BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1996
MATHEMATICS
3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON)

Time allowed—Two hours
(Plus 5 minutes' reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a *separate* Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

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QUESTION 1. Use a *separate* Writing Booklet.

Marks

(a) $(x-2)$ is a factor of the polynomial $P(x) = 2x^3 + x + a$. Find the value of a. **1**

(b) Find the acute angle between the lines $2x + y = 4$ and $x - y = 2$, to the nearest degree. **2**

(c) Let $A(-1, 2)$ and $B(3, 5)$ be points in the plane. Find the coordinates of the point C which divides the interval AB externally in the ratio 3 : 1. **2**

(d) A committee of 3 men and 4 women is to be formed from a group of 8 men and 6 women. Write an expression for the number of ways this can be done. **1**

(e) Solve the inequality $\dfrac{2}{x-1} \leq 1$. **3**

(f) Using the substitution $u = e^x$, find $\displaystyle\int \frac{e^x}{1+e^{2x}}\,dx$. **3**

QUESTION 2. Use a *separate* Writing Booklet.

Marks

(a) The function $f(x) = x^3 - \ln(x+1)$ has one root between 0·5 and 1. **4**

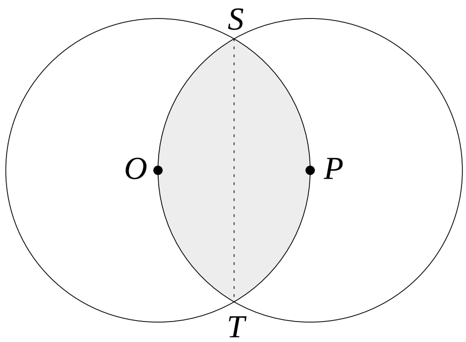
(i) Show that the root lies between 0·8 and 0·9.

(ii) Hence use the halving-the-interval method to find the value of the root, correct to one decimal place.

(b) Use the table of standard integrals on page 12 to show that **3**

$$\int_6^{15} \frac{dx}{\sqrt{x^2+64}} = \ln 2.$$

(c) The points O and P in the plane are d cm apart. A circle centre O is drawn to pass through P, and another circle centre P is drawn to pass through O. The two circles meet at S and T, as in the diagram. **5**



(i) Show that triangle SOP is equilateral.

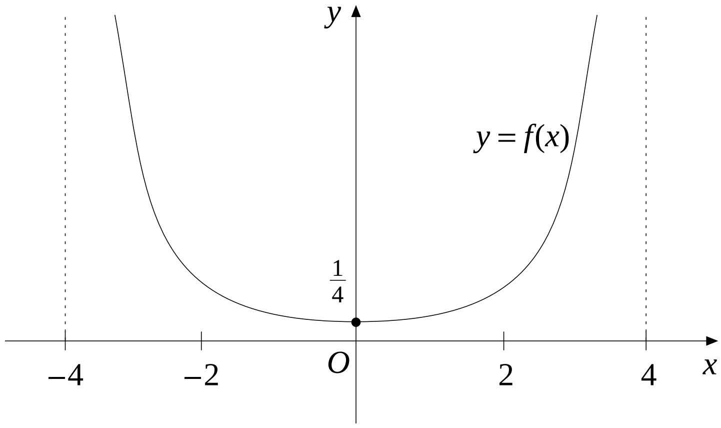
(ii) Show that the size of angle SOT is $\dfrac{2\pi}{3}$.

(iii) Hence find the area of the shaded region in terms of d.

QUESTION 3. Use a *separate* Writing Booklet.

Marks

(a) **4**



Let $f(x)=\dfrac{1}{\sqrt{16-x^2}}$. The graph of $y=f(x)$ is sketched above.

(i) Show that $f(x)$ is an even function.

(ii) Find the area enclosed by $y=f(x)$, the x axis, $x=-2$, and $x=2$.

(b) Show that $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\cos^2 x\,dx=\frac{\pi}{8}-\frac{1}{4}$. **2**

(c) The function $g(x)$ is given by $g(x)=2+\cos x$. The graph $y=g(x)$ for $\dfrac{\pi}{4}\le x\le\dfrac{\pi}{2}$ is rotated about the x axis. **3**

Find the volume of the solid generated. (You may use the result of part (b).)

(d) The function $h(x)$ is given by **3**

$$h(x)=\sin^{-1}x+\cos^{-1}x,\quad 0\le x\le 1.$$

(i) Find $h'(x)$.

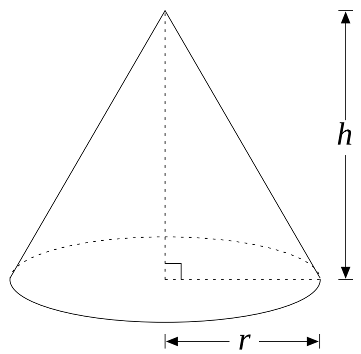
(ii) Sketch the graph of $y=h(x)$.

QUESTION 4. Use a *separate* Writing Booklet.

Marks

(a) Prove that $\dfrac{\sin 3\theta}{\sin\theta} - \dfrac{\cos 3\theta}{\cos\theta} = 2 \quad (\text{for } \sin\theta \neq 0,\ \cos\theta \neq 0)$. **3**

(b) **4**



Grain is poured at a constant rate of 0·5 cubic metres per second. It forms a conical pile, with the angle at the apex of the cone equal to 60°. The height of the pile is h metres, and the radius of the base is r metres.

(i) Show that $r = \dfrac{h}{\sqrt{3}}$.

(ii) Show that V, the volume of the pile, is given by

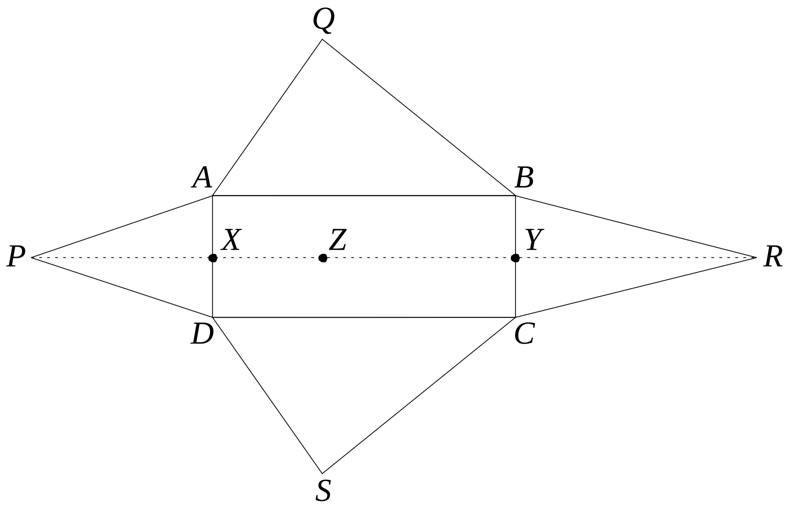
$$V = \frac{\pi h^3}{9}.$$

(iii) Hence find the rate at which the height of the pile is increasing when the height of the pile is 3 metres.

QUESTION 4. (Continued)

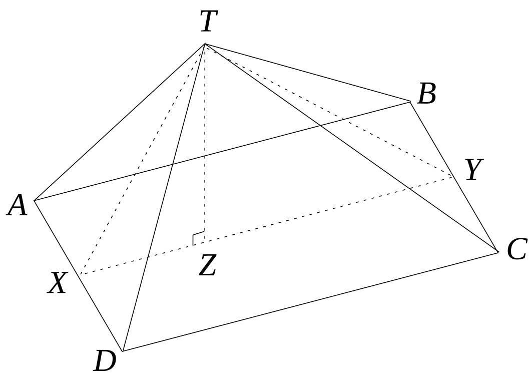
Marks

(c) **5**



The figure shows the net of an oblique pyramid with a rectangular base. In this figure, $PXZYR$ is a straight line, $PX = 15$ cm, $RY = 20$ cm, $AB = 25$ cm, and $BC = 10$ cm. Further, $AP = PD$ and $BR = RC$.

When the net is folded, points P, Q, R, and S all meet at the apex T, which lies vertically above the point Z in the horizontal base, as shown below.



(i) Show that triangle TXY is right-angled.

(ii) Hence show that T is 12 cm above the base.

(iii) Hence find the angle that the face DCT makes with the base.

QUESTION 5. Use a *separate* Writing Booklet.

Marks

(a) A cup of hot coffee at temperature T°C loses heat when placed in a cooler environment. It cools according to the law **5**

$$\frac{dT}{dt} = k(T - T_0),$$

where t is time elapsed in minutes, and T_0 is the temperature of the environment in degrees Celsius.

(i) A cup of coffee at 100°C is placed in an environment at –20°C for 3 minutes, and cools to 70°C. Find k.

(ii) The same cup of coffee, at 70°C, is then placed in an environment at 20°C. Assuming k remains the same, find the temperature of the coffee after a further 15 minutes.

(b) A particle is moving along the x axis. Its velocity v at position x is given by **2**

$$v = \sqrt{10x - x^2}.$$

Find the acceleration of the particle when $x = 4$.

(c) Mice are placed in the centre of a maze which has five exits. Each mouse is equally likely to leave the maze through any one of the five exits. Thus, the probability of any given mouse leaving by a particular exit is $\frac{1}{5}$. **5**

Four mice, A, B, C, and D, are put into the maze and behave independently.

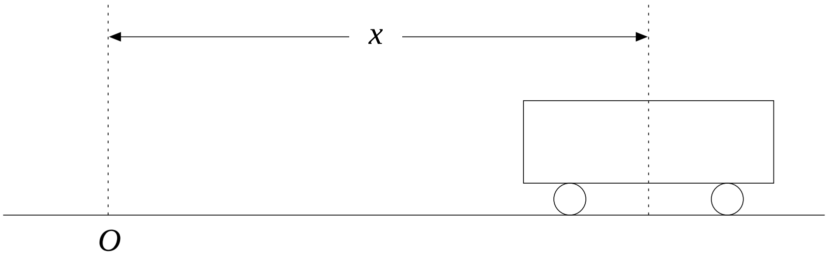
(i) What is the probability that A, B, C, and D all come out the same exit?

(ii) What is the probability that A, B, and C come out the same exit, and D comes out a different exit?

(iii) What is the probability that *any* three of the four mice come out the same exit, and the other comes out a different exit?

(iv) What is the probability that no more than two mice come out the same exit?

QUESTION 6. Use a *separate* Writing Booklet. **Marks**



A trolley is moving in simple harmonic motion about the origin O. The displacement, x metres, of the centre of the trolley from O at time t seconds is given by

$$x = 6\sin\left(2t + \frac{\pi}{4}\right).$$

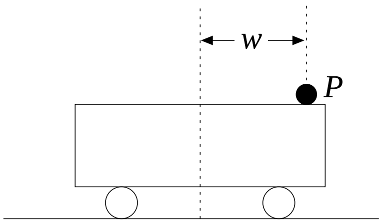
(a) State the period and amplitude of the motion. **2**

(b) Sketch the graph of $x = 6\sin\left(2t + \frac{\pi}{4}\right)$ for $0 \le t \le 2\pi$. **2**

(c) Find the velocity of the trolley when $t = 0$. **2**

(d) Find the first time after $t = 0$ when the centre of the trolley is at $x = 3$. **2**

(e) **4**



A particle P, on top of the trolley, is moving in simple harmonic motion about the centre of the trolley. Its displacement, w metres, from the centre of the trolley at time t seconds, is given by

$$w = \sin(2t).$$

The displacement, y metres, of P from the origin is the sum of the two displacements x and w, so that

$$y = 6\sin\left(2t + \frac{\pi}{4}\right) + \sin(2t).$$

(i) Show that P is moving in simple harmonic motion about O.

(ii) Find the amplitude of this motion.

QUESTION 7. Use a *separate* Writing Booklet.

Marks

(a) Using the fact that $(1+x)^4(1+x)^9 = (1+x)^{13}$, show that **2**

$$ {}^4C_0\,{}^9C_4 + {}^4C_1\,{}^9C_3 + {}^4C_2\,{}^9C_2 + {}^4C_3\,{}^9C_1 + {}^4C_4\,{}^9C_0 = {}^{13}C_4. $$

(b) Consider the function $f(x) = \frac{1}{4}\left[(x-1)^2 + 7\right]$. **10**

(i) Sketch the parabola $y = f(x)$, showing clearly any intercepts with the axes, and the coordinates of its vertex. Use the same scale on both axes.

(ii) What is the largest domain containing the value $x = 3$, for which the function has an inverse function $f^{-1}(x)$?

(iii) Sketch the graph of $y = f^{-1}(x)$ on the same set of axes as your graph in part (i). Label the two graphs clearly.

(iv) What is the domain of the inverse function?

(v) Let a be a real number not in the domain found in part (ii). Find $f^{-1}(f(a))$.

(vi) Find the coordinates of any points of intersection of the two curves $y = f(x)$ and $y = f^{-1}(x)$.

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STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$