BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1997

MATHEMATICS

3 UNIT (ADDITIONAL)
AND
3/4 UNIT (COMMON)

Time allowed—Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a *separate* Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

Marks

BLANK PAGE

QUESTION 1. Use a *separate* Writing Booklet.

Marks

(a) Differentiate $e^{3x}\cos x$. **2**

(b) Find the perpendicular distance from the point $(1, 2)$ to the line $y = 3x + 4$. **2**

(c) Given that $\log_a b = 2{\cdot}8$ and $\log_a c = 4{\cdot}1$, find $\log_a\left(\frac{b}{c}\right)$. **1**

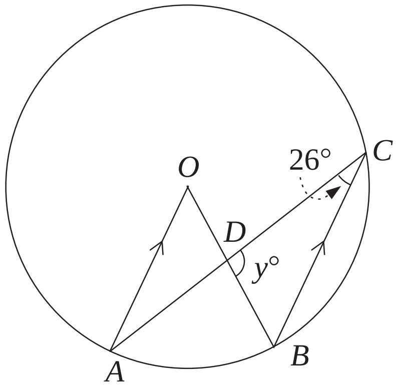
(d) Evaluate $\displaystyle\int_0^2 \frac{dx}{4+x^2}$. **3**

(e) Using the substitution $u = 2x + 1$, or otherwise, find $\displaystyle\int_0^1 \frac{4x}{2x+1}\,dx$. **4**

QUESTION 2. Use a *separate* Writing Booklet.

Marks

(a) **3**



The points A, B, and C lie on a circle with centre O. The lines AO and BC are parallel, and OB and AC intersect at D. Also, $\angle ACB = 26°$ and $\angle BDC = y°$, as shown in the diagram.

Copy or trace the diagram into your Writing Booklet.

(i) State why $\angle AOB = 52°$.

(ii) Find y. Justify your answer.

(b) A particle is moving in simple harmonic motion. Its displacement x at time t is given by **4**

$$x = 3\sin(2t + 5).$$

(i) Find the period of the motion.

(ii) Find the maximum acceleration of the particle.

(iii) Find the speed of the particle when $x = 2$.

(c) The polynomial $P(x) = x^3 + bx^2 + cx + d$ has roots 0, 3, and –3. **5**

(i) Find b, c, and d.

(ii) Without using calculus, sketch the graph of $y = P(x)$.

(iii) Hence, or otherwise, solve the inequality $\dfrac{x^2 - 9}{x} > 0$.

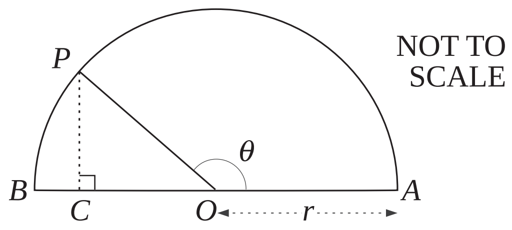
QUESTION 3. Use a *separate* Writing Booklet.

Marks

(a) (i) On the same set of axes, sketch the graphs of $y = 2\sin\theta$ and $y = \theta$ for $-\pi \le \theta \le \pi$. **3**

(ii) Use your sketch to find the number of solutions of the equation $2\sin\theta = \theta$ for $-\pi \le \theta \le \pi$.

(b) **4**



The point P lies on the circumference of a semicircle of radius r and diameter AB, as shown. The point C lies on AB and PC is perpendicular to AB.

The arc AP subtends an angle θ at the centre O, and the length of the arc AP is twice the length of PC.

(i) Show that $2\sin\theta = \theta$.

(ii) Taking $\theta = 1{\cdot}8$ as an approximation for the solution to the equation $2\sin\theta = \theta$ between $\frac{\pi}{2}$ and π, use one application of Newton's method to give a better approximation.

(c) In each game of Sic Bo, three regular, six-sided dice are thrown once. **5**

(i) In a single game, what is the probability that all three dice show 2?

(ii) What is the probability that exactly two of the dice show 2?

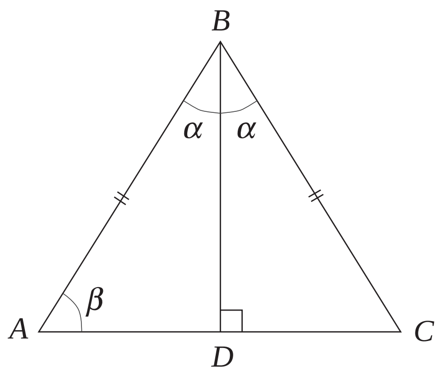
(iii) What is the probability that exactly two of the dice show the same number?

(iv) A player claims that you expect to see three different numbers on the dice in at least half of the games. Is the player correct? Justify your answer.

QUESTION 4. Use a *separate* Writing Booklet.

Marks

(a) **5**



The triangle ABC is isosceles, with $AB = BC$, and BD is perpendicular to AC.

Let $\angle ABD = \angle CBD = \alpha$ and $\angle BAD = \beta$, as shown in the diagram.

(i) Show that $\sin\beta = \cos\alpha$.

(ii) By applying the sine rule in ΔABC, show that $\sin 2\alpha = 2\sin\alpha\cos\alpha$.

(iii) Given that $0 < \alpha < \dfrac{\pi}{2}$, show that the limiting sum of the geometric series

$$\sin 2\alpha + \sin 2\alpha\cos^2\alpha + \sin 2\alpha\cos^4\alpha + \sin 2\alpha\cos^6\alpha + \cdots$$

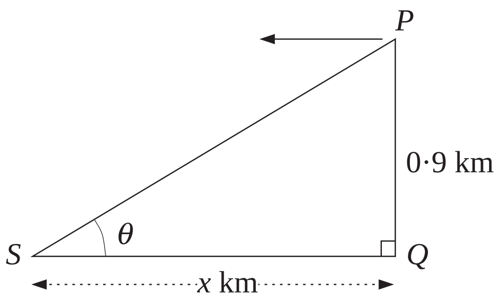
is equal to $2\cot\alpha$.

(b) By using the substitution $x = \sin t$, or otherwise, evaluate $\displaystyle\int_0^{\frac{1}{2}} \sqrt{1 - x^2}\, dx$. **3**

QUESTION 4. (Continued)

Marks

(c) **4**



A searchlight on the ground at S detects and tracks a plane P that is due east of the searchlight. The plane is flying due west at a constant velocity of 240 kilometres per hour and maintains a constant height of 900 metres above ground level.

Let $\theta(t)$ radians be the angle of elevation of the plane at time t seconds and let $x(t)$ kilometres be the distance from S to the point Q on the ground directly below P.

(i) Show that $\dfrac{dx}{d\theta} = -\dfrac{0{\cdot}9}{\sin^2\theta}$.

(ii) Show that the rate of change of the angle of elevation of the plane when $\theta = \dfrac{\pi}{4}$ is equal to $\dfrac{1}{27}$ radians per second.

QUESTION 5. Use a *separate* Writing Booklet.

Marks

(a) A particle moves along the x axis, starting at $x = 0{\cdot}1$ at time $t = 0$. The velocity of the particle is described by **7**

$$v = \sqrt{2x}\, e^{-x^2}, \quad x \geq 0{\cdot}1,$$

where x is the displacement of the particle from the origin.

(i) Show that the particle has acceleration given by

$$a = e^{-2x^2}\left(1 - 4x^2\right), \quad x \geq 0{\cdot}1.$$

(ii) Hence find the fastest speed attained by the particle.

(iii) Show that T, the time taken to travel from $x = 1$ to $x = 2$, can be expressed as

$$T = \int_1^2 \frac{1}{\sqrt{2x}} e^{x^2}\, dx.$$

(iv) Use the trapezoidal rule with three function values to obtain an approximate value for T.

(b) (i) For positive integers n and r, with $r < n$, show that **5**

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1},$$

where $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$. Do NOT use induction.

(ii) Use mathematical induction to prove that, for $n \geq 3$,

$$\sum_{j=3}^{n} \binom{j-1}{2} = \binom{n}{3}.$$

QUESTION 6. Use a *separate* Writing Booklet.

Marks

(a) The function $f(x) = \sec x$ for $0 \le x < \frac{\pi}{2}$, and is not defined for other values of x. **4**

(i) State the domain of the inverse function $f^{-1}(x)$.

(ii) Show that $f^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right)$.

(iii) Hence find $\frac{d}{dx} f^{-1}(x)$.

(b) An amount $\$A$ is borrowed at $r\%$ per annum reducible interest, calculated monthly. The loan is to be repaid in equal monthly instalments of $\$M$. **8**

Let $R = \left(1 + \frac{r}{1200}\right)$ and let $\$B_n$ be the amount owing after n monthly repayments have been made.

(i) Show that $B_n = AR^n - M\left(\frac{R^n - 1}{R - 1}\right)$.

Pat borrows \$30 000 at 9% per annum reducible interest, calculated monthly. The loan is to be repaid in 60 equal monthly instalments.

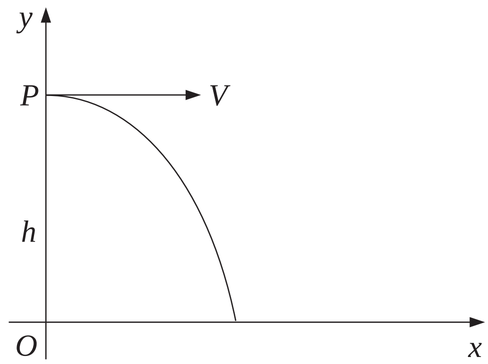
(ii) Show that the monthly repayments should be \$622·75.

(iii) With the twelfth repayment, Pat pays an additional \$5000, so this payment is \$5622·75. After this, repayments continue at \$622·75 per month. How many more repayments will be needed?

QUESTION 7. Use a *separate* Writing Booklet.

Marks

(a) **9**



A particle is projected horizontally from a point P, h metres above O, with a velocity of V metres per second. The equations of motion of the particle are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g.$$

(i) Using calculus, show that the position of the particle at time t is given by

$$x = Vt, \quad y = h - \frac{1}{2}gt^2.$$

A canister containing a life raft is dropped from a plane to a stranded sailor. The plane is travelling at a constant velocity of 216 km/h, at a height of 120 metres above sea level, along a path that passes above the sailor.

(ii) How long will the canister take to hit the water? $\left(\text{Take } g = 10 \text{ m/s}^2.\right)$

(iii) A current is causing the sailor to drift at a speed of 3·6 km/h in the same direction as the plane is travelling. The canister is dropped from the plane when the horizontal distance from the plane to the sailor is D metres. What values can D take if the canister lands at most 50 metres from the stranded sailor?

(b) (i) Simplify $n\binom{n-1}{1} + n\binom{n-1}{2} + \cdots + n\binom{n-1}{n-2}$. **3**

(ii) Find the smallest positive integer n such that

$$n\binom{n-1}{1} + n\binom{n-1}{2} + \cdots + n\binom{n-1}{n-2} > 20\,000.$$

BLANK PAGE

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$