BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2015 HIGHER SCHOOL CERTIFICATE EXAMINATION

# Mathematics Extension 1

**General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

**Total marks – 70**

**Section I** Pages 2–5

**10 marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II** Pages 6–13

**60 marks**

- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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**1** What is the remainder when $x^3 - 6x$ is divided by $x + 3$?

(A) $-9$

(B) $9$

(C) $x^2 - 2x$

(D) $x^2 - 3x + 3$

**2** Given that $N = 100 + 80e^{kt}$, which expression is equal to $\frac{dN}{dt}$?

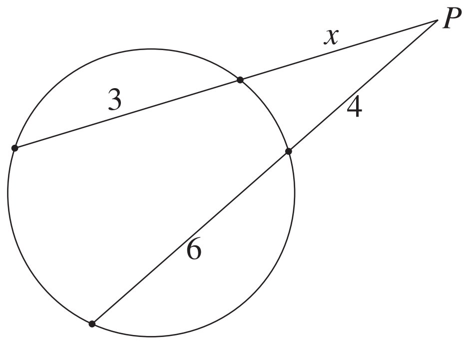
(A) $k(100 - N)$

(B) $k(180 - N)$

(C) $k(N - 100)$

(D) $k(N - 180)$

**3** Two secants from the point $P$ intersect a circle as shown in the diagram.

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What is the value of $x$?

(A) 2

(B) 5

(C) 7

(D) 8

**4** A rowing team consists of 8 rowers and a coxswain.

The rowers are selected from 12 students in Year 10.

The coxswain is selected from 4 students in Year 9.

In how many ways could the team be selected?

(A) ${}^{12}C_8 + {}^{4}C_1$

(B) ${}^{12}P_8 + {}^{4}P_1$

(C) ${}^{12}C_8 \times {}^{4}C_1$

(D) ${}^{12}P_8 \times {}^{4}P_1$

**5** What are the asymptotes of $y = \dfrac{3x}{(x+1)(x+2)}$ ?

(A) $y = 0, \quad x = -1, \quad x = -2$

(B) $y = 0, \quad x = 1, \quad x = 2$

(C) $y = 3, \quad x = -1, \quad x = -2$

(D) $y = 3, \quad x = 1, \quad x = 2$

**6** What is the domain of the function $f(x) = \sin^{-1}(2x)$?

(A) $-\pi \le x \le \pi$

(B) $-2 \le x \le 2$

(C) $-\dfrac{\pi}{4} \le x \le \dfrac{\pi}{4}$

(D) $-\dfrac{1}{2} \le x \le \dfrac{1}{2}$

**7** What is the value of $k$ such that $\int_0^k \frac{1}{\sqrt{4-x^2}}\,dx = \frac{\pi}{3}$ ?

(A) 1

(B) $\sqrt{3}$

(C) 2

(D) $2\sqrt{3}$

**8** What is the value of $\lim_{x \to 3} \frac{\sin(x-3)}{(x-3)(x+2)}$ ?

(A) 0

(B) $\frac{1}{5}$

(C) 5

(D) Undefined

**9** Two particles oscillate horizontally. The displacement of the first is given by $x = 3\sin 4t$ and the displacement of the second is given by $x = a\sin nt$. In one oscillation, the second particle covers twice the distance of the first particle, but in half the time.

What are the values of $a$ and $n$?

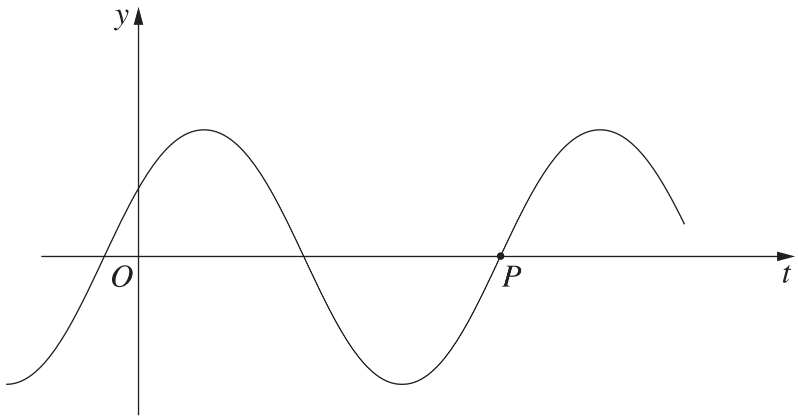
(A) $a = 1.5, \quad n = 2$

(B) $a = 1.5, \quad n = 8$

(C) $a = 6, \quad n = 2$

(D) $a = 6, \quad n = 8$

**10** The graph of the function $y = \cos\left(2t - \frac{\pi}{3}\right)$ is shown below.




What are the coordinates of the point $P$?

(A) $\left(\frac{5\pi}{12}, 0\right)$

(B) $\left(\frac{2\pi}{3}, 0\right)$

(C) $\left(\frac{11\pi}{12}, 0\right)$

(D) $\left(\frac{7\pi}{6}, 0\right)$

## Section II

**60 marks**
**Attempt Questions 11–14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \sin^2 x \, dx$. **2**

(b) Calculate the size of the acute angle between the lines $y = 2x + 5$ and $y = 4 - 3x$. **2**

(c) Solve the inequality $\dfrac{4}{x+3} \geq 1$. **3**

(d) Express $5\cos x - 12\sin x$ in the form $A\cos(x + \alpha)$, where $0 \leq \alpha \leq \dfrac{\pi}{2}$. **2**

(e) Use the substitution $u = 2x - 1$ to evaluate $\displaystyle\int_1^2 \frac{x}{(2x-1)^2}\,dx$. **3**

(f) Consider the polynomials $P(x) = x^3 - kx^2 + 5x + 12$ and $A(x) = x - 3$.

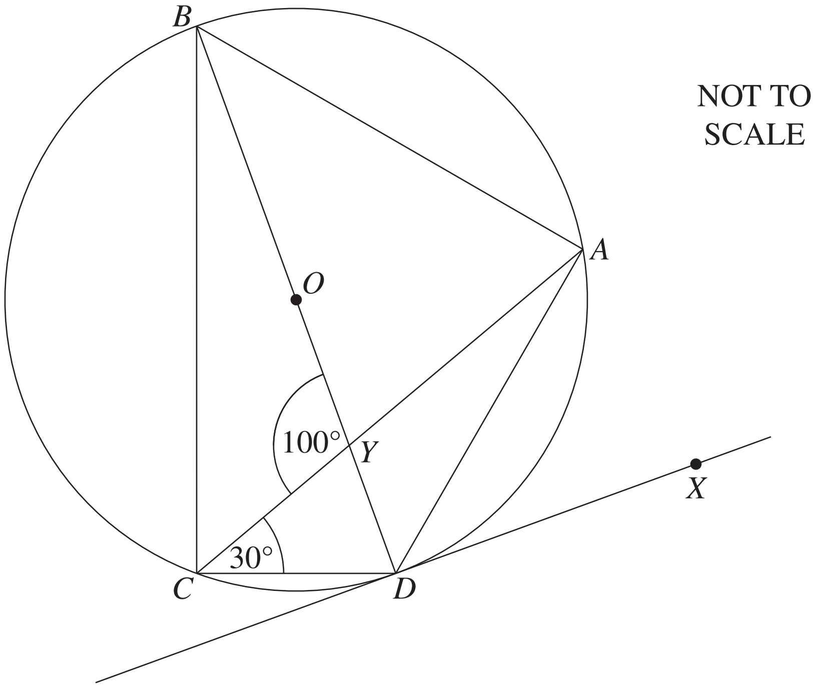
(i) Given that $P(x)$ is divisible by $A(x)$, show that $k = 6$. **1**

(ii) Find all the zeros of $P(x)$ when $k = 6$. **2**

**Question 12** (15 marks) Use a SEPARATE writing booklet.

(a) In the diagram, the points $A$, $B$, $C$ and $D$ are on the circumference of a circle, whose centre $O$ lies on $BD$. The chord $AC$ intersects the diameter $BD$ at $Y$. The tangent at $D$ passes through the point $X$.

It is given that $\angle CYB = 100°$ and $\angle DCY = 30°$.




Copy or trace the diagram into your writing booklet.

(i) What is the size of $\angle ACB$? **1**

(ii) What is the size of $\angle ADX$? **1**

(iii) Find, giving reasons, the size of $\angle CAB$. **2**

**Question 12 continues on page 8**

Question 12 (continued)

(b) The points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola $x^2 = 4ay$.

The equation of the chord $PQ$ is given by $(p + q)x - 2y - 2apq = 0$. (Do NOT prove this.)

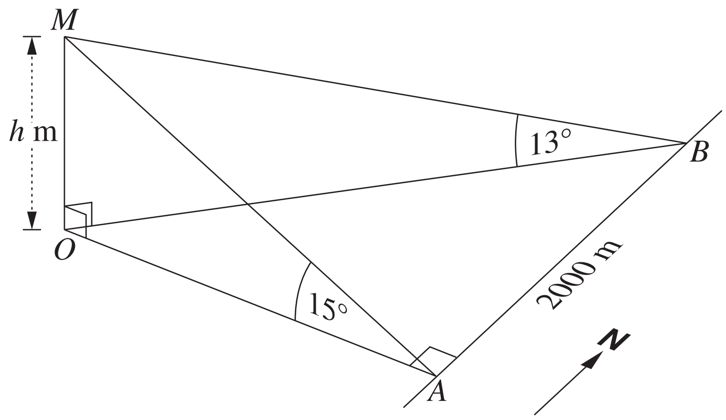
(i) Show that if $PQ$ is a focal chord then $pq = -1$. **1**

(ii) If $PQ$ is a focal chord and $P$ has coordinates $(8a, 16a)$, what are the coordinates of $Q$ in terms of $a$? **2**

(c) A person walks 2000 metres due north along a road from point $A$ to point $B$. The point $A$ is due east of a mountain $OM$, where $M$ is the top of the mountain. The point $O$ is directly below point $M$ and is on the same horizontal plane as the road. The height of the mountain above point $O$ is $h$ metres.

From point $A$, the angle of elevation to the top of the mountain is 15°.

From point $B$, the angle of elevation to the top of the mountain is 13°.

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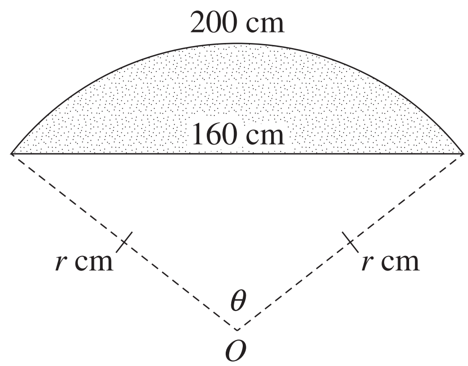
(i) Show that $OA = h \cot 15°$. **1**

(ii) Hence, find the value of $h$. **2**

**Question 12 continues on page 9**

(d) A kitchen bench is in the shape of a segment of a circle. The segment is bounded by an arc of length 200 cm and a chord of length 160 cm. The radius of the circle is $r$ cm and the chord subtends an angle $\theta$ at the centre $O$ of the circle.

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(i) Show that $160^2 = 2r^2(1 - \cos\theta)$. **1**

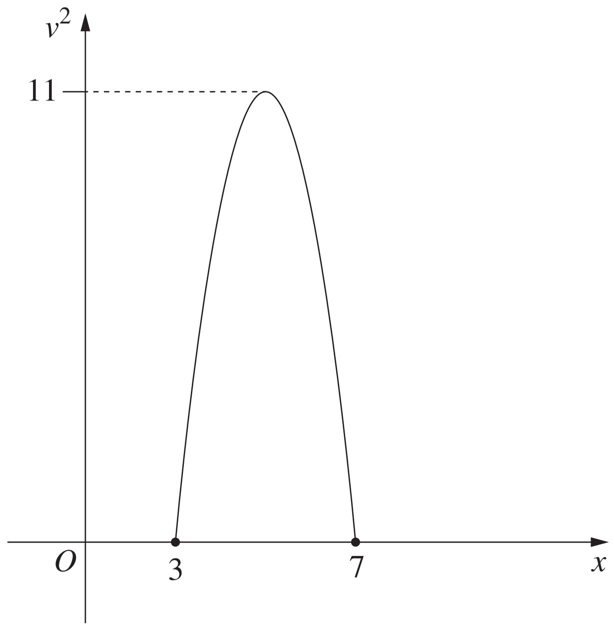
(ii) Hence, or otherwise, show that $8\theta^2 + 25\cos\theta - 25 = 0$. **2**

(iii) Taking $\theta_1 = \pi$ as a first approximation to the value of $\theta$, use one application of Newton's method to find a second approximation to the value of $\theta$. Give your answer correct to two decimal places. **2**

**End of Question 12**

**Question 13** (15 marks) Use a SEPARATE writing booklet.

(a) A particle is moving along the $x$-axis in simple harmonic motion. The displacement of the particle is $x$ metres and its velocity is $v$ m s$^{-1}$. The parabola below shows $v^2$ as a function of $x$.




(i) For what value(s) of $x$ is the particle at rest? **1**

(ii) What is the maximum speed of the particle? **1**

(iii) The velocity $v$ of the particle is given by the equation **3**

$$v^2 = n^2\left(a^2 - (x - c)^2\right)$$

where $a$, $c$ and $n$ are positive constants.

What are the values of $a$, $c$ and $n$?

**Question 13 continues on page 11**

Question 13 (continued)

(b) Consider the binomial expansion

$$\left(2x + \frac{1}{3x}\right)^{18} = a_0 x^{18} + a_1 x^{16} + a_2 x^{14} + \cdots$$

where $a_0, a_1, a_2, \ldots$ are constants.

(i) Find an expression for $a_2$. **2**

(ii) Find an expression for the term independent of $x$. **2**

(c) Prove by mathematical induction that for all integers $n \geq 1$, **3**

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

(d) Let $f(x) = \cos^{-1}(x) + \cos^{-1}(-x)$, where $-1 \leq x \leq 1$.

(i) By considering the derivative of $f(x)$, prove that $f(x)$ is constant. **2**

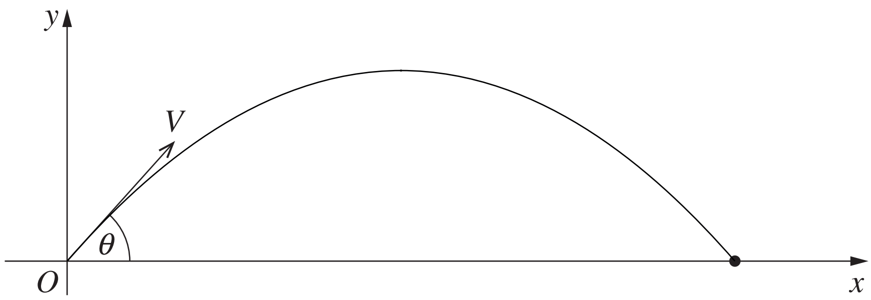
(ii) Hence deduce that $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$. **1**

**End of Question 13**

**Question 14** (15 marks) Use a SEPARATE writing booklet.

(a) A projectile is fired from the origin $O$ with initial velocity $V$ m s$^{-1}$ at an angle $\theta$ to the horizontal. The equations of motion are given by

$$x = Vt\cos\theta, \quad y = Vt\sin\theta - \frac{1}{2}gt^2. \quad \text{(Do NOT prove this.)}$$




(i) Show that the horizontal range of the projectile is $\dfrac{V^2\sin 2\theta}{g}$. **2**

A particular projectile is fired so that $\theta = \dfrac{\pi}{3}$.

(ii) Find the angle that this projectile makes with the horizontal when $t = \dfrac{2V}{\sqrt{3}\,g}$. **2**

(iii) State whether this projectile is travelling upwards or downwards when $t = \dfrac{2V}{\sqrt{3}\,g}$. Justify your answer. **1**

**Question 14 continues on page 13**

Question 14 (continued)

(b) A particle is moving horizontally. Initially the particle is at the origin $O$ moving with velocity $1\ \text{m s}^{-1}$.

The acceleration of the particle is given by $\ddot{x} = x - 1$, where $x$ is its displacement at time $t$.

(i) Show that the velocity of the particle is given by $\dot{x} = 1 - x$. **3**

(ii) Find an expression for $x$ as a function of $t$. **2**

(iii) Find the limiting position of the particle. **1**

(c) Two players $A$ and $B$ play a series of games against each other to get a prize. In any game, either of the players is equally likely to win.

To begin with, the first player who wins a total of 5 games gets the prize.

(i) Explain why the probability of player $A$ getting the prize in exactly 7 games is $\binom{6}{4}\left(\frac{1}{2}\right)^7$. **1**

(ii) Write an expression for the probability of player $A$ getting the prize in at most 7 games. **1**

(iii) Suppose now that the prize is given to the first player to win a total of $(n + 1)$ games, where $n$ is a positive integer. **2**

By considering the probability that $A$ gets the prize, prove that

$$\binom{n}{n}2^n + \binom{n+1}{n}2^{n-1} + \binom{n+2}{n}2^{n-2} + \cdots + \binom{2n}{n} = 2^{2n}.$$

**End of paper**

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## STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$