quad t = \frac{1}{\sqrt{3}}$

Question 13 (a) Sample answer – continued:

$$I = \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right) + 5} \times \frac{1}{\left(1+t^2\right)} dt$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2dt}{6t - 4 + 4t^2 + 5 + 5t^2}$$

$$= \int_{\frac{1}{\sqrt{3}}}^{1} \frac{2dt}{9t^2 + 6t + 1}$$

$$= 2\int_{\frac{1}{\sqrt{3}}}^{1} \frac{dt}{(3t+1)^2}$$

$$= -\frac{2}{3}\left[\frac{1}{3t+1}\right]_{\frac{1}{\sqrt{3}}}^{1}$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{1}{\sqrt{3}+1}\right)$$

$$= -\frac{2}{3}\left(\frac{1}{4} - \frac{\sqrt{3}-1}{2}\right)$$

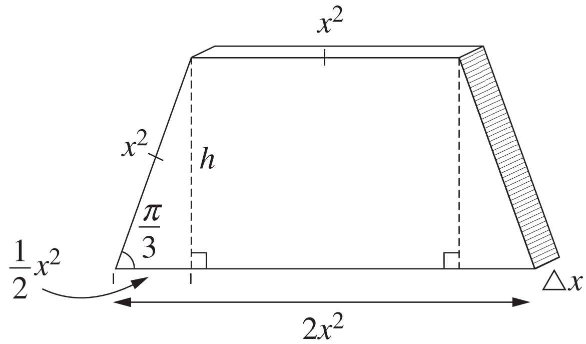
$$= -\frac{1}{6} + \frac{\sqrt{3}-1}{3}$$

$$= \frac{2\sqrt{3}-3}{6}$$

Question 13 (b)

Criteria	Marks
• Provides a correct solution	4
• Finds an expression for the volume of the solid	3
• Finds the area of each cross-section, or equivalent merit	2
• Finds the height of each trapezium, or equivalent merit	1

Sample answer:



$$h = x^2 \sin\frac{\pi}{3} = x^2 \frac{\sqrt{3}}{2}$$

$$\text{Area of cross-section } = \frac{1}{2} \times x^2 \frac{\sqrt{3}}{2}\left(x^2 + 2x^2\right)$$

$$= \frac{\sqrt{3}}{4} x^2 \cdot 3x^2$$

$$= \frac{3\sqrt{3}}{4} x^4$$

$$\text{Volume of the slice } = \frac{3\sqrt{3}}{4} x^4 \triangle x$$

$$\therefore \text{Volume of the solid} = \int_0^2 \frac{3\sqrt{3}}{4} x^4 \, dx$$

$$= \frac{3\sqrt{3}}{4}\left[\frac{x^5}{5}\right]_0^2$$

$$= \frac{3\sqrt{3}}{4 \times 5}\left(2^5\right)$$

$$= \frac{24\sqrt{3}}{5} \text{ unit}^3$$

Question 13 (c) (i)

Criteria	Marks
• Provides a correct solution	1

Sample answer:

$$M\left(\frac{a}{2}\left(t+\frac{1}{t}\right), \frac{b}{2}\left(t-\frac{1}{t}\right)\right)$$

Substituting for x and y,

$$\begin{aligned}\frac{x^2}{a^2}-\frac{y^2}{b^2} &= \frac{1}{4}\left(t+\frac{1}{t}\right)^2-\frac{1}{4}\left(t-\frac{1}{t}\right)^2 \\ &= \frac{1}{4}\left(t+\cancel{\frac{1}{t}}+t-\cancel{\frac{1}{t}}\right)\left(\cancel{t}+\frac{1}{t}-\cancel{t}+\frac{1}{t}\right) \\ &= \frac{1}{4}2t\cdot\frac{2}{t} \\ &= 1\end{aligned}$$

So M lies on the hyperbola.

Question 13 (c) (ii)

Criteria	Marks
• Provides a correct proof	3
• Finds the slope of the tangent, or equivalent merit	2
• Finds slope of PQ, or attempts to find the slope of the tangent, or equivalent merit	1

Sample answer:

Slope of PQ is $\dfrac{bt + \frac{b}{t}}{at - \frac{a}{t}} = \dfrac{b\left(t + \frac{1}{t}\right)}{a\left(t - \frac{1}{t}\right)}$

Differentiating $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ with respect to x:

$$\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dt} = 0$$

$$\frac{dy}{dt} = \frac{xb^2}{ya^2}$$

at M

$$\frac{dy}{dt} = \frac{\frac{a}{2}\left(t + \frac{1}{t}\right)b^2}{\frac{b}{2}\left(t - \frac{1}{t}\right)a^2}$$

$$= \frac{b\left(t + \frac{1}{t}\right)}{a\left(t - \frac{1}{t}\right)}$$

$=$ slope of PQ

so PQ is a tangent to hyperbola at M

Question 13 (c) (iii)

Criteria	Marks
• Provides a correct solution	2
• Finds $OP \times OQ$ in terms of a and b, or equivalent merit	1

Sample answer:

$$OP = \sqrt{(a^2 + b^2)t^2}$$

$$OQ = \sqrt{\frac{a^2 + b^2}{t^2}}$$

$$\therefore OP \times OQ = a^2 + b^2$$

$$\begin{aligned} OS^2 &= a^2 e^2 \\ &= a^2\left(1 + \frac{b^2}{a^2}\right) \\ &= a^2 + b^2 \\ &= OP \times OQ \end{aligned}$$

Question 13 (c) (iv)

Criteria	Marks
• Provides a correct solution	2
• Finds one coordinate of M in terms of a, b and e, or equivalent merit	1

Sample answer:

Equations of asymptotes are $y = \pm\dfrac{b}{a}x$

If $at = ae$ then $t = e$.

The coords of M are then $\left(\dfrac{a}{2}\left(e + \dfrac{1}{e}\right), \dfrac{b}{2}\left(e - \dfrac{1}{e}\right)\right)$

slope of MS is

$$\frac{\frac{b}{2}\left(e - \frac{1}{e}\right)}{\frac{a}{2}\left(e + \frac{1}{e}\right) - ae} = \frac{b\left(e - \frac{1}{e}\right)}{ae + \frac{a}{e} - 2ae}$$

$$= \frac{b\left(e - \frac{1}{e}\right)}{a\left(\frac{1}{e} - e\right)}$$

$$= -\frac{b}{a}$$

which is the slope of one of the asymptotes and so MS is parallel to it.

Question 14 (a) (i)

Criteria	Marks
• Provides a correct solution	2
• Shows that $P''(1) = 0$, or equivalent merit	1

Sample answer:

$P'(x) = 5x^4 - 20x + 15$

$P''(x) = 20x^3 - 20$

$P(1) = 1 - 10 + 15 - 6 = 0$

$P'(1) = 5 - 20 + 15 = 0$

$P''(1) = 20 - 20 = 0$ $\qquad\therefore x = 1$ is a root of multiplicity 3

Question 14 (a) (ii)

Criteria	Marks
• Provides a correct solution	2
• Finds product and sum of remaining two roots, or equivalent merit	1

Sample answer:

Let other roots be α, β

$\alpha + \beta + 1 + 1 + 1 = 0 \quad , \quad \alpha\beta(1)(1)(1) = 6$

$\alpha + \beta = -3 \qquad \alpha\beta = 6$

Hence, α and β are the solutions to $x^2 + 3x + 6 = 0$

$$x = \frac{-3 \pm \sqrt{9 - 4(1)(6)}}{2}$$

$$= \frac{-3 \pm \sqrt{-15}}{2}$$

$$= \frac{-3 \pm \sqrt{15}i}{2}$$

The complex roots are $\dfrac{-3 + \sqrt{15}i}{2}, \dfrac{-3 - \sqrt{15}i}{2}$

Question 14 (b) (i)

Criteria	Marks
• Provides a correct solution	3
• Uses the slopes of *OP* and the normal in a correct expression for $\tan\phi$, or equivalent merit	2
• Finds the slope of the tangent at *P*, or equivalent merit	1

Sample answer:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Differentiating with respect to x:

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=-\frac{xb^2}{ya^2}$$

slope of tangent at P is $-\dfrac{b\cos\theta}{a\sin\theta}$

slope of normal at P is $m_1=\dfrac{a\sin\theta}{b\cos\theta}$

slope of OP is $m_2=\dfrac{b\sin\theta}{a\cos\theta}$

Hence, $\tan\phi=\left|\dfrac{m_1-m_2}{1+m_1m_2}\right|$

$$=\left|\frac{\frac{a}{b}\frac{\sin\theta}{\cos\theta}-\frac{b}{a}\frac{\sin\theta}{\cos\theta}}{1+\frac{\sin^2\theta}{\cos^2\theta}}\right|$$

$$=\left|\frac{\frac{\sin\theta}{\cos\theta}\left(\frac{a}{b}-\frac{b}{a}\right)}{\frac{\cos^2\theta+\sin^2\theta}{\cos^2\theta}}\right|$$

$$=\frac{\left(\frac{a^2-b^2}{ab}\right)\frac{\sin\theta}{\cos\theta}\cdot\cos^2\theta}{\cos^2\theta+\sin^2\theta},a>b$$

$$=\frac{a^2-b^2}{ab}\sin\theta\cos\theta$$

Question 14 (b) (ii)

Criteria	Marks
• Provides a correct solution	2
• Uses double angle formula, or equivalent merit	1

Sample answer:

$$\tan\phi = \left(\frac{a^2 - b^2}{ab}\right)\frac{\sin 2\theta}{2}$$

as $\sin x$ has maximum at $x = \frac{\pi}{2}$

$\tan\phi$ has maximum at $2\theta = \frac{\pi}{2}$

$\therefore \phi$ is a maximum when $\theta = \frac{\pi}{4}$

Question 14 (c) (i)

Criteria	Marks
• Provides a correct solution	2
• Finds $m\ddot{x} = F - kv^2$, or equivalent merit	1

Sample answer:

From Newton's second law

$m\ddot{x} = F - kv^2$

The terminal velocity ($v = 300$) occurs when $\ddot{x} = 0$, so

$$F = k \times (300)^2, \text{ or } k = \frac{F}{(300)^2}$$

Hence $m\ddot{x} = F - \dfrac{F}{(300)^2}v^2$

$$= F\left(1 - \left(\frac{v}{300}\right)^2\right)$$

Question 14 (c) (ii)

Criteria	Marks
• Provides a correct solution	4
• Obtains an expression for t in terms of v, or equivalent merit	3
• Attempts to use partial fractions to find t in terms of v, or equivalent merit	2
• Obtains an equation relating t and v involving integrals, or equivalent merit	1

Sample answer:

$$m\frac{dv}{dt} = F\left(1-\left(\frac{v}{300}\right)^2\right)$$

$$\int \frac{dv}{1-\left(\frac{v}{300}\right)^2} = \int \frac{F}{m}\,dt$$

$$\frac{F}{m}t = \int \frac{1}{\left(1-\frac{v}{300}\right)\left(1+\frac{v}{300}\right)}\,dv$$

$$= \frac{1}{2}\int \frac{1}{1-\frac{v}{300}} + \frac{1}{1+\frac{v}{300}}\,dv$$

$$= \frac{300}{2}\left[\ln\left(1+\frac{v}{300}\right) - \ln\left(1-\frac{v}{300}\right)\right] + c$$

$v = 0$ when $t = 0 \Rightarrow c = 0$

Thus $t = \dfrac{m}{F} \times 150 \ln\left(\dfrac{1+\frac{v}{300}}{1-\frac{v}{300}}\right)$

$$= \frac{150m}{F}\ln\left(\frac{300+v}{300-v}\right)$$

Put $v = 200$, $t = \dfrac{150m}{F}\ln 5$

Question 15 (a)

Criteria	Marks
• Provides a correct solution	2
• Expands $(a+b+c)^2$ and attempts to use the relative sizes of *a*, *b* or *c*, or equivalent merit	1

Sample answer:

$a+b+c=1, \quad a \le b \le c$

So $(a+b+c)^2 = 1$

$$\begin{aligned} LHS &= a^2+b^2+c^2+2(ab+bc+ac) \\ &\ge a^2+b^2+c^2+2(a^2+b^2+a^2) \quad (\text{since } b \ge a, c \ge b, c \ge a) \end{aligned}$$

$\therefore LHS \ge 5a^2+3b^2+c^2$

$\therefore 5a^2+3b^2+c^2 \le 1$

Question 15 (b) (i)

Criteria	Marks
• Provides a correct solution	2
• Writes 1 + *i* or 1 – *i* in modulus argument form, or equivalent merit	1

Sample answer:

Write (1 + *i*) and (1 – *i*) in modulus-argument form:

$$1+i=\sqrt{2}\left(\frac{1}{\sqrt{2}}+\frac{i}{\sqrt{2}}\right)=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)$$

and

$$1-i=\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{i}{\sqrt{2}}\right)=\sqrt{2}\left(\cos\frac{\pi}{4}-i\sin\frac{\pi}{4}\right)=\sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right)+i\sin\left(-\frac{\pi}{4}\right)\right)$$

Using de Moivre's theorem

$$(1+i)^n=\left[\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]^n$$

$$=\left(\sqrt{2}\right)^n\left(\cos\frac{n\pi}{4}+i\sin\frac{n\pi}{4}\right)$$

and

$$(1-i)^n=\left[\sqrt{2}\left(\cos\left(\frac{-\pi}{4}\right)+i\sin\left(\frac{-\pi}{4}\right)\right)\right]^n$$

$$=\left(\sqrt{2}\right)^n\left(\cos\left(\frac{-n\pi}{4}\right)+i\sin\left(\frac{-n\pi}{4}\right)\right)$$

$$=\sqrt{2}^n\left(\cos\frac{n\pi}{4}-i\sin\frac{n\pi}{4}\right)$$

adding

$$(1+i)^n+(1-i)^n=2\left(\sqrt{2}\right)^n\cos\frac{n\pi}{4}$$

Question 15 (b) (ii)

Criteria	Marks
• Provides a correct solution	3
• Attempts to add two binomial expressions, or equivalent merit	2
• Applies the binomial theorem to $(1 + i)^n$ or $(1 - i)^n$, or equivalent merit	1

Sample answer:

Using the binomial theorem

$$(1+i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k$$

$$= \binom{n}{0} + \binom{n}{1} i - \binom{n}{2} - \binom{n}{3} i + \binom{n}{4} + \ldots + \binom{n}{n}$$

and

$$(1-i)^n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} i^k$$

$$= \binom{n}{0} - \binom{n}{1} i - \binom{n}{2} + \binom{n}{3} i + \binom{n}{4} - \ldots + \binom{n}{n}$$

The last terms are positive since n is divisible by 4

Adding the identities

$$(1+i)^n + (1-i)^n$$

$$= 2\left[\binom{n}{0} - \binom{n}{2} + \ldots + \binom{n}{n}\right]$$

Combining this with the expressions from (i) for $(1+i)^n + (1-i)^n$

$$\binom{n}{0} - \binom{n}{2} + \ldots + \binom{n}{n} = \left(\sqrt{2}\right)^n \cos\frac{n\pi}{4}$$

$$= \sqrt{2}^n (-1)^{\frac{n}{4}}$$

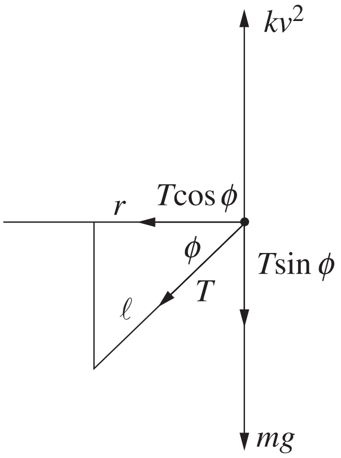
since $\cos\frac{n\pi}{4} = -1$ for odd multiples of π $\left(\frac{n}{4} \text{ odd}\right)$

$= 1$ for even multiples of π $\left(\frac{n}{4} \text{ even}\right)$

Question 15 (c) (i)

Criteria	Marks
• Provides a correct solution	3
• Resolves forces in both directions and eliminates *r*, or equivalent merit	2
• Resolves forces in one direction, or equivalent merit	1

Sample answer:



Vertical $\quad kv^2 = T\sin\phi + mg$

$$T\sin\phi = kv^2 - mg$$

Horizontal $\quad T\cos\phi = \dfrac{mv^2}{r}$

$$= \frac{mv^2}{\ell\cos\phi}$$

$$T\cos^2\phi = \frac{mv^2}{\ell}$$

$$\therefore \frac{T\sin\phi}{T\cos^2\phi} = \frac{kv^2 - mg}{\frac{mv^2}{\ell}}$$

$$\text{ie } \frac{\sin\phi}{\cos^2\phi} = \frac{k\ell}{m} - \frac{g\ell}{v^2}$$

Question 15 (c) (ii)

Criteria	Marks
• Provides a correct solution	2
• Obtains a quadratic inequality in sinϕ and attempts to solve it, or equivalent merit	1

Sample answer:

$$\cos^2\phi > 0 \quad \text{so}$$

$$\begin{aligned}\sin\phi &< \frac{\ell k}{m}\cos^2\phi \\ &= \frac{\ell k}{m}\left(1-\sin^2\phi\right) \\ \therefore \frac{m}{\ell k}\sin\phi &< 1-\sin^2\phi\end{aligned}$$

m, ℓ, k all positive

$$\sin^2\phi + \frac{m}{\ell k}\sin\phi - 1 < 0$$

$$\begin{aligned}\sin\phi \ &< \ \frac{-\frac{m}{\ell k}+\sqrt{\frac{m^2}{\ell^2k^2}+4}}{2} \quad \left(\text{as } \phi < \frac{\pi}{2}\right) \\ &= \frac{\frac{-m}{\ell k}+\frac{1}{\ell k}\sqrt{m^2+4\ell^2k^2}}{2}\end{aligned}$$

$$\therefore \sin\phi < \frac{-m+\sqrt{m^2+4\ell^2k^2}}{2\ell k}$$

Question 15 (c) (iii)

Criteria	Marks
• Provides a correct solution	2
• Differentiates the given expression with respect to ϕ, or equivalent merit	1

Sample answer:

$$F(\phi) = \frac{\sin\phi}{\cos^2\phi}$$

$$F'(\phi) = \frac{\cos^3\phi + 2\sin^2\phi\cos\phi}{\cos^4\phi}$$

$$> 0\left(\text{as, for } -\frac{\pi}{2} < \phi < \frac{\pi}{2},\ \cos^3\phi > 0, \cos\phi > 0, \sin^2\phi > 0 \text{ and } \cos^4\phi > 0\right)$$

$\therefore F$ is increasing.

Question 15 (c) (iv)

Criteria	Marks
• Provides a correct explanation	1

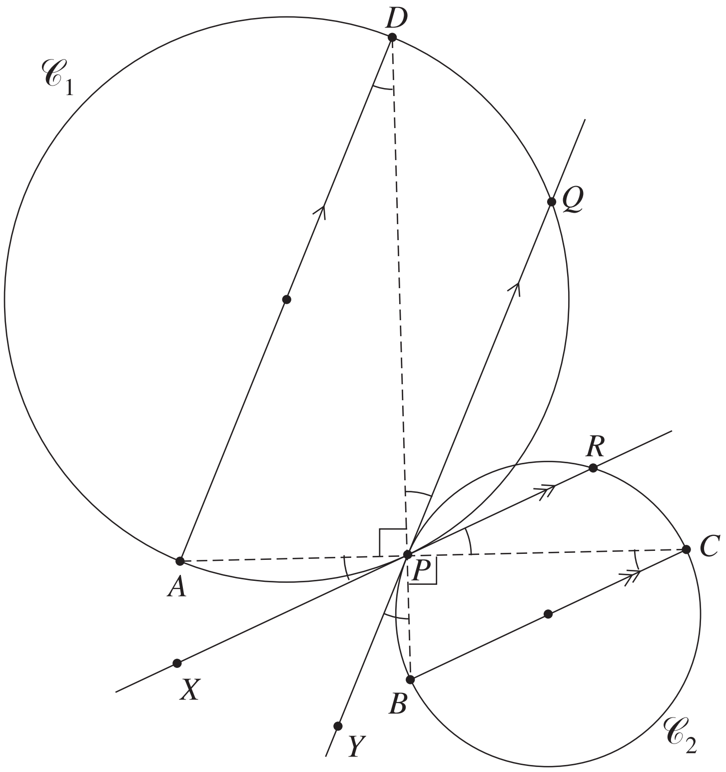
Sample answer:

As v increases $\dfrac{\ell g}{v^2}$ decreases, so $\dfrac{\ell k}{m} - \dfrac{\ell g}{v^2}$ increases.

That is, $\dfrac{\sin\phi}{\cos^2\phi}$ increases.

By (iii) $\dfrac{\sin\phi}{\cos^2\phi}$ is an increasing function and so ϕ increases as v increases.

Question 16 (a)



Question 16 (a) (i)

Criteria	Marks
• Provides a correct solution	2
• Uses angles on alternate segment, or equivalent merit	1

Sample answer:

$\angle APX = \angle ADP$ (The $\angle$ between a tangent and a chord equals the angle in the alternate segment).

$\angle ADP = \angle DPQ$ (Alternate $\angle$s, $AD \parallel PQ$)

$\therefore \angle APX = \angle DPQ$

Question 16 (a) (ii)

Criteria	Marks
• Provides a correct solution	3
• Makes significant progress towards the solution	2
• Observes result similar to that in part (i) in the other circle, or uses angle in semicircle, or equivalent merit	1

Sample answer:

Using a similar argument to that in part (i), it can be shown that

$\angle BPY = \angle BCP = \angle CPR$

$\angle APD = 90°$ (Angle in a semi-circle)

Similarly $\angle BPC = 90°$

$\angle XPQ = \angle YPR$ (Vertically opposite $\angle$s)

$\therefore\ \angle APX + \angle APD + \angle DPQ = \angle BPY + \angle BPC + \angle CPR$

$\therefore\ 2.\angle APX + 90° = 90° + 2.\angle CPR$

$\therefore \angle APX = \angle CPR$

$\therefore$ APC is a straight line (equal vertically opposite $\angle$s)

$\therefore$ A, P, C are collinear.

Question 16 (a) (iii)

Criteria	Marks
• Provides a correct solution	1

Sample answer:

From (i) and (ii)

$\angle APX = \angle CPR$ implies that

$\angle ADP = \angle BCP$

$\therefore$ A, B, C and D are concyclic points (equal $\angle$s subtended at C and D by interval AB on the same side)

$\therefore ABCD$ is a cyclic quadrilateral.

Question 16 (b) (i)

Criteria	Marks
• Provides a correct solution	3
• Correctly sums and simplifies the middle term, or equivalent merit	2
• Applies formula for geometric series, or equivalent merit	1

Sample answer:

$$\frac{1}{1+x^2} - \left(1 - x^2 + x^4 - x^6 + \ldots + (-1)^{n-1} x^{2(n-1)}\right)$$

$$= \frac{1}{1+x^2} - \frac{1-\left(-x^2\right)^n}{1+x^2} = \frac{1-1+\left(-x^2\right)^n}{1+x^2}, \text{ using sum of geometric series.}$$

$$= \frac{\left(-x^2\right)^n}{1+x^2}$$

Since $1+x^2 \geq 1$ for all x, $-x^{2n} \leq \dfrac{\left(-x^2\right)^n}{1+x^2} \leq x^{2n}$

We have

$$-x^{2n} \leq \frac{1}{1+x^2} - \left(1 - x^2 + x^4 \ldots + (-1)^{n-1} x^{2(n-1)}\right) \leq x^{2n}$$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Integrates some terms in the inequality in (i), or equivalent merit	1

Sample answer:

Inequalities are preserved by integration:

$$\pm \int_0^1 x^{2n}\, dx = \pm \left. \frac{x^{2n+1}}{2n+1} \right|_0^1 = \pm \frac{1}{2n+1}$$

$$\int_0^1 \frac{1}{1+x^2}\, dx = \tan^{-1} x \Big|_0^1 = \frac{\pi}{4} - 0 = \frac{\pi}{4}$$

$$\int_0^1 1 - x^2 + x^4 \ldots + (-1)^{n-1} x^{2n-2}\, dx$$

$$= \left[x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots + (-1)^{n-1} \frac{x^{2n-1}}{2n-1} \right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^{n-1} \frac{1}{2n-1}$$

so $-\dfrac{1}{2n+1} \le \dfrac{\pi}{4} - \left(1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots + (-1)^{n-1} \dfrac{1}{2n-1} \right) \le \dfrac{1}{2n+1}$

Question 16 (b) (iii)

Criteria	Marks
• Provides a correct explanation	1

Sample answer:

Since $\lim_{x \to \infty} -\frac{1}{2n+1} = \lim_{x \to \infty} \frac{1}{2n+1} = 0$,

then as $n \to \infty$

$$\frac{\pi}{4} - \left(1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots + (-1)^{n-1}\frac{1}{2n-1}\right) \to 0$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$$

Question 16 (c)

Criteria	Marks
• Provides a correct solution	3
• Makes substantial progress	2
• Does substitution $u = 1 + \ln x$ or $u = \ln x$, or equivalent merit	1

Sample answer:

$$I = \int \frac{\ln x}{(1+\ln x)^2}\,dx$$

Put $t = \ln x$, then $\dfrac{dt}{dx} = \dfrac{1}{x} = \dfrac{1}{e^t}$

$$I = \int \frac{te^t}{(1+t)^2}\,dt$$

$$= \int \frac{(1+t)e^t}{(1+t)^2}\,dt - \int \frac{e^t}{(1+t)^2}\,dt$$

$$= \int \frac{e^t}{1+t}\,dt - \left[\frac{-e^t}{1+t} + \int \frac{e^t}{1+t}\,dt\right] \quad \text{, by parts,}$$

$$= \frac{e^t}{1+t} + c$$

$$= \frac{x}{1+\ln x} + c$$

Mathematics Extension 2

2014 HSC Examination Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	7.6	E4
2	1	7.4	E4
3	1	3.1	E4
4	1	2.1, 2.4	E3
5	1	1.7	E6
6	1	5.1	E7
7	1	4.1	E8
8	1	2.3	E3
9	1	8	HE3, E9
10	1	8	E8, HE6

Section II

Question	Marks	Content	Syllabus outcomes
11 (a) (i)	2	2.2	E3
11 (a) (ii)	2	2.2	E3
11 (b)	3	4.1	E8
11 (c)	3	2.5	E3
11 (d)	2	1.1, 1.2	E6
11 (e)	3	5.1	E7
12 (a) (i)	2	1.3	E6
12 (a) (ii)	2	1.5	E6
12 (b) (i)	1	8.0	E2
12 (b) (ii)	2	8.0	E2
12 (c)	3	1.8	E4, E9
12 (d) (i)	1	4.1	E8
12 (d) (ii)	2	4.1	E8
12 (d) (iii)	2	4.1	E8
13 (a)	3	4.1	E8
13 (b)	4	5.1	E7
13 (c) (i)	1	3.2	E4
13 (c) (ii)	3	3.2	E4, E9
13 (c) (iii)	2	3.2	E4
13 (c) (iv)	2	3.2	E4

Question	Marks	Content	Syllabus outcomes
14 (a) (i)	2	7.2	E3
14 (a) (ii)	2	2.1,7.2	E3
14 (b) (i)	3	3.1	E3, E9
14 (b) (ii)	2	8.0	E2
14 (c) (i)	2	6.2.1	E5
14 (c) (ii)	4	6.2.1	E5
15 (a)	2	8.3	E2
15 (b) (i)	2	2.4, 8.0	E2, E3
15 (b) (ii)	3	2.4, 8.0	E2
15 (c) (i)	3	6.3.3	E5
15 (c) (ii)	2	8.3	E2
15 (c) (iii)	2	8.0	E2
15 (c) (iv)	1	6.3.3, 8.0	E5
16 (a) (i)	2	8.1	PE3, E2, E9
16 (a) (ii)	3	8.1	PE3, E2, E9
16 (a) (iii)	1	8.1	PE3, E2, E9
16 (b) (i)	3	8.3	E2
16 (b) (ii)	2	8.0	E2
16 (b) (iii)	1	8.0	E2
16 (c)	3	4.1	E8