BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1997

MATHEMATICS

2/3 UNIT (COMMON)

Time allowed—Three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a *separate* Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1. Use a *separate* Writing Booklet.

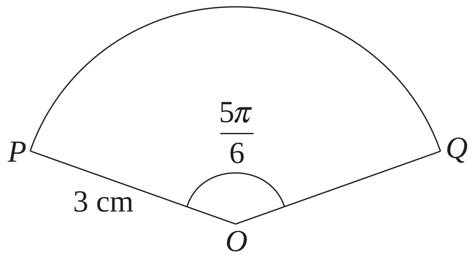
Marks

(a) Find the value of $\dfrac{1}{7+5\times 3}$ correct to three significant figures. **2**

(b) Simplify $(2-3x)-(5-4x)$. **2**

(c) Write down the exact value of $135°$ in radians. **1**

(d) **1**



In the diagram, PQ is an arc of a circle with centre O. The radius $OP = 3$ cm and the angle POQ is $\dfrac{5\pi}{6}$ radians.

Find the length of the arc PQ.

(e) Using the table of standard integrals, find $\int \sec 3x \tan 3x \, dx$. **1**

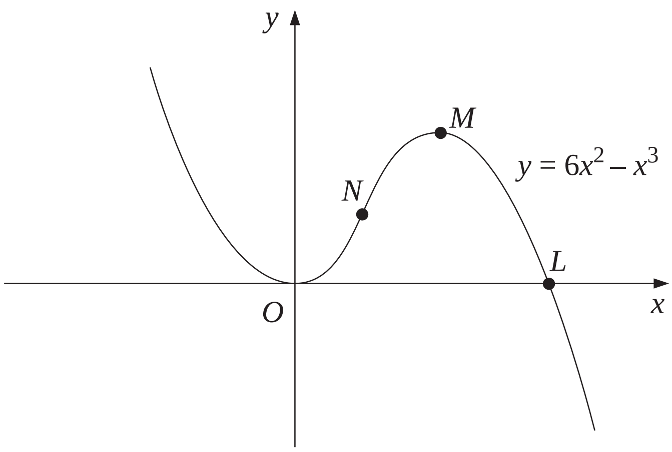
(f) Forty-five balls, numbered 1 to 45, are placed in a barrel, and one ball is drawn at random. What is the probability that the number on the ball drawn is even? **2**

(g) By rationalising the denominator, express $\dfrac{8}{3-\sqrt{5}}$ in the form $a + b\sqrt{5}$. **3**

QUESTION 2. Use a *separate* Writing Booklet.

Marks

(a) **6**



The diagram shows a sketch of the curve $y = 6x^2 - x^3$. The curve cuts the x axis at L, and has a local maximum at M and a point of inflection at N.

(i) Find the coordinates of L.

(ii) Find the coordinates of M.

(iii) Find the coordinates of N.

(b) The graph of $y = f(x)$ passes through the point $(1, 4)$ and $f'(x) = 2x + 7$. Find $f(x)$. **2**

(c) Find a primitive of $\dfrac{2x}{x^2 + 1}$. **1**

(d) Consider the parabola with equation $x^2 = 4(y - 1)$. **3**

(i) Find the coordinates of the vertex of the parabola.

(ii) Find the coordinates of the focus of the parabola.

QUESTION 3. Use a *separate* Writing Booklet.

Marks

(a) Differentiate the following functions: **6**

(i) $\left(x^2+5\right)^3$

(ii) $\dfrac{\cos x}{x}$

(iii) $x^2 \ln x$.

(b) Let A and B be the points $(0,1)$ and $(2,3)$ respectively. **6**

(i) Find the coordinates of the midpoint of AB.

(ii) Find the slope of the line AB.

(iii) Find the equation of the perpendicular bisector of AB.

(iv) The point P lies on the line $y=2x-9$ and is equidistant from A and B. Find the coordinates of P.

QUESTION 4. Use a *separate* Writing Booklet.

Marks

(a) The table shows the values of a function $f(x)$ for five values of x. **3**

x	1	1·5	2	2·5	3
$f(x)$	5	1	–2	3	7

Use Simpson's rule with these five values to estimate $\int_1^3 f(x)\,dx$.

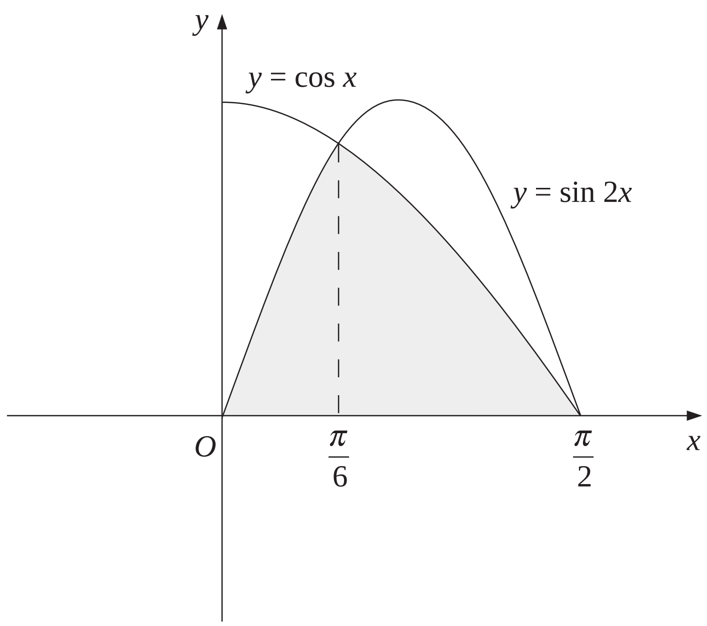
(b) (i) Sketch the graph of $y = x^2 - 6$, and label all intercepts with the axes. **6**

(ii) On the same set of axes, carefully sketch the graph of $y = |x|$.

(iii) Find the x coordinates of the two points where the graphs intersect.

(iv) Hence solve the inequality $x^2 - 6 \leq |x|$.

(c) **3**



The diagram shows the graphs of the functions $y = \cos x$ and $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{2}$. The two graphs intersect at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

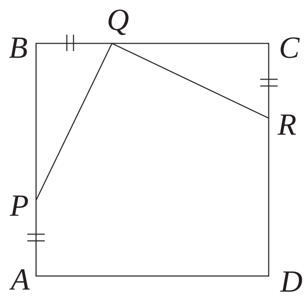
Calculate the area of the shaded region.

QUESTION 5. Use a *separate* Writing Booklet.

Marks

(a) Evaluate $\int_0^{\ln 7} e^{-x}\,dx$. **2**

(b) **4**



In the diagram, $ABCD$ is a square. The points P, Q, and R lie on AB, BC, and CD respectively, such that $AP = BQ = CR$.

(i) Prove that triangles PBQ and QCR are congruent.

(ii) Prove that $\angle PQR$ is a right angle.

(c) The rate of inflation measures the rate of change in prices. Between January 1996 and December 1996, prices were rising but the rate of inflation was falling. Draw a graph of prices as a function of time that fits this description. **2**

(d) A ball is dropped from a height of 2 metres onto a hard floor and bounces. After each bounce, the maximum height reached by the ball is 75% of the previous maximum height. Thus, after it first hits the floor, it reaches a height of 1·5 metres before falling again, and after the second bounce, it reaches a height of 1·125 metres before falling again. **4**

(i) What is the maximum height reached after the third bounce?

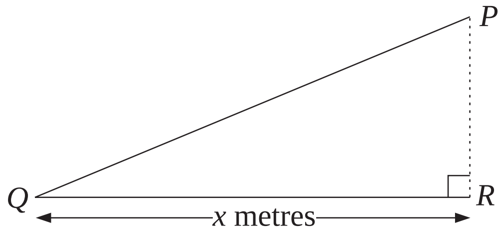
(ii) What kind of sequence is formed by the successive maximum heights?

(iii) What is the total distance travelled by the ball from the time it was first dropped until it eventually comes to rest on the floor?

QUESTION 6. Use a *separate* Writing Booklet.

Marks

(a) **6**



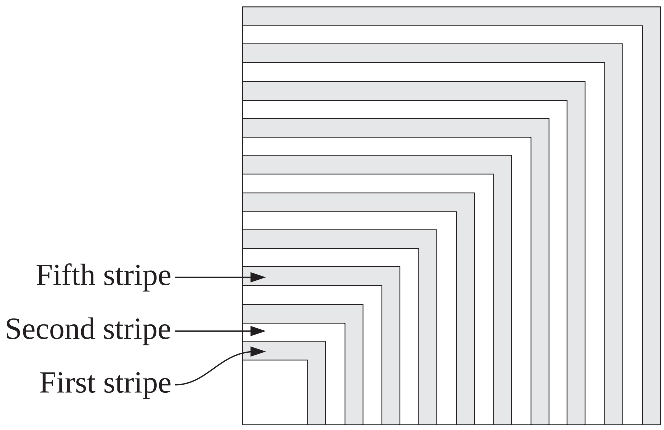
A wire of length 5 metres is to be bent to form the hypotenuse and base of a right-angled triangle PQR, as shown in the diagram. Let the length of the base QR be x metres.

(i) What is the length of the hypotenuse PQ in terms of x?

(ii) Show that the area of the triangle PQR is $\frac{1}{2}x\sqrt{25-10x}$ square metres.

(iii) What is the maximum possible area of the triangle?

(b) **6**



A logo is made of 20 squares with a common corner, as shown in the diagram. The odd-numbered 'stripes' between successive squares are shaded in the diagram. The shaded stripes are painted in gold paint, which costs \$9 per square centimetre.

The side length of the nth square is $(2n+4)$ cm. The nth stripe lies between the nth square and the $(n+1)$th square.

(i) Show that the area of the nth stripe is $(8n+20)$ cm^2.

(ii) Hence find the areas of the first and last stripes.

(iii) Hence find the total cost of the gold paint for the logo.

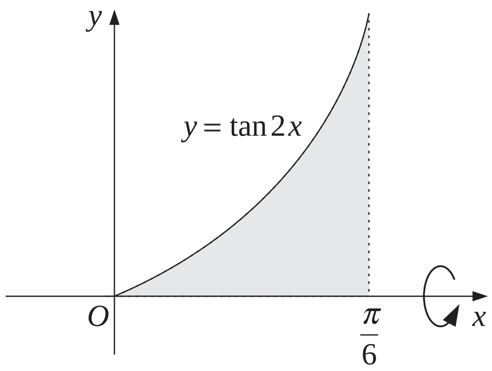
QUESTION 7. Use a *separate* Writing Booklet.

Marks

(a) By expressing $\sec\theta$ and $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, show that **2**

$$\sec^2\theta - \tan^2\theta = 1.$$

(b) **5**



The diagram shows part of the graph of the function $y = \tan 2x$. The shaded region is bounded by the curve, the x axis, and the line $x = \frac{\pi}{6}$. The region is rotated about the x axis to form a solid.

(i) Show that the volume of the solid is given by

$$V = \pi \int_0^{\frac{\pi}{6}} \left(\sec^2 2x - 1\right) dx.$$

You may use the result of part (a).

(ii) Find the exact volume of the solid.

(c) A ball is dropped into a long vertical tube filled with honey. The rate at which the ball decelerates is proportional to its velocity. Thus **5**

$$\frac{dv}{dt} = -kv,$$

where v is the velocity in centimetres per second, t is the time in seconds, and k is a constant.

When the ball first enters the honey, at $t = 0$, $v = 100$. When $t = 0{\cdot}25$, $v = 85$.

(i) Show that $v = Ce^{-kt}$ satisfies the equation $\frac{dv}{dt} = -kv$.

(ii) Find the value of the constant C.

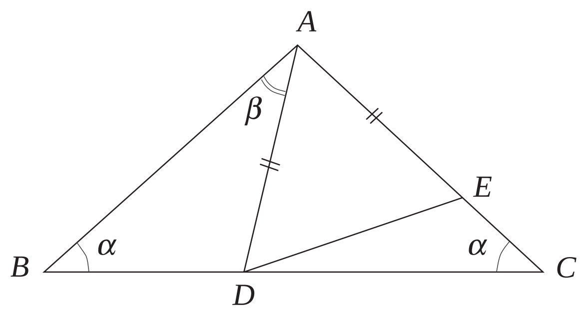
(iii) Find the value of the constant k.

(iv) Find the velocity when $t = 2$.

QUESTION 8. Use a *separate* Writing Booklet.

Marks

(a) **4**



In the isosceles triangle ABC, $\angle ABC = \angle ACB = \alpha$. The points D and E lie on BC and AC, so that $AD = AE$, as shown in the diagram. Let $\angle BAD = \beta$.

(i) Explain why $\angle ADC = \alpha + \beta$.

(ii) Find $\angle DAC$ in terms of α and β.

(iii) Hence, or otherwise, find $\angle EDC$ in terms of β.

(b) A particle is moving along the x axis. Its position at time t is given by **8**

$$x = t + \sin t.$$

(i) At what times during the period $0 < t < 3\pi$ is the particle stationary?

(ii) At what times during the period $0 < t < 3\pi$ is the acceleration equal to 0?

(iii) Carefully sketch the graph of $x = t + \sin t$ for $0 < t < 3\pi$.

Clearly label any stationary points and any points of inflection.

QUESTION 9. Use a *separate* Writing Booklet.

Marks

(a) A bag contains two red balls, one black ball, and one white ball. Andrew selects one ball from the bag and keeps it hidden. He then selects a second ball, also keeping it hidden. **5**

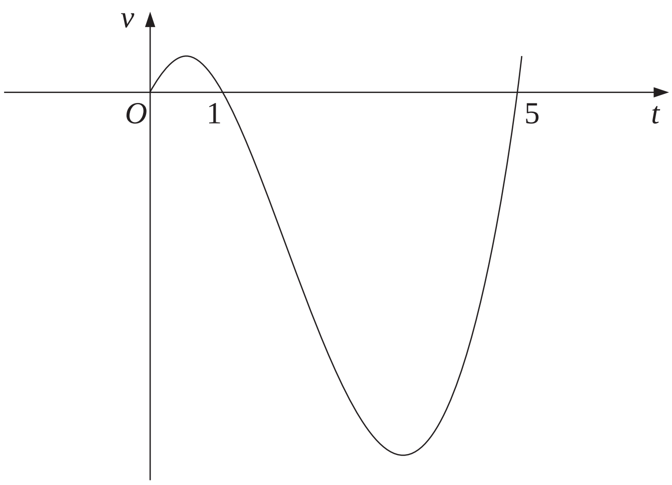
(i) Draw a tree diagram to show all the possible outcomes.

(ii) Find the probability that both the selected balls are red.

(iii) Find the probability that at least one of the selected balls is red.

(iv) Andrew drops one of the selected balls and we can see that it is red. What is the probability that the ball that is still hidden is also red?

(b) **7**



A pen moves along the x axis, ruling a line. The diagram shows the graph of the velocity of the tip of the pen as a function of time.

The velocity, in centimetres per second, is given by the equation

$$v = 4t^3 - 24t^2 + 20t,$$

where t is the time in seconds. When $t = 0$, the tip of the pen is at $x = 3$. That is, the tip is initially 3 centimetres to the right of the origin.

(i) Find an expression for x, the position of the tip of the pen, as a function of time.

(ii) What feature will the graph of x as a function of t have at $t = 1$?

(iii) The pen uses 0·05 milligrams of ink per centimetre travelled.

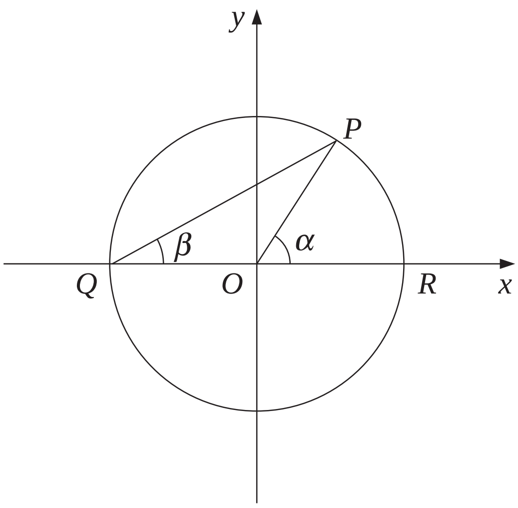
How much ink is used between $t = 0$ and $t = 2$?

QUESTION 10. Use a *separate* Writing Booklet.

Marks

(a) Graph the solution of $4x \leq 15 \leq -9x$ on a number line. **2**

(b) **10**



In the diagram, Q is the point $(-1, 0)$, R is the point $(1, 0)$, and P is another point on the circle with centre O and radius 1. Let $\angle POR = \alpha$ and $\angle PQR = \beta$, and let $\tan \beta = m$.

(i) Explain why ΔOPQ is isosceles, and hence deduce that $\alpha = 2\beta$.

(ii) Find the equation of the line PQ.

(iii) Show that the x coordinates of P and Q are solutions of the equation

$$\left(1+m^2\right)x^2 + 2m^2x + m^2 - 1 = 0.$$

(iv) Using this equation, find the coordinates of P in terms of m.

(v) Hence deduce that $\tan 2\beta = \dfrac{2\tan\beta}{1-\tan^2\beta}$.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$