BOARD OF STUDIES

NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

2000

MATHEMATICS

4 UNIT (ADDITIONAL)

Time allowed—Three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 16.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1 Use a SEPARATE Writing Booklet. **Marks**

(a) Find $\displaystyle\int \frac{\cos x}{\sin^4 x}\,dx$. **2**

(b) Use completion of squares to find $\displaystyle\int \frac{4}{x^2+6x+10}\,dx$. **2**

(c) (i) Find the real numbers a, b and c such that $\displaystyle\frac{9}{x^2(3-x)} \equiv \frac{ax+b}{x^2}+\frac{c}{3-x}$. **4**

(ii) Find $\displaystyle\int \frac{9}{x^2(3-x)}\,dx$.

(d) Find $\displaystyle\int \sqrt{x}\ln x\,dx$. **3**

(e) Use the substitution $t=\tan\dfrac{\theta}{2}$ to find $\displaystyle\int \frac{d\theta}{1+\sin\theta+\cos\theta}$. **4**

QUESTION 2 Use a SEPARATE Writing Booklet. **Marks**

(a) Find all pairs of integers x and y that satisfy $(x + iy)^2 = 24 + 10i$. **3**

(b) Consider the equation $z^2 + az + (1 + i) = 0$. **2**

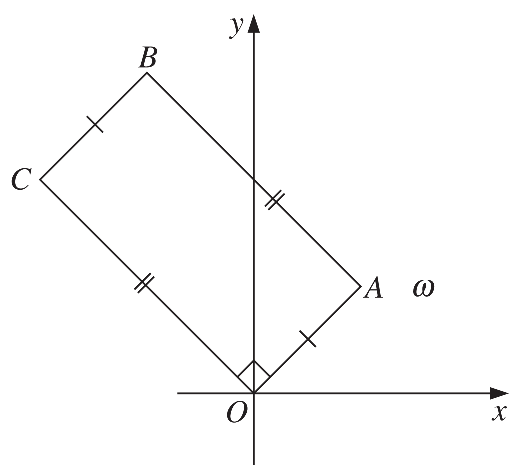
Find the complex number a, given that i is a root of the equation.

(c) (i) Let $z = \cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$. Find z^6. **4**

(ii) Plot, on the Argand diagram, all complex numbers that are solutions of $z^6 = -1$.

(d) Sketch the region in the Argand diagram that satisfies the inequality $z\bar{z} + 2(z + \bar{z}) \leq 0$. **3**

(e) **3**



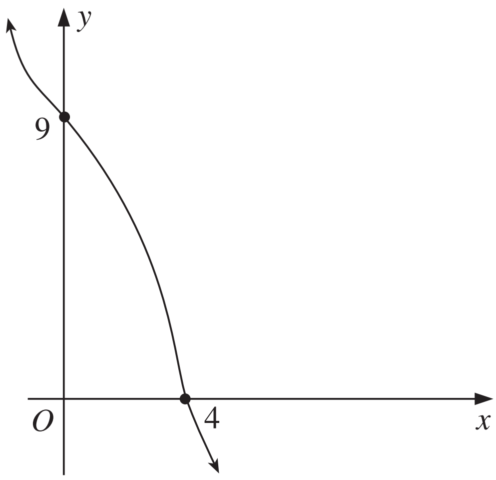
In the Argand diagram, $OABC$ is a rectangle, where $OC = 2OA$. The vertex A corresponds to the complex number ω.

(i) What complex number corresponds to the vertex C?

(ii) What complex number corresponds to the point of intersection D of the diagonals OB and AC?

QUESTION 3 Use a SEPARATE Writing Booklet. **Marks**

(a) The diagram shows the graph of the (decreasing) function $y = f(x)$. **7**



Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$

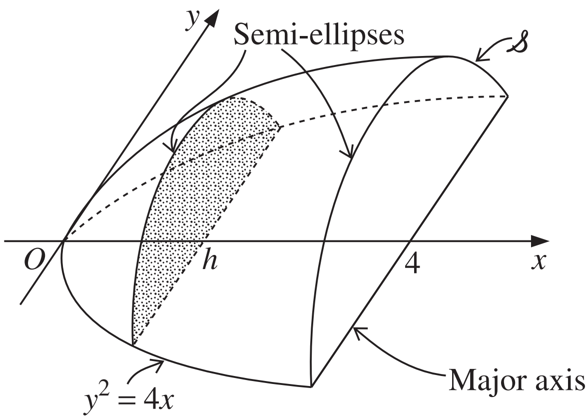
(ii) $y = \dfrac{1}{f(x)}$

(iii) $y^2 = f(x)$

(iv) the inverse function $y = f^{-1}(x)$.

QUESTION 3 (Continued) **Marks**

(b) **6**



The base of a solid $\mathcal{S}$ is the region in the xy plane enclosed by the parabola $y^2 = 4x$ and the line $x = 4$, and each cross-section perpendicular to the x axis is a semi-ellipse with the minor axis one-half of the major axis.

(i) Show that the area of the semi-ellipse at $x = h$ is πh.
(You may assume that the area of an ellipse with semi-axes a and b is πab.)

(ii) Find the volume of the solid $\mathcal{S}$.

(iii) Consider the solid $\mathcal{T}$, which is obtained by rotating the region enclosed by the parabola and the line $x = 4$ about the x axis. What is the relation between the volume of $\mathcal{S}$ and the volume of $\mathcal{T}$?

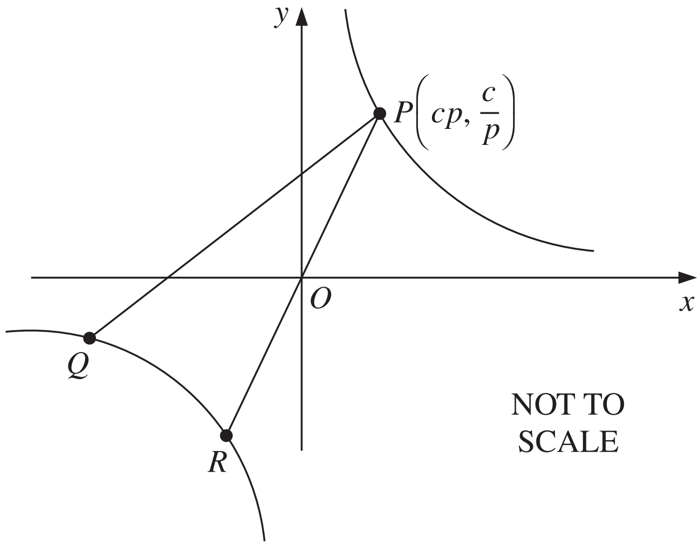
(c) A modern supercomputer can calculate 1000 billion (ie, 10^{12}) basic arithmetical operations per second. Use Stirling's formula to estimate how many years such a computer would take to calculate 100! basic arithmetical operations. Stirling's formula states that $n!$ is approximately equal to **2**

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} .$$

Leave your answer in scientific notation.

QUESTION 4 Use a SEPARATE Writing Booklet. **Marks**

(a) **8**



The point $P\left(cp, \frac{c}{p}\right)$, where $p \neq \pm 1$, is a point on the hyperbola $xy = c^2$, and the normal to the hyperbola at P intersects the second branch at Q. The line through P and the origin O intersects the second branch at R.

(i) Show that the equation of the normal at P is

$$py - c = p^3(x - cp).$$

(ii) Show that the x coordinates of P and Q satisfy the equation

$$x^2 - c\left(p - \frac{1}{p^3}\right)x - \frac{c^2}{p^2} = 0 .$$

(iii) Find the coordinates of Q, and deduce that the $\angle QRP$ is a right angle.

QUESTION 4 (Continued) **Marks**

(b) The temperature T_1 of a beaker of chemical, and the temperature T_2 of a surrounding vat of cooler water, satisfy, in accordance with Newton's law of cooling, the equations **7**

$$\frac{dT_1}{dt} = -k\left(T_1 - T_2\right)$$

$$\frac{dT_2}{dt} = \frac{3}{4}k\left(T_1 - T_2\right)$$

where k is a positive constant.

(i) Show, by differentiation, that $\frac{3}{4}T_1 + T_2 = C$, where C is a constant.

(ii) Find an expression for $\frac{dT_1}{dt}$ in terms of T_1, and show $T_1 = \frac{4}{7}C + Be^{-\frac{7}{4}kt}$ satisfies this differential equation for any constant B.

(iii) Initially, the beaker of chemical had a temperature of 120°C and the vat of water had a temperature of 22°C. Ten minutes later, the temperature of the beaker of chemical had fallen to 90°C.

Find the temperature of the beaker of chemical after a further ten minutes.

QUESTION 5 Use a SEPARATE Writing Booklet. **Marks**

(a) Consider the polynomial **4**

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e$$

where a, b, c, d and e are integers. Suppose α is an integer such that $p(\alpha) = 0$.

(i) Prove that α divides e.

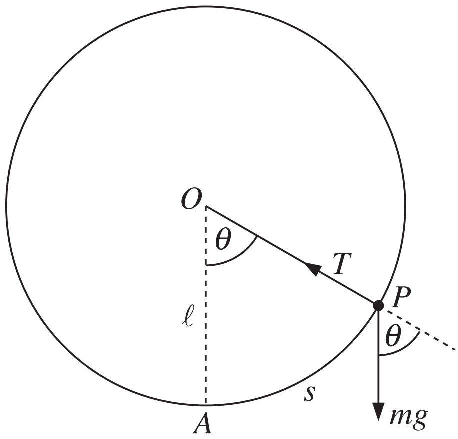
(ii) Prove that the polynomial

$$q(x) = 4x^4 - x^3 + 3x^2 + 2x - 3$$

does not have an integer root.

QUESTION 5 (Continued) **Marks**

(b) **11**



A string of length ℓ is initially vertical and has a mass P of m kg attached to it. The mass P is given a horizontal velocity of magnitude V and begins to move along the arc of a circle in a counterclockwise direction.

Let O be the centre of this circle and A the initial position of P. Let s denote the arc length AP, $v = \dfrac{ds}{dt}$, $\theta = \angle AOP$ and let the tension in the string be T. The acceleration due to gravity is g and there are no frictional forces acting on P.

For parts (i) to (iv), assume that the mass is moving along the circle.

(i) Show that the tangential acceleration of P is given by $\dfrac{d^2s}{dt^2} = \dfrac{1}{\ell}\dfrac{d}{d\theta}\left(\dfrac{1}{2}v^2\right)$.

(ii) Show that the equation of motion of P is $\dfrac{1}{\ell}\dfrac{d}{d\theta}\left(\dfrac{1}{2}v^2\right) = -g\sin\theta$.

(iii) Deduce that $V^2 = v^2 + 2\ell g(1 - \cos\theta)$.

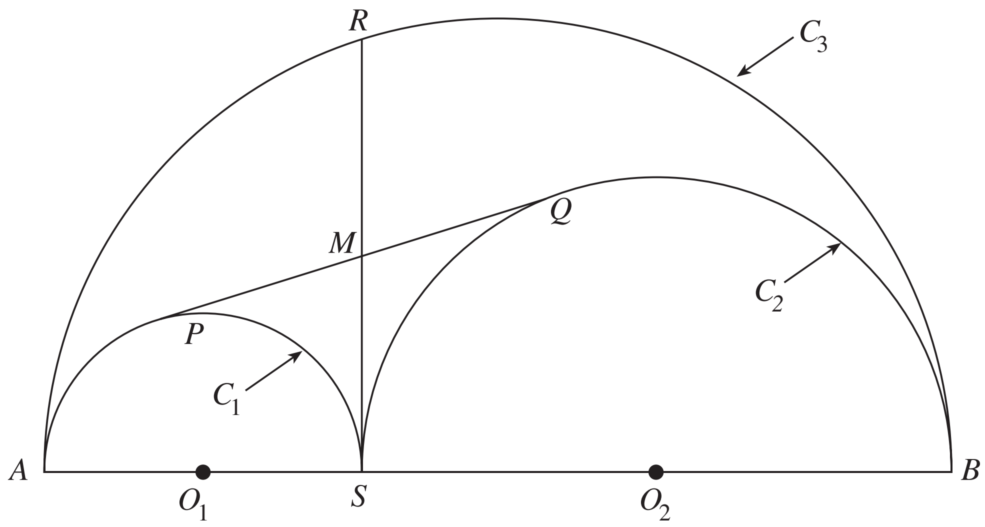
(iv) Explain why $T - mg\cos\theta = \dfrac{1}{\ell}mv^2$.

(v) Suppose that $V^2 = 3g\ell$. Find the value of θ at which $T = 0$.

(vi) Consider the situation in part (v). Briefly describe, in words, the path of P after the tension T becomes zero.

QUESTION 6 Use a SEPARATE Writing Booklet. **Marks**

(a) **9**



In the diagram, C_1 and C_2 are semicircles of radii r_1 and r_2, with centres O_1 and O_2 on AB. The two semicircles touch at the point S on AB. The semicircle C_3 has diameter AB, and R is the point on C_3 such that RS is tangential to both C_1 and C_2 (so RS is perpendicular to AB). The other common tangent to C_1 and C_2 touches C_1 at P and C_2 at Q. The tangents PQ and RS intersect at M.

(i) State why $MP = MS = MQ$.

(ii) By using the 'intersecting chords theorem' (applied to C_3), or otherwise, prove that $RS^2 = 4r_1r_2$.

(The intersecting chords theorem states that the products of the intercepts of two intersecting chords are equal.)

(iii) Show that $\angle O_1MO_2$ is a right angle, and deduce that $MS^2 = r_1r_2$.

(iv) Deduce that $PSQR$ is a rectangle.

(b) (i) Evaluate $\displaystyle\int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^2}}$. **6**

(ii) Explain carefully why, for $n \geq 2$,

$$\frac{1}{2} \leq \int_0^{\frac{1}{2}} \frac{dx}{\sqrt{1-x^n}} \leq \frac{\pi}{6}.$$

BLANK PAGE

Please turn over

QUESTION 7 Use a SEPARATE Writing Booklet. **Marks**

(a) (i) Show that, for $x > 0$, **9**

$$\ln x \le x - 1, \text{ with equality only at } x = 1.$$

(ii) From (i) deduce that

$$\sum_{i=1}^{n} x_i \ln \frac{y_i}{x_i} \le 0$$

whenever $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i = 1$, where $x_i > 0$, $y_i > 0$ for $i = 1, 2, \ldots, n$.

Show also that equality occurs only if $x_i = y_i$ for $i = 1, 2, \ldots, n$.

(iii) By considering part (ii) with equal values of y_i for $i = 1, 2, \ldots, n$, prove that the maximum value of

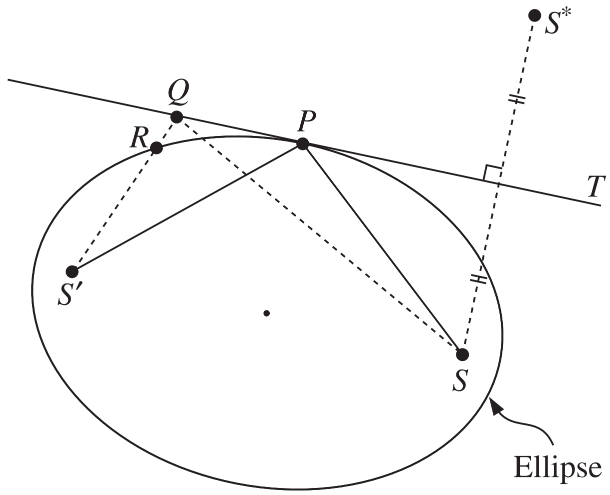
$$\sum_{i=1}^{n} x_i \ln \frac{1}{x_i} \text{ is } \ln n,$$

where $\sum_{i=1}^{n} x_i = 1$ and $x_i > 0$ for $i = 1, 2, \ldots, n$.

(iv) Does the result of part (iii) hold if $\ln$ is replaced by $\log_2$? Give reasons for your answer.

QUESTION 7 (Continued) **Marks**

(b) **6**



In the diagram, P is an arbitrary point on the ellipse, and QPT is the tangent to the ellipse at P. The points S' and S are the foci of the ellipse, and S^* is the reflection of S across the tangent, as shown. Let the line $S'Q$ intersect the ellipse at R.

(i) Assuming $Q \neq P$, prove that

$$S'Q + QS > S'R + RS.$$

(ii) Deduce that the shortest path from S' to S passing through a point on the tangent is that through P, having length $S'P + PS$.

(iii) By considering the point S^*, deduce that $\angle QPS' = \angle TPS$.

Please turn over

QUESTION 8 Use a SEPARATE Writing Booklet. **Marks**

(a) (i) Use the formula for the sum of a geometric series to show that **6**

$$\sum_{k=1}^{n}\left(z+z^2+\ldots+z^k\right)=\frac{nz}{1-z}-\frac{z^2}{(1-z)^2}\left(1-z^n\right),\ z\neq 1.$$

(ii) Let $z=\cos\theta+i\sin\theta$, where $0<\theta<2\pi$. By considering the imaginary part of the left-hand side of the equation of part (i), deduce that

$$\sum_{k=1}^{n}(\sin\theta+\sin 2\theta+\ldots+\sin k\theta)=\frac{(n+1)\sin\theta-\sin(n+1)\theta}{4\sin^2\frac{\theta}{2}}.$$

$\left(\text{You may assume that } \dfrac{z}{1-z}=\dfrac{i}{2\sin\frac{\theta}{2}}\left(\cos\frac{\theta}{2}+i\sin\frac{\theta}{2}\right).\right)$

(b) A fair coin is to be tossed repeatedly. For integers r and s, not both zero, let $P(r,s)$ be the probability that a total of r heads are tossed before a total of s tails are tossed so that $P(0,1)=1$ and $P(1,0)=0$. **9**

(i) Explain why, for $r,\ s\geq 1$,

$$P(r,s)=\frac{1}{2}P(r-1,s)+\frac{1}{2}P(r,s-1)$$

(ii) Find $P(2,3)$ by using part (i).

(iii) By using induction on $n = r + s - 1$, or otherwise, prove that

$$P(r,s)=\frac{1}{2^n}\left\{\binom{n}{0}+\binom{n}{1}+\ldots+\binom{n}{s-1}\right\}\ \text{for}\ s\geq 1.$$

End of paper

BLANK PAGE

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$