BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1995
MATHEMATICS
4 UNIT (ADDITIONAL)

Time allowed—Three hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Each question attempted is to be returned in a *separate* Writing Booklet clearly marked Question 1, Question 2, etc. on the cover. Each booklet must show your Student Number and the Centre Number.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1. Use a *separate* Writing Booklet. **Marks**

(a) Find $\displaystyle\int \frac{dx}{x(\ln x)^2}$. **2**

(b) Find $\displaystyle\int xe^x\,dx$. **2**

(c) Show that $\displaystyle\int_1^4 \frac{6t+23}{(2t-1)(t+6)}\,dt = \ln 70$. **4**

(d) Find $\displaystyle\frac{d}{dx}\left(x\sin^{-1}x\right)$, and hence find $\displaystyle\int \sin^{-1}x\,dx$. **3**

(e) Using the substitution $t=\tan\dfrac{x}{2}$, or otherwise, calculate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5+3\sin x+4\cos x}$. **4**

QUESTION 2. Use a *separate* Writing Booklet. **Marks**

(a) Let $w_1 = 8 - 2i$ and $w_2 = -5 + 3i$. **1**

Find $w_1 + \overline{w}_2$.

(b) (i) Show that $(1 - 2i)^2 = -3 - 4i$. **3**

(ii) Hence solve the equation

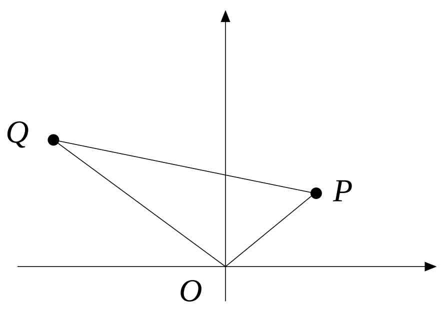
$$z^2 - 5z + (7 + i) = 0.$$

(c) Sketch the locus of z satisfying: **5**

(i) $\arg(z - 4) = \dfrac{3\pi}{4}$;

(ii) $\operatorname{Im} z = |z|$.

(d) **6**



The diagram shows a complex plane with origin O. The points P and Q represent arbitrary non-zero complex numbers z and w respectively. Thus the length of PQ is $|z - w|$.

(i) Copy the diagram into your Writing Booklet, and use it to show that

$$|z - w| \le |z| + |w|.$$

(ii) Construct the point R representing $z + w$.

What can be said about the quadrilateral $OPRQ$?

(iii) If $|z - w| = |z + w|$, what can be said about the complex number $\dfrac{w}{z}$?

QUESTION 3. Use a *separate* Writing Booklet. **Marks**

(a) Let $f(x) = -x^2 + 6x - 8$. **10**

On separate diagrams, and without using calculus, sketch the following graphs. Indicate clearly any asymptotes and intercepts with the axes.

(i) $y = f(x)$

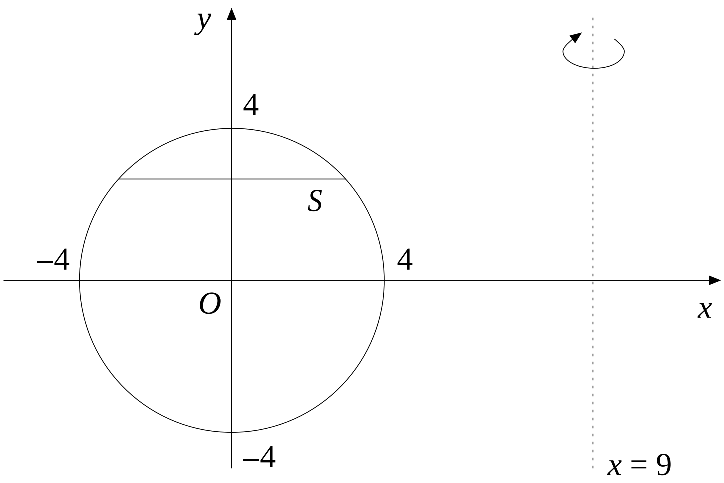
(ii) $y = |f(x)|$

(iii) $y^2 = f(x)$

(iv) $y = \dfrac{1}{f(x)}$

(v) $y = e^{f(x)}$

(b) **5**



The circle $x^2 + y^2 = 16$ is rotated about the line $x = 9$ to form a ring.

When the circle is rotated, the line segment S at height y sweeps out an annulus.

The x coordinates of the end-points of S are x_1 and $-x_1$, where $x_1 = \sqrt{16 - y^2}$.

(i) Show that the area of the annulus is equal to

$$36\pi\sqrt{16 - y^2}.$$

(ii) Hence find the volume of the ring.

QUESTION 4. Use a *separate* Writing Booklet. **Marks**

(a) (i) Find the least positive integer k such that **4**

$$\cos\left(\frac{4\pi}{7}\right)+i\sin\left(\frac{4\pi}{7}\right)$$

is a solution of $z^k = 1$.

(ii) Show that if the complex number w is a solution of $z^n = 1$, then so is w^m, where m and n are arbitrary integers.

(b) (i) Solve $x^2 > 2x+1$. **4**

(ii) Prove by mathematical induction that $2^n > n^2$ for all integers $n \geq 5$.

(c) (i) Show that, if $0 < x < \frac{\pi}{2}$, then **7**

$$\frac{\sin(2m+1)x}{\sin x} - \frac{\sin(2m-1)x}{\sin x} = 2\cos(2mx).$$

(ii) Show that, for any positive integer m,

$$\int_0^{\frac{\pi}{2}} \cos(2mx)\,dx = 0.$$

(iii) Deduce that, if m is any positive integer,

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2m+1)x}{\sin x}\,dx = \int_0^{\frac{\pi}{2}} \frac{\sin(2m-1)x}{\sin x}\,dx.$$

(iv) Show that, if $m = 1$, then

$$\int_0^{\frac{\pi}{2}} \frac{\sin(2m-1)x}{\sin x}\,dx = \frac{\pi}{2}.$$

(v) Hence show that

$$\int_0^{\frac{\pi}{2}} \frac{\sin 5x}{\sin x}\,dx = \frac{\pi}{2}.$$

QUESTION 5. Use a *separate* Writing Booklet. **Marks**

(a) (i) Show that $\sin x + \sin 3x = 2\sin 2x \cos x$. **4**

(ii) Hence or otherwise, find all solutions of

$$\sin x + \sin 2x + \sin 3x = 0 \quad \text{for} \quad 0 \le x < 2\pi.$$

(b) Let $f(t) = t^3 + ct + d$, where c and d are constants. **6**

Suppose that the equation $f(t) = 0$ has three distinct real roots, t_1, t_2, and t_3.

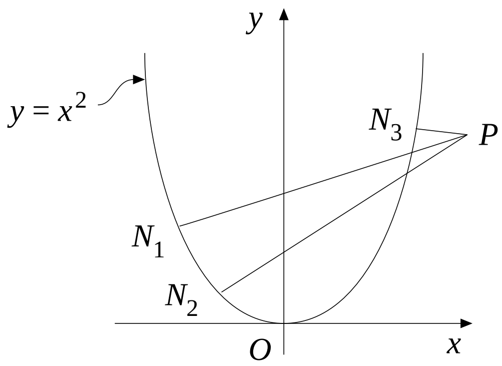
(i) Find $t_1 + t_2 + t_3$.

(ii) Show that $t_1^2 + t_2^2 + t_3^2 = -2c$.

(iii) Since the roots are real and distinct, the graph of $y = f(t)$ has two turning points, at $t = u$ and $t = v$, and $f(u).f(v) < 0$.

Show that $27d^2 + 4c^3 < 0$.

(c) **5**



Consider the parabola $y = x^2$.

Some points (e.g. P) lie on three distinct normals (PN_1, PN_2, and PN_3) to the parabola.

(i) Show that the equation of the normal to $y = x^2$ at the point (t, t^2) may be written as

$$t^3 + \left(\frac{1-2y}{2}\right)t + \left(\frac{-x}{2}\right) = 0.$$

(ii) Suppose that the normals to $y = x^2$ at three distinct points $N_1(t_1, t_1^2)$, $N_2(t_2, t_2^2)$, and $N_3(t_3, t_3^2)$ all pass through $P(x_0, y_0)$.

Using the result of part (b) (iii), show that the coordinates of P satisfy

$$y_0 > 3\left(\frac{x_0}{4}\right)^{\frac{2}{3}} + \frac{1}{2}.$$

QUESTION 6. Use a *separate* Writing Booklet. **Marks**

(a) Pat observed an aeroplane flying at a constant height, h, and with constant velocity. Pat first sighted it due east, at an angle of elevation of 45°. A short time later it was exactly north-east, at an angle of elevation of $60°$. **6**

(i) Draw a diagram to represent this information.

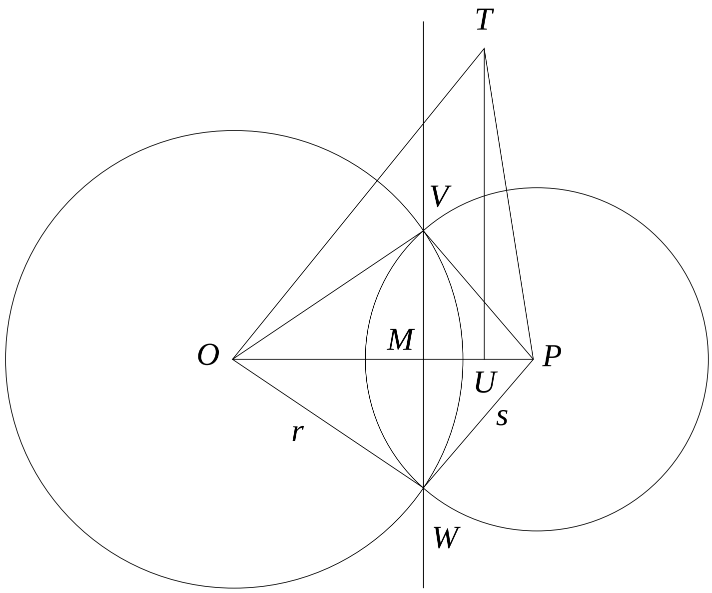
(ii) Find an expression in terms of h for the initial horizontal distance between Pat and the point directly below the aeroplane.

(iii) In what direction was the aeroplane flying? Give your answer as a bearing to the nearest degree.

Question 6 continues on page 8

QUESTION 6. (Continued) **Marks**

(b) **9**



In the above diagram, a circle with centre O and radius r meets a circle with centre P and radius s at the points V and W. The straight lines VW and OP meet at M. The point T is arbitrary, and U is the point on the line OP such that TU is perpendicular to OP.

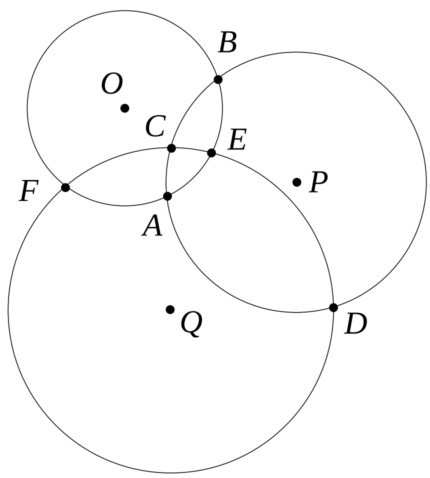
(i) Prove that OP and VW are perpendicular.

(ii) Show that $OT^2 - PT^2 = OU^2 - PU^2$ and that $OM^2 - PM^2 = r^2 - s^2$.

(iii) Hence show that T lies on the line VW exactly when

$$OT^2 - PT^2 = r^2 - s^2.$$

(iv)



$FAEB$, $BCAD$, and $DECF$ are circles with centres O, P, and Q, and radii r, s, and t, respectively.

Using the result of part (iii), or otherwise, show that the straight lines AB, CD, and EF are concurrent.

QUESTION 7. Use a *separate* Writing Booklet. **Marks**

(a) Let $I_n = \int_0^{\frac{\pi}{2}} (\sin x)^n \, dx$, where n is an integer, $n \geq 0$. **8**

(i) Using integration by parts, show that, for $n \geq 2$,

$$I_n = \left(\frac{n-1}{n}\right) I_{n-2}.$$

(ii) Deduce that

$$I_{2n} = \frac{2n-1}{2n} \cdot \frac{2n-3}{2n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

and

$$I_{2n+1} = \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1.$$

(iii) Explain why $I_k > I_{k+1}$.

(iv) Hence, using the fact that $I_{2n-1} > I_{2n}$ and $I_{2n} > I_{2n+1}$, show that

$$\frac{\pi}{2}\left(\frac{2n}{2n+1}\right) < \frac{2^2 . 4^2 \ldots (2n)^2}{1.3^2.5^2 \ldots (2n-1)^2 (2n+1)} < \frac{\pi}{2}.$$

(b) A fair coin is tossed $2n$ times. The probability of observing k heads and $(2n-k)$ tails is given by **7**

$$P_k = \binom{2n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{2n-k}.$$

(i) Show that the most likely outcome is $k = n$. That is, show that P_k is greatest when $k = n$.

(ii) Show that $P_n = \frac{(2n)!}{2^{2n}(n!)^2}$.

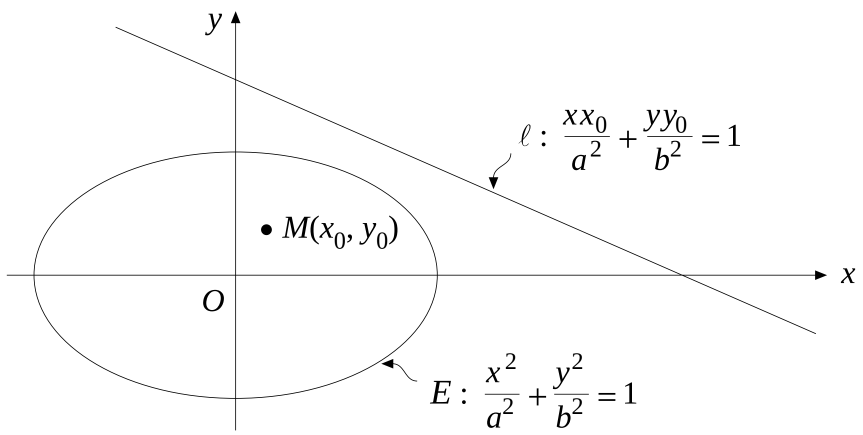
(iii) Using the result of part (a) (iii), show that

$$\frac{1}{\sqrt{\pi\left(n + \frac{1}{2}\right)}} < P_n < \frac{1}{\sqrt{\pi n}}.$$

QUESTION 8. Use a *separate* Writing Booklet. **Marks**

(a) Suppose that p and q are real numbers. Show that $pq \le \dfrac{p^2+q^2}{2}$. **1**

(b) **6**



The ellipse E is given by the equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

The point $M(x_0, y_0)$ lies inside E, so that $\dfrac{x_0^{\,2}}{a^2} + \dfrac{y_0^{\,2}}{b^2} < 1$.

The line l is given by the equation $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$.

(i) Using the result of part (a), or otherwise, show that the line l lies entirely outside E. That is, show that if $P(x_1, y_1)$ is any point on l, then

$$\frac{x_1^{\,2}}{a^2} + \frac{y_1^{\,2}}{b^2} > 1.$$

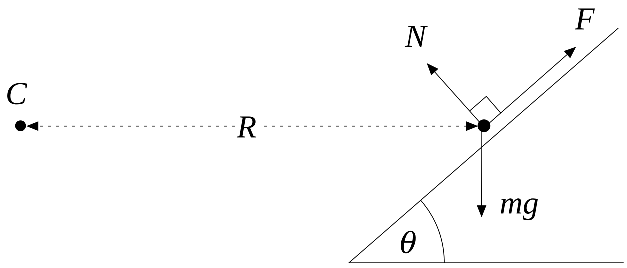
(ii) The chord of contact to E from any point $Q(x_2, y_2)$ outside E has equation

$$\frac{xx_2}{a^2} + \frac{yy_2}{b^2} = 1.$$

Show that M lies on the chord of contact to E from any point on l.

QUESTION 8. (Continued) **Marks**

(c) **8**



A particle of mass m travels at constant speed v round a circular track of radius R, centre C. The track is banked inwards at an angle θ, and the particle does not move up or down the bank.

The reaction exerted by the track on the particle has a normal component N, and a component F due to friction, directed up or down the bank. The force F lies in the range from $-\mu N$ to μN, where μ is a positive constant and N is the normal component; the sign of F is positive when F is directed up the bank.

The acceleration due to gravity is g.

The acceleration related to the circular motion is of magnitude $\frac{v^2}{R}$, and is directed towards the centre of the track.

(i) By resolving forces horizontally and vertically, show that

$$\frac{v^2}{Rg} = \frac{N\sin\theta - F\cos\theta}{N\cos\theta + F\sin\theta}.$$

(ii) Show that the maximum speed $v_{\max}$ at which the particle can travel without slipping up the track is given by

$$\frac{v_{\max}^2}{Rg} = \frac{\tan\theta + \mu}{1 - \mu\tan\theta}.$$

[You may suppose that $\mu\tan\theta < 1$.]

(iii) Show that if $\mu \geq \tan\theta$, then the particle will not slide down the track, regardless of its speed.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$