BOSTES | BOARD OF STUDIES TEACHING & EDUCATIONAL STANDARDS NSW

2015 HSC Mathematics Extension 2 Marking Guidelines

Section I

Multiple-choice Answer Key

Question	Answer
1	D
2	A
3	A
4	C
5	B
6	C
7	A
8	B
9	D
10	C

Section II

Question 11 (a)

Criteria	Marks
• Provides correct solution	2
• Attempts to use conjugate of denominator, or equivalent merit	1

Sample answer:

$$\frac{4+3i}{2-i} = \frac{(4+3i)}{(2-i)} \times \frac{(2+i)}{(2+i)} = \frac{8+6i+4i-3}{4+1}$$

$$= \frac{5+10i}{5}$$

$$= 1+2i$$

Question 11 (b) (i)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$|z| = \left|-\sqrt{3}+i\right|$$

$$= \sqrt{\left(-\sqrt{3}\right)^2 + 1}$$

$$= \sqrt{3+1}$$

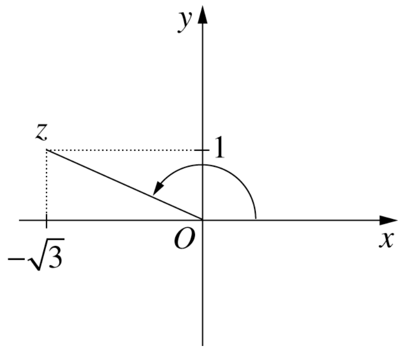
$$= 2$$

Question 11 (b) (ii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\arg z = \pi + \tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)$$
$$= \pi - \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$
$$= \frac{5\pi}{6}$$



Question 11 (b) (iii)

Criteria	Marks
• Provides correct answer	1

Sample answer:

$$\arg\left(\frac{z}{w}\right) = \arg z - \arg w$$
$$= \frac{5\pi}{6} - \frac{\pi}{7}$$
$$= \frac{29\pi}{42}$$

Question 11 (c)

Criteria	Marks
• Provides correct solution	2
• Finds A, or equivalent merit	1

Sample answer:

$$\frac{1}{x\left(x^2+2\right)}=\frac{A}{x}+\frac{Bx+C}{x^2+2}$$

$$1=A\left(x^2+2\right)+(Bx+C)x$$

$$1=Ax^2+2A+Bx^2+Cx$$

Equating constant terms,

$1=2A$

$A=\frac{1}{2}$

Equating coefficients of x^2,

$O=A+B$

$O=\frac{1}{2}+B$

$B=-\frac{1}{2}$

Equating coefficients of x,

$C=0$

Question 11 (d)

Criteria	Marks
• Provides correct sketch	2
• Finds eccentricity, or equivalent merit	1

Sample answer:

For an ellipse $b^2 = a^2\left(1-e^2\right)$

$$\Rightarrow \quad 16 = 25\left(1-e^2\right)$$

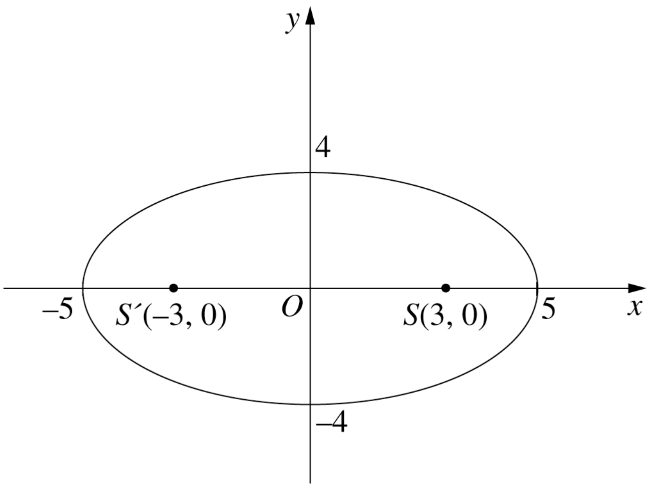
$$\frac{16}{25} = 1-e^2$$

$$e^2 = 1-\frac{16}{25}$$

$$e = \frac{3}{5}$$

$$\therefore \text{Foci } (\pm ae,\ 0) = \left(\pm 5\times\frac{3}{5},\ 0\right)$$

$$= (\pm 3,\ 0)$$



Question 11 (e)

Criteria	Marks
• Provides correct solution	3
• Correctly differentiates, or equivalent merit	2
• Uses the chain rule, or equivalent merit	1

Sample answer:

Differentiating with respect to x

$$1 + 2xy^3 + x^2 3y^2 \frac{dy}{dx} = 0$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{-1 - 2xy^3}{x^2 3y^2}$$

At $(2,\ -1)$

$$\frac{dy}{dx} = \frac{-1 - 2 \times 2 \times (-1)^3}{2^2 \times 3 \times (-1)^2}$$

$$= \frac{1}{4}$$

Question 11 (f) (i)

Criteria	Marks
• Provides correct solution	2
• Expresses $\cot\theta$ or $\operatorname{cosec}\theta$ in terms of $\tan\dfrac{\theta}{2}$, or equivalent merit	1

Sample answer:

$$LHS = \cot\theta + \operatorname{cosec}\theta \qquad \text{Let } t = \tan\frac{\theta}{2}$$

$$= \frac{1-t^2}{2t} + \frac{1+t^2}{2t}$$

$$= \frac{2}{2t}$$

$$= \frac{1}{t}$$

$$= \cot\left(\frac{\theta}{2}\right)$$

$$= RHS$$

Question 11 (f) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\int (\cot\theta + \operatorname{cosec}\theta)\,d\theta$$

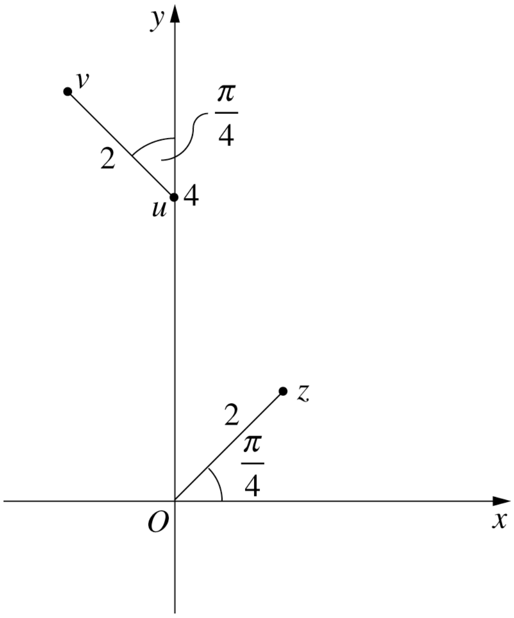
$$= \int \cot\frac{\theta}{2}\,d\theta$$

$$= \int \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta$$

$$= \frac{1}{\frac{1}{2}}\ln\left|\sin\frac{\theta}{2}\right| + c$$

$$= 2\ln\left|\sin\frac{\theta}{2}\right| + c$$

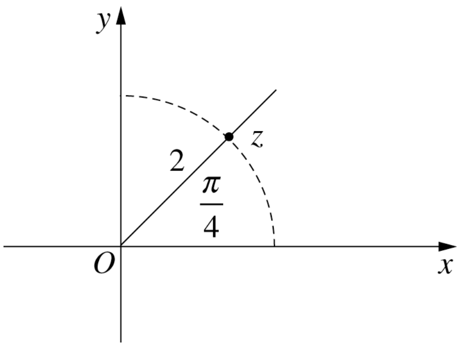
Question 12 (a) (i), (ii), (iii)



Question 12 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:



Question 12 (a) (ii)

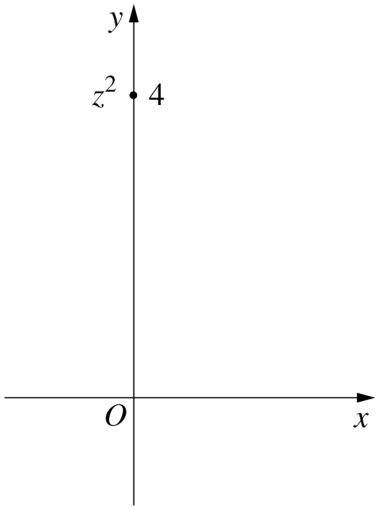
Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\arg\left(z^2\right) = \frac{\pi}{2}$$

$$\left|z^2\right| = 4$$

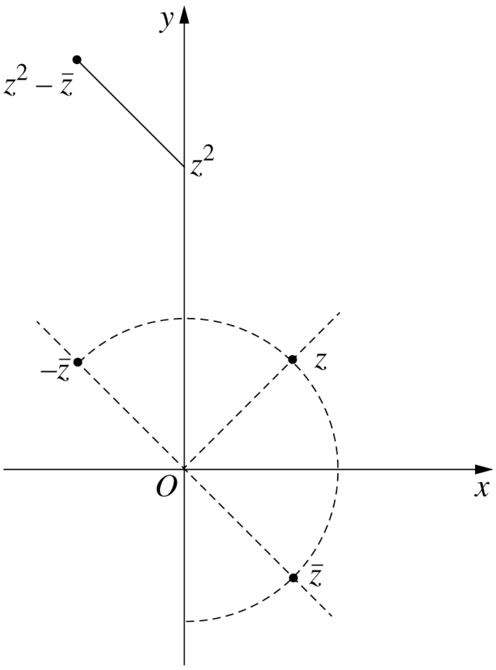
$$\text{so } z^2 = 4i$$



Question 12 (a) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:



Question 12 (b) (i)

Criteria	Marks
• Provides correct solution	3
• Finds a, or equivalent merit	2
• Recognises that the conjugates are also roots, or equivalent merit	1

Sample answer:

The roots are $a \pm ib$ and $a \pm 2ib$

$\therefore$ The sum of the roots is

$(a+2ib)+(a-2ib)+(a+ib)+(a-ib)=4$

$\Rightarrow 4a=4$

$a=1$

The product of the roots is

$(a+2ib)(a-2ib)(a+ib)(a-ib)=10$

$\Rightarrow \left(a^2+4b^2\right)\left(a^2+b^2\right)=10$

But $a=1$,

$\therefore \left(1+4b^2\right)\left(1+b^2\right)=10$

$4b^4+5b^2+1-10=0$

$4b^4+5b^2-9=0$

$\left(4b^2+9\right)\left(b^2-1\right)=0$

$b^2=1$

$b=\pm 1$

$\therefore$ Roots are $1 \pm i,\ 1 \pm 2i$

Question 12 (b) (ii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

The roots are $x = 1 \pm i$ and $x = 1 \pm 2i$

Polynomials with real coefficients have roots that occur in complex conjugate pairs.

$\therefore$ Using the sum and product of the roots, the required quadratics are:

$x^2-(1+i+1-i)x+(1+i)(1-i)$ and $x^2-(1+2i+1-2i)x+(1+2i)(1-zi)$

$= x^2-2x+2$ $\qquad$ $= x^2-2x+5$

Question 12 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts a division of polynomials, or equivalent merit	1

Sample answer:

$$\frac{(x-2)(x-5)}{(x-1)} = \frac{x^2-7x+10}{x-1}$$

Using long division,

$$\begin{array}{r} x-6 \\ x-1\overline{)x^2-7x+10} \\ \underline{x^2-x} \\ -6x+10 \\ \underline{-6x+6} \\ 4 \end{array}$$

$$\therefore \frac{(x-2)(x-5)}{x-1} = x-6+\frac{4}{x-1}$$

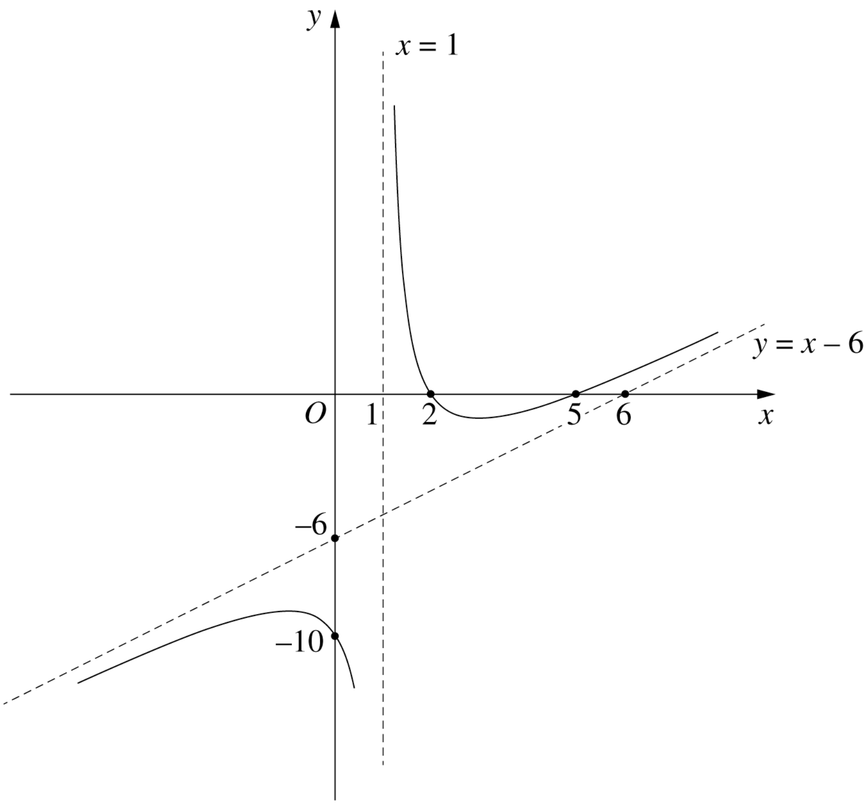
where $m=1,\ b=-6,\ a=4$

$\therefore$ The oblique asymptote is $y=x-6$

Question 12 (c) (ii)

Criteria	Marks
• Provides correct sketch	2
• Identifies any TWO intercepts and the vertical asymptote, or equivalent merit	1

Sample answer:



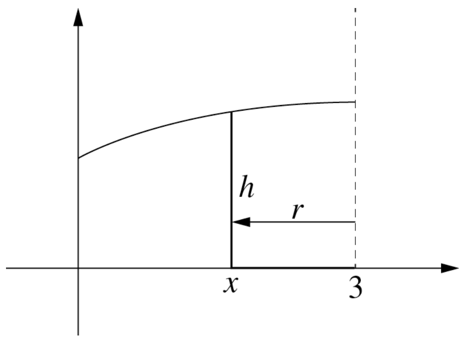
Question 12 (d)

Criteria	Marks
• Provides correct solution	4
• Attempts to evaluate the correct integral using a suitable method, or equivalent merit	3
• Provides correct integral expression for the volume, or equivalent merit	2
• Recognises the radius is $3 - x$, or equivalent merit	1

Sample answer:

Radius of a typical shell is $3 - x$

$$\therefore V = 2\pi \int_0^3 (3-x)\sqrt{x+1}\,dx$$



area of a typical shell = $2\pi rh$

Let $u^2 = x + 1$

$$x = u^2 - 1$$

$$\frac{dx}{du} = 2u$$

$x = 0, \quad u = 1$

$x = 3 \quad u = 2$

$$\therefore V = 2\pi \int_1^2 \left(4 - u^2\right) u.2u\,du$$

$$= 4\pi \int_1^2 \left(4u^2 - u^4\right) du$$

$$= 4\pi \left[\frac{4u^3}{3} - \frac{u^5}{5}\right]_1^2$$

$$= 4\pi \left(\frac{32}{3} - \frac{32}{5} - \left(\frac{4}{3} - \frac{1}{5}\right)\right)$$

$$= \frac{188\pi}{15} \text{ units}^3$$

Question 13 (a) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

At Q, $x = a\tan\theta,\ y = b\sec\theta,$ so substituting into H_2

$$
\begin{aligned}
LHS &= \frac{(a\tan\theta)^2}{a^2} - \frac{(b\sec\theta)^2}{b^2} \\
&= \frac{a^2\tan^2\theta}{a^2} - \frac{b^2\sec^2\theta}{b^2} \\
&= \frac{a^2b^2\left(\tan^2\theta - \sec^2\theta\right)}{a^2b^2} \\
&= \frac{a^2b^2(-1)}{a^2b^2} \qquad (\text{since } \tan^2\theta + 1 = \sec^2\theta) \\
&= -1 \\
&= RHS
\end{aligned}
$$

Question 13 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds the slope of PQ, or equivalent merit	1

Sample answer:

The gradient of PQ is $\dfrac{b\sec\theta - b\tan\theta}{a\tan\theta - a\sec\theta}$

$$= \frac{-b(\tan\theta - \sec\theta)}{a(\tan\theta - \sec\theta)}$$

$$= \frac{-b}{a}$$

$\therefore$ equation of PQ is

$$y - b\tan\theta = \frac{-b}{a}(x - a\sec\theta)$$

$$\Rightarrow ay - ab\tan\theta = -bx + ab\sec\theta$$

$$\Rightarrow bx + ay = ab(\tan\theta + \sec\theta)$$

Question 13 (a) (iii)

Criteria	Marks
• Provides correct proof	3
• Finds a correct expression for the area, or equivalent merit	2
• Finds a correct expression for one relevant distance, or equivalent merit	1

Sample answer:

$$PQ^2 = a^2(\tan\theta - \sec\theta)^2 + b^2(\sec\theta - \tan\theta)^2$$
$$= \left(a^2 + b^2\right)(\tan\theta - \sec\theta)^2$$
$$\therefore PQ = \sqrt{a^2 + b^2}\,|\tan\theta - \sec\theta|$$
$$= \sqrt{a^2 + b^2}\,(\sec\theta - \tan\theta)$$

Since for $0 \le \theta \le \dfrac{\pi}{2}$, $\sec\theta \ge \tan\theta$

line PQ: $bx + ay - ab(\tan\theta + \sec\theta) = 0$

Perpendicular distance from O to PQ is

$$= \frac{|b \times 0 + a \times O - ab(\tan\theta + \sec\theta)|}{\sqrt{a^2 + b^2}}$$
$$= \frac{ab(\tan\theta + \sec\theta)}{\sqrt{a^2 + b^2}}$$

Now, area of $\triangle OPQ$ is

$$\frac{1}{2} \times \sqrt{a^2 + b^2}\,(\sec\theta - \tan\theta) \times \frac{ab(\tan\theta + \sec\theta)}{\sqrt{a^2 + b^2}}$$
$$= \frac{ab}{2}\left(\sec^2\theta - \tan^2\theta\right)$$
$$= \frac{ab}{2}\ (1) \text{ which is independent of } \theta$$

Question 13 (b) (i)

Criteria	Marks
• Provides correct solution	1

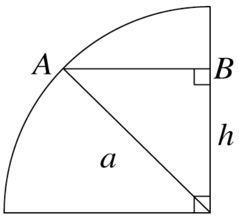
Sample answer:

The cylinder is bounded by a circle $x^2 + y^2 = a^2$

$\therefore$ when $y = h,\ x = \sqrt{a^2 - h^2}$

Or

Using the right hand quarter cylinder



Using Pythagoras' Theorem, $AB = \sqrt{a^2 - h^2}$

Question 13 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Finds correct integral expression for the volume, or equivalent merit	1

Sample answer:

Area of the slice $ABCD = a^2 - h^2$

Hence, the volume is given by

$$V = \int_0^a \left(a^2 - h^2\right) dh$$

$$= \left[a^2 h - \frac{h^3}{3}\right]_0^a$$

$$= \frac{2a^3}{3} \text{ units}^3$$

Question 13 (c) (i)

Criteria	Marks
• Provides correct solution	2
• Correctly finds $\frac{dS}{dt}$ in terms of $\frac{dr}{dt}$, or equivalent merit	1

Sample answer:

$$S = 4\pi r^2 \text{ and } \frac{dS}{dt} = \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \text{ and } \frac{dS}{dt} = \frac{dS}{dr} \cdot \frac{dr}{dt}$$

$$\therefore \frac{dS}{dt} = 8\pi r \frac{dr}{dt}$$

$$\text{so } \left(\frac{4\pi}{3}\right)^{\frac{1}{3}} = 8\pi r \frac{dr}{dt}$$

$$\text{giving } \frac{dr}{dt} = \frac{1}{8\pi r}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$$

Question 13 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Correctly finds $\frac{dV}{dt}$ in terms of $\frac{dr}{dt}$, or equivalent merit	1

Sample answer:

$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

$$= 4\pi r^2 \cdot \frac{1}{8\pi r}\left(\frac{4\pi}{3}\right)^{\frac{1}{3}} \text{ by (i)}$$

$$= \frac{1}{2} r \left(\frac{4\pi}{3}\right)^{\frac{1}{3}}$$

$$= \frac{1}{2}\left(r^3 \cdot \frac{4\pi}{3}\right)^{\frac{1}{3}}$$

$$\frac{dV}{dt} = \frac{1}{2} V^{\frac{1}{3}}$$

Question 13 (c) (iii)

Criteria	Marks
• Provides correct solution	2
• Finds correct primitive for t in terms of V, or equivalent merit	1

Sample answer:

As $$\frac{dV}{dt} = \frac{1}{2}V^{\frac{1}{3}}$$

so $$\int \frac{dV}{V^{\frac{1}{3}}} = \frac{1}{2}\int dt$$

$$\int_{8\,000}^{64\,000} V^{-\frac{1}{3}}\,dV = \frac{1}{2}\int_0^t dt$$

$$\therefore \quad \frac{1}{2}t = \left[\frac{3}{2}.V^{\frac{2}{3}}\right]_{8\,000}^{64\,000}$$

$$t = 3\left(40^2 - 20^2\right)$$

$$\therefore \quad \text{time} = 3\ 600 \text{ seconds}$$

$$(= 1 \text{ hour})$$

Question 14 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Attempts to differentiate and obtains an expression involving $\cos^2\theta$, or equivalent merit	1

Sample answer:

$y = \sin^{n-1}\theta\cos\theta$

$y' = u'v + v'u$ Let $u = \sin^{n-1}\theta$ $v = \cos\theta$

$u' = (n-1)\sin^{n-2}\theta\cos\theta$ $v' = -\sin\theta$

$$\begin{aligned} y' &= (n-1)\sin^{n-2}\theta\cos^2\theta + \sin^{n-1}\theta(-\sin\theta) \\ &= (n-1)\sin^{n-2}\theta\left(1-\sin^2\theta\right) - \sin^n\theta \\ &= (n-1)\sin^{n-2}\theta - (n-1)\sin^n\theta - \sin^n\theta \\ &= (n-1)\sin^{n-2}\theta - n\sin^n\theta \end{aligned}$$

Question 14 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Attempts to evaluate the integral of the expression from part (i), or equivalent merit	1

Sample answer:

$$y' = \frac{dy}{d\theta} = (n-1)\sin^{n-2}\theta - n\sin^n\theta$$

$$\int_0^{\frac{\pi}{2}} dy = \int_0^{\frac{\pi}{2}} (n-1)\sin^{n-2}\theta - n\sin^n\theta \, d\theta$$

$$\Big[\ y\ \Big]_0^{\frac{\pi}{2}} = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta - n\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$$

$$\Big[\ \sin^{n-1}\theta\cos\theta\ \Big]_0^{\frac{\pi}{2}} = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta - n\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$$

$$0 = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta - n\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta$$

$$n\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = (n-1)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta$$

$$\int_0^{\frac{\pi}{2}} \sin^n\theta \, d\theta = \left(\frac{n-1}{n}\right)\int_0^{\frac{\pi}{2}} \sin^{n-2}\theta \, d\theta$$

Question 14 (a) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\int_0^{\frac{\pi}{2}} \sin^4\theta\, d\theta = \frac{4-1}{4}\int_0^{\frac{\pi}{2}} \sin^2\theta\, d\theta$$

$$= \frac{3}{4}\left[\left(\frac{2-1}{2}\right)\int_0^{\frac{\pi}{2}} d\theta \right]$$

$$= \frac{3}{4}\times\frac{1}{2}\left[\ \theta\ \right]_0^{\frac{\pi}{2}}$$

$$= \frac{3}{8}\left(\frac{\pi}{2}-0\right)$$

$$= \frac{3\pi}{16}$$

Question 14 (b) (i)

Criteria	Marks
• Provides correct solution	1

Sample answer:

$$\alpha+\beta+\gamma = -\frac{b}{a} = 0$$

$$\alpha\beta+\alpha\gamma+\beta\gamma = \frac{c}{a} = -p$$

$$(\alpha+\beta+\gamma)^2 = 0$$

ie $\alpha^2+\beta^2+\gamma^2+2(\alpha\beta+\alpha\gamma+\beta\gamma) = 0$

$$16-2p = 0$$

$$\therefore \quad p = 8$$

Question 14 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains a correct equation involving q and at least one of α^3, β^3, γ^3, or equivalent merit	1

Sample answer:

α is a root so

$$\alpha^3 = p\alpha - q \qquad (1)$$

similarly $\beta^3 = p\beta - q \qquad (2)$

$$\gamma^3 = p\gamma - q \qquad (3)$$

adding (1) + (2) + (3)

$$\alpha^3 + \beta^3 + \gamma^3 = p(\alpha + \beta + \gamma) - 3q$$

$$-9 = p \times 0 - 3q$$

$$\therefore \; q = 3$$

Question 14 (b) (iii)

Criteria	Marks
• Provides correct solution	2
• Obtains an equation involving $\alpha^4 + \beta^4 + \gamma^4$, or equivalent merit	1

Sample answer:

Since $\alpha^3 = p\alpha - q$

so $\alpha^4 = p\alpha^2 - q\alpha$

similarly $\beta^4 = p\beta^2 - q\beta$

$$\gamma^4 = p\gamma^2 - q\gamma$$

adding $\alpha^4 + \beta^4 + \gamma^4 = p(\alpha^2 + \beta^2 + \gamma^2) - q(\alpha + \beta + \gamma)$

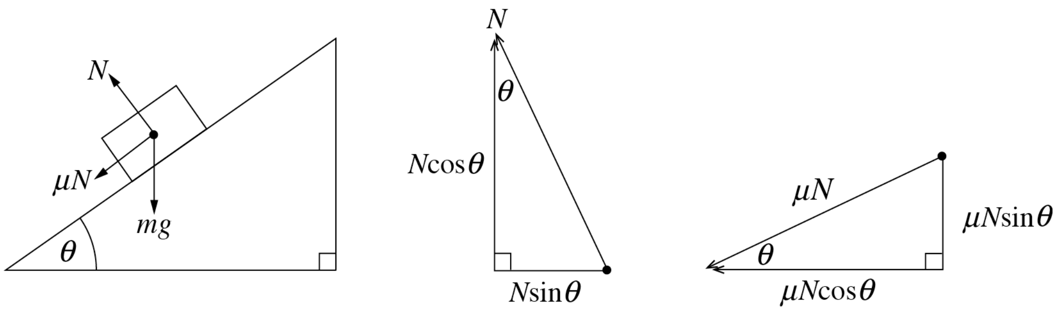
$$= 8 \times 16 - q \times 0$$

$$= 128$$

Question 14 (c) (i)

Criteria	Marks
• Provides correct solution	3
• Provides correct resolution of forces, or equivalent merit	2
• Resolves forces in vertical direction, or equivalent merit	1

Sample answer:



Resolving forces:

Vertically $N\cos\theta - \mu N\sin\theta = mg$ (1)

Horizontally $N\sin\theta + \mu N\cos\theta = \dfrac{mv^2}{r}$ (2)

Dividing (2) by (1)

$$\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta} = \frac{v^2}{rg}$$

$$\therefore\ v^2 = rg\left(\frac{\sin\theta + \mu\cos\theta}{\cos\theta - \mu\sin\theta}\right)$$

Dividing by $\dfrac{\cos\theta}{\cos\theta}$:

$$v^2 = rg\left(\frac{\tan\theta + \mu}{1 - \mu\tan\theta}\right)$$

Question 14 (c) (ii)

Criteria	Marks
• Provides correct solution	2
• Obtains correct expression for $\tan\theta$ in terms of μ, or equivalent merit	1

Sample answer:

Since $v = V$

$V^2 = rg$ and $V^2 = rg\left(\dfrac{\tan\theta + \mu}{1 - \mu\tan\theta}\right)$

so $\dfrac{\tan\theta + \mu}{1 - \mu\tan\theta} = 1$

hence

$$\tan\theta + \mu = 1 - \mu\tan\theta$$

$$\mu + \mu\tan\theta = 1 - \tan\theta$$

so $$\mu = \frac{1 - \tan\theta}{1 + \tan\theta}$$

since $0 < \theta < \dfrac{\pi}{2}$, $\tan\theta > 0$

so $1 - \tan\theta < 1$ and $1 + \tan\theta > 1$

$$\therefore \mu < 1$$

Question 15 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains $\ddot{x} = -kv^2$, or equivalent merit	1

Sample answer:

$$\ddot{x} = -kv^2$$

$$\frac{dv}{dt} = -kv^2$$

$$k\,dt = -\frac{dv}{v^2}$$

$$\Rightarrow kt = -\int \frac{dv}{v^2}$$

$$kt = \frac{1}{v} + C$$

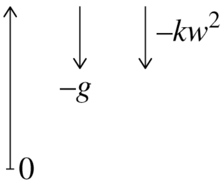
Now $t = 0,\ v = u \Rightarrow C = -\frac{1}{u}$

$$\therefore \quad \frac{1}{v} = kt + \frac{1}{u}$$

Question 15 (a) (ii)

Criteria	Marks
• Provides correct solution	3
• Correctly integrates to find t, or equivalent merit	2
• Obtains $\ddot{x} = -g - kw^2$, or equivalent merit	1

Sample answer:



$$\ddot{x} = -g - kw^2$$

$$\therefore \frac{dw}{dt} = -\left(g + kw^2\right)$$

$$\frac{dt}{dw} = -\frac{1}{g + kw^2}$$

$$t = -\int \frac{1}{g + kw^2}\,dw$$

$$= -\frac{1}{k}\int \frac{1}{\frac{g}{k} + w^2}\,dw$$

$$= -\frac{1}{k}\left[\sqrt{\frac{k}{g}}\tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right] + C$$

$$\therefore t = -\frac{1}{\sqrt{gk}}\tan^{-1}\left(w\sqrt{\frac{k}{g}}\right) + C$$

Now $t = 0,\ w = u,$ so

$$C = \frac{1}{\sqrt{gk}}\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

$$\therefore t = \frac{1}{\sqrt{gk}}\left(\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) - \tan^{-1}\left(w\sqrt{\frac{k}{g}}\right)\right)$$

Question 15 (a) (iii)

Criteria	Marks
• Provides correct solution	1

Sample answer:

The second particle is at rest when $w = 0$

$$\therefore t = \frac{1}{\sqrt{gk}} \tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

At this time the velocity of the first particle is

$$\frac{1}{V} = \frac{1}{u} + k\left(\frac{1}{\sqrt{gk}} \tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)\right) \text{ from (i)}$$

$$\therefore \frac{1}{V} = \frac{1}{u} + \sqrt{\frac{k}{g}} \tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$$

Question 15 (a) (iv)

Criteria	Marks
• Provides correct solution	1

Sample answer:

From (iii), $\frac{1}{V} = \frac{1}{u} + \sqrt{\frac{k}{g}} \tan^{-1}\left(u\sqrt{\frac{k}{g}}\right)$

for u range, $\frac{1}{u} \approx 0$, $\tan^{-1}\left(u\sqrt{\frac{k}{g}}\right) \approx \frac{\pi}{2}$

$$\therefore \frac{1}{V} \approx \sqrt{\frac{k}{g}} \cdot \frac{\pi}{2} \Rightarrow V \approx \frac{2}{\pi}\sqrt{\frac{g}{k}}$$

Question 15 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Obtains one of the required inequalities	1

Sample answer:

Since for $x \geq 0$

$$1 - x^2 \leq 1 \qquad \text{ie } (1-x)(1+x) \leq 1$$

since $(1+x) > 0$ we have $1 - x \leq \dfrac{1}{1+x}$

and since $x \geq 0$ then $\quad 1 + x \geq 1$

$$\therefore \frac{1}{1+x} \leq 1$$

$$\therefore 1 - x \leq \frac{1}{1+x} \leq 1$$

Question 15 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Integrates the inequalities from part (i), or equivalent merit	1

Sample answer:

Since $\int_a^b \frac{1}{1+x}\,dx = \left[\ln(1+x)\right]_a^b$

$$= \ln(1+b) - \ln(1+a)$$

We want $\ln\left(1+\frac{1}{n}\right)$ so let $a = 0$ and $b = \frac{1}{n}$

$$\therefore \int_0^{\frac{1}{n}} 1 - x\,dx \le \int_0^{\frac{1}{n}} \frac{1}{1+x}\,dx \le \int_0^{\frac{1}{n}} 1\,dx$$

$$\therefore \left[x - \frac{x^2}{2}\right]_0^{\frac{1}{n}} \le \left[\ln(1+x)\right]_0^{\frac{1}{n}} \le \left[x\right]_0^{\frac{1}{n}}$$

$$\therefore \frac{1}{n} - \frac{1}{2n^2} \le \ln\left(1+\frac{1}{n}\right) \le \frac{1}{n}$$

multiplying through by n:

$$1 - \frac{1}{2n} \le n\ln\left(1+\frac{1}{n}\right) \le 1$$

Question 15 (b) (iii)

Criteria	Marks
• Provides correct explanation	1

Sample answer:

From (ii), $1-\frac{1}{2n} \le n\ln\left(1+\frac{1}{n}\right) \le 1$

$\Rightarrow \quad 1-\frac{1}{2n} \le \ln\left(1+\frac{1}{n}\right)^n \le 1$

$$\lim_{n\to\infty}\left(1-\frac{1}{2n}\right) \le \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n \le \lim_{n\to\infty} 1$$

ie $\quad 1 \le \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n \le 1$

$\Rightarrow \quad \lim_{n\to\infty}\ln\left(1+\frac{1}{n}\right)^n = 1$

$\Rightarrow \quad \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$

Question 15 (c) (i)

Criteria	Marks
• Provides correct proof	1

Sample answer:

Replacing x with x^2, y with y^2 in the given inequality we have

$\sqrt{x^2y^2} \le \frac{x^2+y^2}{2}$

ie $\quad xy \le \frac{x^2+y^2}{2}$

so $\sqrt{xy} \le \sqrt{\frac{x^2+y^2}{2}}$, since $x, y > 0$

Question 15 (c) (ii)

Criteria	Marks
• Provides correct proof	2
• Uses (i) to obtain a valid expression in a^2, b^2, c^2 and d^2, or equivalent merit	1

Sample answer:

$$\sqrt{(ab)(cd)} = \sqrt{ab}\sqrt{cd}$$

$$\leq \sqrt{\frac{a^2+b^2}{2}}\sqrt{\frac{c^2+d^2}{2}} \qquad \text{by (i)}$$

$$\leq \sqrt{\left(\frac{a^2+b^2}{2}\right)\left(\frac{c^2+d^2}{2}\right)}$$

$$\leq \frac{1}{2}\left(\frac{a^2+b^2+c^2+d^2}{2}\right) \qquad \text{by given inequality}$$

$$= \frac{a^2+b^2+c^2+d^2}{4}$$

Taking positive square roots,

$$\sqrt[4]{(abcd)} \leq \sqrt{\frac{a^2+b^2+c^2+d^2}{4}}$$

Question 16 (a) (i)

Criteria	Marks
• Provides correct solution	2
• Shows total number of arrangements is $\binom{15}{5}$, or equivalent merit	1

Sample answer:

There are $\binom{15}{5}$ ways to place 5 black counters in the grid.

There are three choices to place a black counter in each column, so there are 3^5 ways to place the black counters with one in each column.

$$\therefore \text{Probability} = \frac{3^5}{\binom{15}{5}} = \frac{81}{1001}$$

Question 16 (a) (ii)

Criteria	Marks
• Provides correct solution	2
• Shows total number of arrangements is $\binom{nq}{q}$, or equivalent merit	1

Sample answer:

There are nq places and we choose q places for black counters, giving $\binom{nq}{q}$ arrangements.

There are n places for each black counter in each of q columns.

So there are n^q arrangements with one black counter in each column.

$$P_n = \frac{n^q}{\binom{nq}{q}}$$

Question 16 (a) (iii)

Criteria	Marks
• Provides correct solution	2
• Obtains an expression for P_n with q terms in denominator, or equivalent merit	1

Sample answer:

$$
\begin{aligned}
P_n &= \frac{n^q}{\binom{nq}{q}} \\
&= \frac{n^q}{nq(nq-1)\ \dots\ (nq-q+1)/q!} \\
&= \frac{q!n^q}{nq(nq-1)\ \dots\ (nq-q+1)} \\
&= \frac{q!}{q\left(q-\frac{1}{n}\right)\left(q-\frac{2}{n}\right)\ \dots\ \left(q-\frac{q}{n}+\frac{1}{n}\right)}
\end{aligned}
$$

As $n \to \infty$,

$$P_n \to \frac{q!}{q^q}$$

Question 16 (b) (i)

Criteria	Marks
• Provides correct solution	2
• Applies De Moivre's theorem, or equivalent merit	1

Sample answer:

By De Moivre's theorem

$$(\cos\alpha + i\sin\alpha)^{2n} = \cos(2n\alpha) + i\sin(2n\alpha)$$

By the bionomial theorem,

$$(\cos\alpha + i\sin\alpha)^{2n} = \cos^{2n}\alpha + \binom{2n}{1}\cos^{2n-1}\alpha . i\sin\alpha + \binom{2n}{2}\cos^{2n-2}\alpha(i\sin\alpha)^2 + \ldots$$
$$+\binom{2n}{2n-2}\cos^2\alpha(i\sin\alpha)^{2n-2} + \binom{2n}{2n-1}\cos\alpha(i\sin\alpha)^{2n-1} + \binom{2n}{2n}(i\sin\alpha)^{2n}$$

Equating real parts,

$$\cos(2n\alpha) = \cos^{2n}\alpha - \binom{2n}{2}\cos^{2n-2}\alpha\sin^2\alpha + \ldots + \binom{2n}{2n-2}\cos^2\alpha\sin^{2n-2}\alpha(-1)^{n-1} + (-1)^n\sin^{2n}\alpha$$

Question 16 (b) (ii)

Criteria	Marks
• Provides correct solution	2
• Explains the connection between $T_{2n}(x)$ and $\cos(2n\alpha)$, or equivalent merit	1

Sample answer:

$T_{2n}(x) = \cos\left(2n\cos^{-1}x\right)$

put $\alpha = \cos^{-1}x$, so $x = \cos\alpha$ in (i)

noting that $\sin^{2k}\alpha = \left(\sin^2\alpha\right)^k$

$= \left(1 - \cos^2\alpha\right)^k$

Then

$$T_{2n}(x) = x^{2n} - \binom{2n}{2}x^{2n-2}\left(1 - x^2\right) + \ldots + (-1)^n\left(1 - x^2\right)^n$$

Question 16 (b) (iii)

Criteria	Marks
• Provides correct solution	3
• Shows that the roots of $T_{2n}(x)$ are $\cos\left(\frac{\pi}{4n}\right)$, $\cos\left(\frac{3\pi}{4n}\right)$, … , $\cos\left(\frac{(4n-1)\pi}{4n}\right)$, or equivalent merit	2
• Obtains (or verifies) that $\cos\left(\frac{(2k-1)\pi}{4n}\right)$ are all roots of $T_{2n}(x)$, or equivalent merit	1

Sample answer:

$0 = T_{2n}(x) = \cos\left(2n\cos^{-1} x\right)$

This occurs when $2n\cos^{-1} x = \frac{(2k-1)\pi}{2}$

$$\text{ie } \cos^{-1} x = \frac{(2k-1)\pi}{4n}$$

$$x = \cos\left(\frac{(2k-1)\pi}{4n}\right)$$

Consider the $2n$ solutions with $k = 1, 2, \ldots, 2n$ ie

$\cos\left(\frac{\pi}{4n}\right), \cos\left(\frac{3\pi}{4n}\right), \ldots, \cos\left(\frac{(4n-1)\pi}{4n}\right)$

As the angles are in the range 0 to π and are distinct these solutions are distinct. (Given that $\cos x$ is decreasing for $0 \le x \le \pi$.)

We have $2n$ distinct solutions of a polynomial of degree $2n$, and so these are all the roots.

As we have a polynomial of even degree, the product of the roots is the constant term divided by the leading term. We may find the constant term by setting $x = 0$, obtaining $(-1)^n$.

The coefficients of x^{2n} are

$$1, \quad \binom{2n}{2}, \quad \binom{2n}{4}, \ldots, \quad 1$$

The leading term is the sum of these and hence

$$\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{3\pi}{4n}\right)\cos\left(\frac{(4n-1)\pi}{4n}\right) = \frac{(-1)^n}{1+\binom{2n}{2}+\binom{2n}{4}+\ldots+1}$$

In fact the denominator on the right may be simplified

$$\cos\left(\frac{\pi}{4n}\right)\cos\left(\frac{3\pi}{4n}\right)\cos\left(\frac{(4n-1)\pi}{4n}\right) = \frac{(-1)^n}{2^{2n-1}}$$

Question 16 (b) (iv)

Criteria	Marks
• Provides correct proof	2
• Evaluates $T_{2n}\left(\frac{1}{\sqrt{2}}\right)$, or equivalent merit	1

Sample answer:

Put $x = \frac{1}{\sqrt{2}}$ then $1 - x^2 = \frac{1}{2}$

$$T_{2n}\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2^n} - \binom{2n}{2} \cdot \frac{1}{2^n} + \binom{2n}{4} \cdot \frac{1}{2^n} + \ldots + (-1)^n \binom{2n}{2n} \cdot \frac{1}{2^n}$$

$$= \cos\left(2n \cos^{-1} \frac{1}{\sqrt{2}}\right) = \cos\left(\frac{n\pi}{2}\right)$$

$$\therefore 1 - \binom{2n}{2} + \binom{2n}{4} - \binom{2n}{6} + \ldots + (-1)^n \binom{2n}{2n} = 2^n \cos\left(\frac{n\pi}{2}\right)$$

2015 HSC Mathematics Extension 2 Mapping Grid

Section I

Question	Marks	Content	Syllabus outcomes
1	1	3.2	E3
2	1	2.1	E3
3	1	1.4	E6
4	1	7.2	E4
5	1	2.4	E3
6	1	4.1	E8
7	1	8	HE3, E9
8	1	1.7	E6
9	1	2.2	E3
10	1	8	E2

Section II

Question	Marks	Content	Syllabus outcomes
11 (a)	2	2.1	E3
11 (b) (i)	1	2.2	E3
11 (b) (ii)	1	2.2	E3
11 (b) (iii)	1	2.2	E3
11 (c)	2	7.6	E4
11 (d)	2	3.1	E3
11 (e)	3	1.8	E6
11 (f) (i)	2	4.1	PE6, E2
11 (f) (ii)	1	4.1	E8
12 (a) (i)	1	2.2	E3
12 (a) (ii)	1	2.2	E3
12 (a) (iii)	1	2.3	E3
12 (b) (i)	3	7.4, 7.5	E3, E4
12 (b) (ii)	1	7.4, 7.5	E3, E4
12 (c) (i)	2	1.8	E6
12 (c) (ii)	2	1.8	E6
12 (d)	4	5.1	E7
13 (a) (i)	1	3.2	E3
13 (a) (ii)	2	3.2	E3
13 (a) (iii)	3	3.2	E3
13 (b) (i)	1	8	E2

Question	Marks	Content	Syllabus outcomes
13 (b) (ii)	2	5.1	E7
13 (c) (i)	2	14.1E, 8	HE3, E8
13 (c) (ii)	2	14.1E, 8	HE3, E8
13 (c) (iii)	2	14.1E, 8, 4.1	HE3, E8
14 (a) (i)	2	4.1	E8
14 (a) (ii)	2	4.1	E8
14 (a) (iii)	1	4.1	E8
14 (b) (i)	1	7.5	E4
14 (b) (ii)	2	7.5	E4
14 (b) (iii)	2	7.5	E4
14 (c) (i)	3	6.3.4	E5
14 (c) (ii)	2	6.3.4	E5
15 (a) (i)	2	6.2.1	E5
15 (a) (ii)	3	6.2.2	E5
15 (a) (iii)	1	6.2.1, 6.2.2	E5
15 (a) (iv)	1	6.2.1	E5
15 (b) (i)	2	8, 3, 1.4E	E2, HE7
15 (b) (ii)	2	8, 1.4E, 12.5	PE3, HE7, E2
15 (b) (iii)	1	8, 1.4E	PE3, E2
15 (c) (i)	1	8.3	E2
15 (c) (ii)	2	8.3	E2
16 (a) (i)	2	8	E2
16 (a) (ii)	2	8	E2
16 (a) (iii)	2	8	E2
16 (b) (i)	2	2.4	E3
16 (b) (ii)	2	8	E4
16 (b) (iii)	3	7.5	E4
16 (b) (iv)	2	7.5	E9