BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2015 HIGHER SCHOOL CERTIFICATE EXAMINATION

# Mathematics Extension 2

**General Instructions**

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

**Total marks – 100**

**Section I** Pages 2–6

**10 marks**

- Attempt Questions 1–10
- Allow about 15 minutes for this section

**Section II** Pages 7–18

**90 marks**

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

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## Section I

**10 marks**
**Attempt Questions 1–10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1–10.

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**1** Which conic has eccentricity $\frac{\sqrt{13}}{3}$ ?

(A) $\frac{x^2}{3}+\frac{y^2}{2}=1$

(B) $\frac{x^2}{3^2}+\frac{y^2}{2^2}=1$

(C) $\frac{x^2}{3}-\frac{y^2}{2}=1$

(D) $\frac{x^2}{3^2}-\frac{y^2}{2^2}=1$

**2** What value of $z$ satisfies $z^2 = 7 - 24i$?

(A) $4 - 3i$

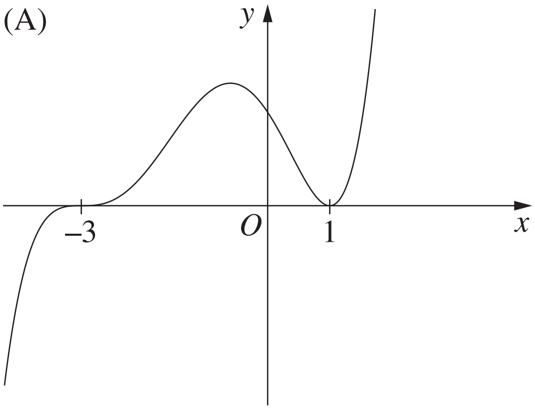
(B) $-4 - 3i$

(C) $3 - 4i$

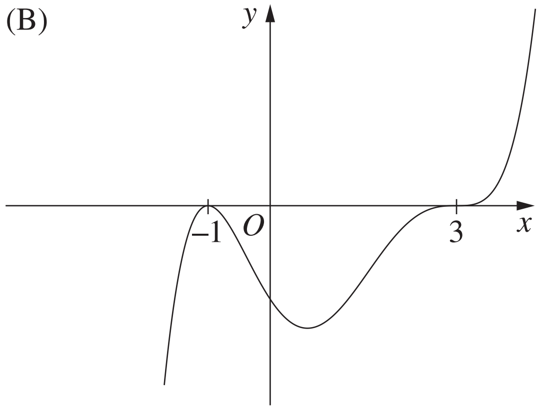
(D) $-3 - 4i$

**3** Which graph best represents the curve $y = (x-1)^2(x+3)^5$ ?

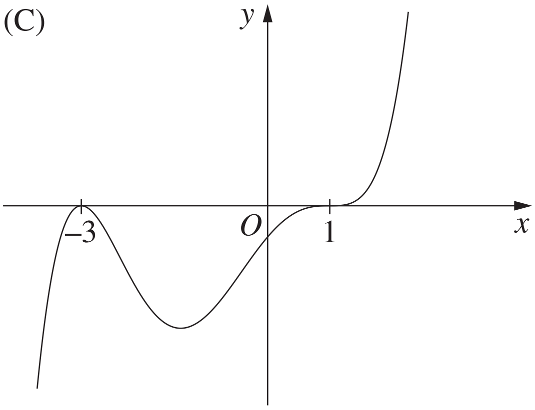
(A)



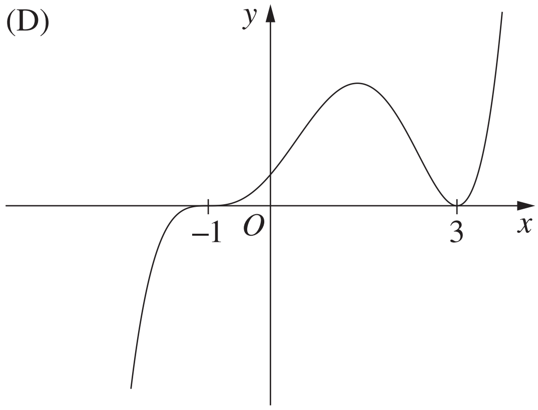

(B)




(C)




(D)




**4** The polynomial $x^3 + x^2 - 5x + 3$ has a double root at $x = \alpha$.

What is the value of $\alpha$?

(A) $-\dfrac{5}{3}$

(B) $-1$

(C) $1$

(D) $\dfrac{5}{3}$

**5** Given that $z = 1 - i$, which expression is equal to $z^3$?

(A) $\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

(B) $2\sqrt{2}\left(\cos\left(\frac{-3\pi}{4}\right) + i\sin\left(\frac{-3\pi}{4}\right)\right)$

(C) $\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

(D) $2\sqrt{2}\left(\cos\left(\frac{3\pi}{4}\right) + i\sin\left(\frac{3\pi}{4}\right)\right)$

**6** Which expression is equal to $\int x^2 \sin x \, dx$?

(A) $-x^2 \cos x - \int 2x \cos x \, dx$

(B) $-2x \cos x + \int x^2 \cos x \, dx$

(C) $-x^2 \cos x + \int 2x \cos x \, dx$

(D) $-2x \cos x - \int x^2 \cos x \, dx$

**7** The numbers 1, 2, … $n$, for $n \geq 4$, are randomly arranged in a row.

What is the probability that the number 1 is somewhere to the left of the number 2?

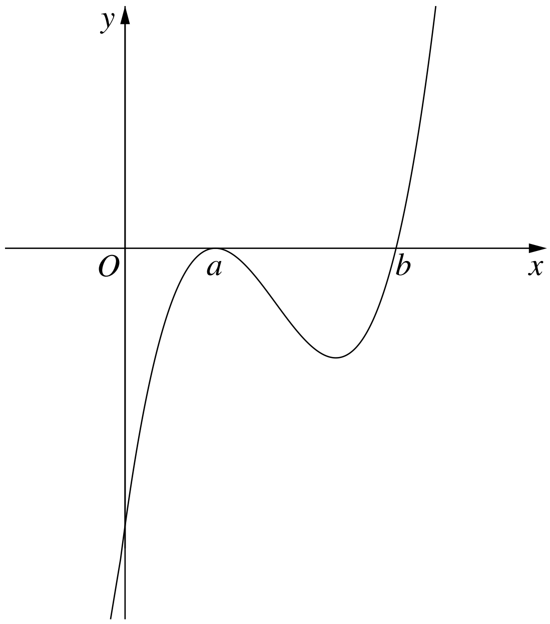
(A) $\frac{1}{2}$

(B) $\frac{1}{n}$

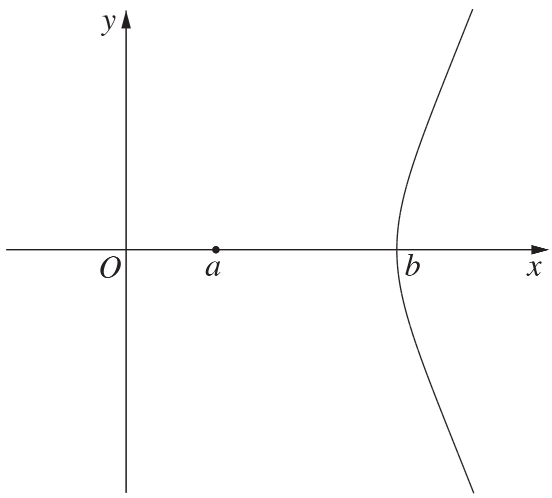
(C) $\frac{1}{2(n-2)!}$

(D) $\frac{1}{2(n-1)!}$

**8** The graph of the function $y = f(x)$ is shown.




A second graph is obtained from the function $y = f(x)$.




Which equation best represents the second graph?

(A) $y^2 = |f(x)|$

(B) $y^2 = f(x)$

(C) $y = \sqrt{f(x)}$

(D) $y = f\left(\sqrt{x}\right)$

**9** The complex number $z$ satisfies $|z-1|=1$.

What is the greatest distance that $z$ can be from the point $i$ on the Argand diagram?

(A) $1$

(B) $\sqrt{5}$

(C) $2\sqrt{2}$

(D) $\sqrt{2}+1$

**10** Consider the expansion of

$$\left(1+x+x^2+\cdots+x^n\right)\left(1+2x+3x^2+\cdots+(n+1)x^n\right).$$

What is the coefficient of $x^n$ when $n = 100$?

(A) 4950

(B) 5050

(C) 5151

(D) 5253

## Section II

**90 marks**
**Attempt Questions 11–16**
**Allow about 2 hours and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a SEPARATE writing booklet.

(a) Express $\dfrac{4+3i}{2-i}$ in the form $x+iy$, where $x$ and $y$ are real. **2**

(b) Consider the complex numbers $z=-\sqrt{3}+i$ and $w=3\left(\cos\dfrac{\pi}{7}+i\sin\dfrac{\pi}{7}\right)$.

(i) Evaluate $|z|$. **1**

(ii) Evaluate $\arg(z)$. **1**

(iii) Find the argument of $\dfrac{z}{w}$. **1**

(c) Find $A$, $B$ and $C$ such that $\dfrac{1}{x\left(x^2+2\right)}=\dfrac{A}{x}+\dfrac{Bx+C}{x^2+2}$. **2**

(d) Sketch $\dfrac{x^2}{25}+\dfrac{y^2}{16}=1$ indicating the coordinates of the foci. **2**

(e) Find the value of $\dfrac{dy}{dx}$ at the point $(2,-1)$ on the curve $x+x^2y^3=-2$. **3**

(f) (i) Show that $\cot\theta+\operatorname{cosec}\theta=\cot\left(\dfrac{\theta}{2}\right)$. **2**

(ii) Hence, or otherwise, find $\displaystyle\int(\cot\theta+\operatorname{cosec}\theta)\,d\theta$. **1**

**Question 12** (15 marks) Use a SEPARATE writing booklet.

(a) The complex number $z$ is such that $|z| = 2$ and $\arg(z) = \dfrac{\pi}{4}$.

Plot each of the following complex numbers on the same half-page Argand diagram.

(i) $z$ **1**

(ii) $u = z^2$ **1**

(iii) $v = z^2 - \overline{z}$ **1**

(b) The polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has roots $a + ib$ and $a + 2ib$ where $a$ and $b$ are real and $b \neq 0$.

(i) By evaluating $a$ and $b$, find all the roots of $P(x)$. **3**

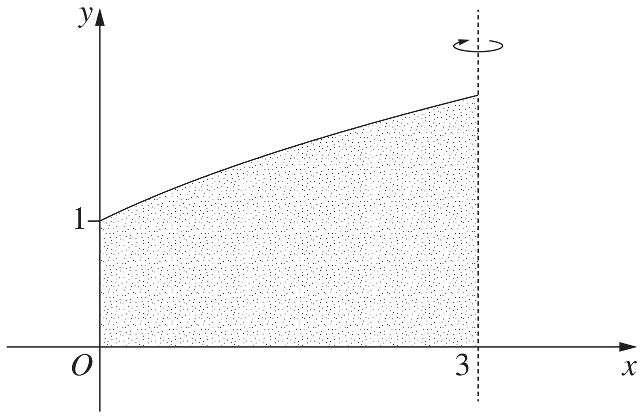
(ii) Hence, or otherwise, find one quadratic polynomial with real coefficients that is a factor of $P(x)$. **1**

(c) (i) By writing $\dfrac{(x-2)(x-5)}{x-1}$ in the form $mx + b + \dfrac{a}{x-1}$, find the equation of the oblique asymptote of $y = \dfrac{(x-2)(x-5)}{x-1}$. **2**

(ii) Hence sketch the graph $y = \dfrac{(x-2)(x-5)}{x-1}$, clearly indicating all intercepts and asymptotes. **2**

**Question 12 continues on page 9**

(d) The diagram shows the graph $y = \sqrt{x+1}$ for $0 \le x \le 3$. The shaded region is rotated about the line $x = 3$ to form a solid. **4**




Use the method of cylindrical shells to find the volume of the solid.

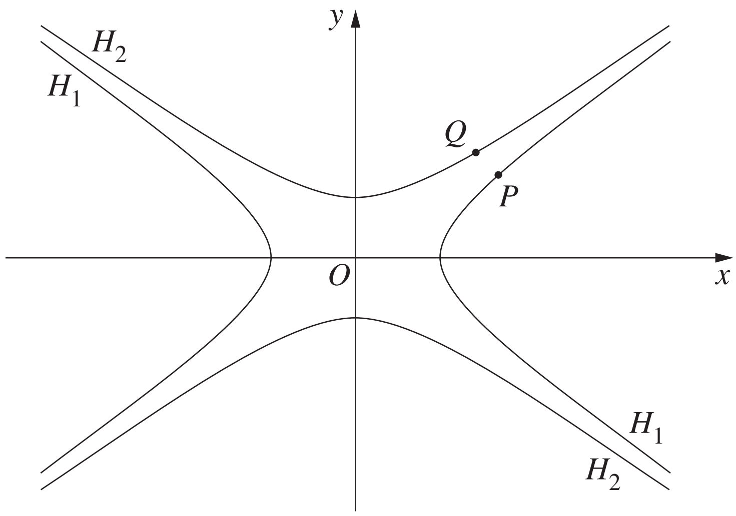
**End of Question 12**

**Question 13** (15 marks) Use a SEPARATE writing booklet.

(a) The hyperbolas $H_1 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $H_2 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ are shown in the diagram.

Let $P(a\sec\theta,\ b\tan\theta)$ lie on $H_1$ as shown on the diagram.

Let $Q$ be the point $(a\tan\theta,\ b\sec\theta)$.




(i) Verify that the coordinates of $Q(a\tan\theta,\ b\sec\theta)$ satisfy the equation for $H_2$. **1**

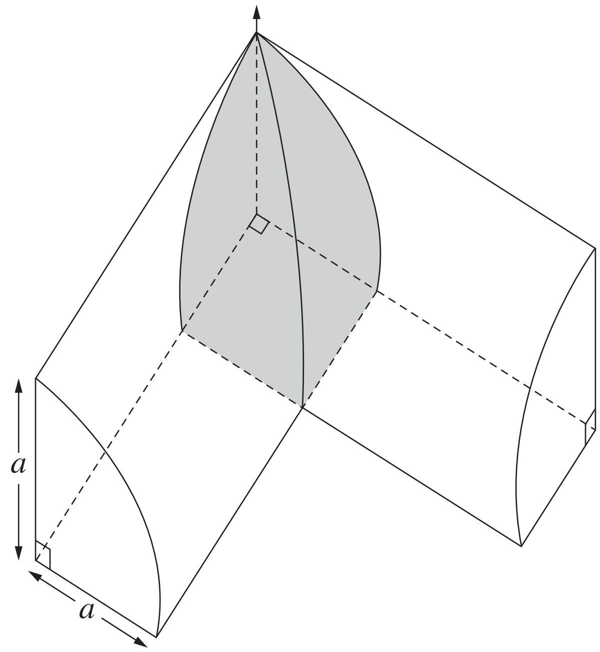
(ii) Show that the equation of the line $PQ$ is $bx + ay = ab(\tan\theta + \sec\theta)$. **2**

(iii) Prove that the area of $\triangle OPQ$ is independent of $\theta$. **3**

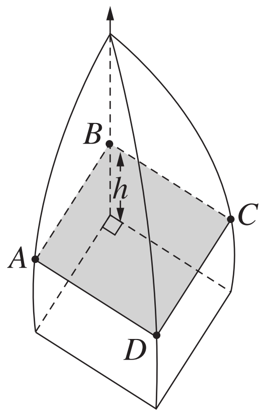
**Question 13 continues on page 11**

Question 13 (continued)

(b) Two quarter cylinders, each of radius $a$, intersect at right angles to form the shaded solid.




A horizontal slice $ABCD$ of the solid is taken at height $h$ from the base. You may assume that $ABCD$ is a square, and is parallel to the base.




(i) Show that $AB = \sqrt{a^2 - h^2}$ . **1**

(ii) Find the volume of the solid. **2**

**Question 13 continues on page 12**

(c) A small spherical balloon is released and rises into the air. At time $t$ seconds, it has radius $r$ cm, surface area $S = 4\pi r^2$ and volume $V = \frac{4}{3}\pi r^3$.

As the balloon rises it expands, causing its surface area to increase at a rate of $\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$ $\text{cm}^2\ \text{s}^{-1}$. As the balloon expands it maintains a spherical shape.

(i) By considering the surface area, show that $\frac{dr}{dt} = \frac{1}{8\pi r}\left(\frac{4}{3}\pi\right)^{\frac{1}{3}}$. **2**

(ii) Show that $\frac{dV}{dt} = \frac{1}{2}V^{\frac{1}{3}}$. **2**

(iii) When the balloon is released its volume is 8000 $\text{cm}^3$. When the volume of the balloon reaches 64 000 $\text{cm}^3$ it will burst. **2**

How long after it is released will the balloon burst?

**End of Question 13**

**Question 14** (15 marks) Use a SEPARATE writing booklet.

(a) (i) Differentiate $\sin^{n-1}\theta\cos\theta$, expressing the result in terms of $\sin\theta$ only. **2**

(ii) Hence, or otherwise, deduce that $\displaystyle\int_0^{\frac{\pi}{2}} \sin^n\theta\, d\theta = \frac{(n-1)}{n}\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta\, d\theta$, **2**

for $n > 1$.

(iii) Find $\displaystyle\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta$. **1**

(b) The cubic equation $x^3 - px + q = 0$ has roots $\alpha$, $\beta$ and $\gamma$.

It is given that $\alpha^2 + \beta^2 + \gamma^2 = 16$ and $\alpha^3 + \beta^3 + \gamma^3 = -9$.

(i) Show that $p = 8$. **1**

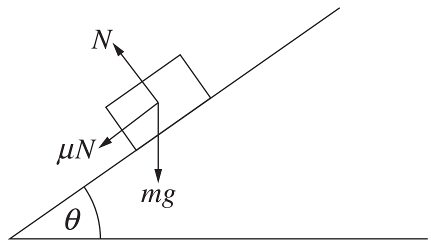
(ii) Find the value of $q$. **2**

(iii) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. **2**

**Question 14 continues on page 14**

Question 14 (continued)

(c) A car of mass $m$ is driven at speed $v$ around a circular track of radius $r$. The track is banked at a constant angle $\theta$ to the horizontal, where $0 < \theta < \frac{\pi}{2}$. At the speed $v$ there is a tendency for the car to slide up the track. This is opposed by a frictional force $\mu N$, where $N$ is the normal reaction between the car and the track, and $\mu > 0$. The acceleration due to gravity is $g$.




(i) Show that $v^2 = rg\left(\dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}\right)$. **3**

(ii) At the particular speed $V$, where $V^2 = rg$, there is still a tendency for the car to slide up the track. **2**

Using the result from part (i), or otherwise, show that $\mu < 1$.

**End of Question 14**

**Question 15** (15 marks) Use a SEPARATE writing booklet.

(a) A particle $A$ of unit mass travels horizontally through a viscous medium. When $t = 0$, the particle is at point $O$ with initial speed $u$. The resistance on particle $A$ due to the medium is $kv^2$, where $v$ is the velocity of the particle at time $t$ and $k$ is a positive constant.

When $t = 0$, a second particle $B$ of equal mass is projected vertically upwards from $O$ with the same initial speed $u$ through the same medium. It experiences both a gravitational force and a resistance due to the medium. The resistance on particle $B$ is $kw^2$, where $w$ is the velocity of the particle $B$ at time $t$. The acceleration due to gravity is $g$.

(i) Show that the velocity $v$ of particle $A$ is given by $\frac{1}{v} = kt + \frac{1}{u}$. **2**

(ii) By considering the velocity $w$ of particle $B$, show that **3**

$$t = \frac{1}{\sqrt{gk}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right).$$

(iii) Show that the velocity $V$ of particle $A$ when particle $B$ is at rest is given by **1**

$$\frac{1}{V} = \frac{1}{u} + \sqrt{\frac{k}{g}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right).$$

(iv) Hence, if $u$ is very large, explain why $V \approx \frac{2}{\pi}\sqrt{\frac{g}{k}}$. **1**

**Question 15 continues on page 16**

(b) Suppose that $x \geq 0$ and $n$ is a positive integer.

(i) Show that $1 - x \leq \frac{1}{1+x} \leq 1$. **2**

(ii) Hence, or otherwise, show that $1 - \frac{1}{2n} \leq n \ln\left(1 + \frac{1}{n}\right) \leq 1$. **2**

(iii) Hence, explain why $\lim_{n \to \infty}\left(1 + \frac{1}{n}\right)^n = e$. **1**

(c) For positive real numbers $x$ and $y$, $\sqrt{xy} \leq \frac{x+y}{2}$. (Do NOT prove this.)

(i) Prove $\sqrt{xy} \leq \sqrt{\frac{x^2 + y^2}{2}}$, for positive real numbers $x$ and $y$. **1**

(ii) Prove $\sqrt[4]{abcd} \leq \sqrt{\frac{a^2 + b^2 + c^2 + d^2}{4}}$, for positive real numbers $a$, $b$, $c$ and $d$. **2**

**End of Question 15**

**Question 16** (15 marks) Use a SEPARATE writing booklet.

(a) (i) A table has 3 rows and 5 columns, creating 15 cells as shown. **2**

| | | | | |
|---|---|---|---|---|
| | | | | |
| | | | | |

Counters are to be placed randomly on the table so that there is one counter in each cell. There are 5 identical black counters and 10 identical white counters.

Show that the probability that there is exactly one black counter in each column is $\dfrac{81}{1001}$.

(ii) The table is extended to have $n$ rows and $q$ columns. There are $nq$ counters, where $q$ are identical black counters and the remainder are identical white counters. The counters are placed randomly on the table with one counter in each cell. **2**

Let $P_n$ be the probability that each column contains exactly one black counter.

Show that $P_n = \dfrac{n^q}{\dbinom{nq}{q}}$.

(iii) Find $\lim\limits_{n \to \infty} P_n$. **2**

**Question 16 continues on page 18**

Question 16 (continued)

(b) Let $n$ be a positive integer.

(i) By considering $(\cos\alpha + i\sin\alpha)^{2n}$, show that **2**

$$\cos(2n\alpha) = \cos^{2n}\alpha - \binom{2n}{2}\cos^{2n-2}\alpha\sin^2\alpha + \binom{2n}{4}\cos^{2n-4}\alpha\sin^4\alpha - \cdots$$

$$+ \cdots + (-1)^{n-1}\binom{2n}{2n-2}\cos^2\alpha\ \sin^{2n-2}\alpha + (-1)^n\sin^{2n}\alpha.$$

Let $T_{2n}(x) = \cos\left(2n\cos^{-1}x\right)$, for $-1 \le x \le 1$.

(ii) Show that **2**

$$T_{2n}(x) = x^{2n} - \binom{2n}{2}x^{2n-2}\left(1-x^2\right) + \binom{2n}{4}x^{2n-4}\left(1-x^2\right)^2 + \cdots + (-1)^n\left(1-x^2\right)^n.$$

(iii) By considering the roots of $T_{2n}(x)$, find the value of **3**

$$\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{3\pi}{4n}\right)\cdots\cos\left(\frac{(4n-1)\pi}{4n}\right).$$

(iv) Prove that **2**

$$1 - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + \cdots + (-1)^n\binom{2n}{2n} = 2^n\cos\left(\frac{n\pi}{2}\right).$$

**End of paper**

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## STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$