BOARD OF STUDIES

NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1997

MATHEMATICS

4 UNIT (ADDITIONAL)

Time allowed—Three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a *separate* Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1. Use a *separate* Writing Booklet.

Marks

(a) Evaluate $\int_0^5 \frac{2}{\sqrt{x+4}}\,dx$. **2**

(b) Evaluate $\int_0^{\frac{\pi}{4}} \frac{\sin\theta}{\cos^4\theta}\,d\theta$. **3**

(c) Find $\int \frac{1}{x^2+2x+3}\,dx$. **2**

(d) Find $\int \frac{4t-6}{(t+1)(2t^2+3)}\,dt$. **4**

(e) Evaluate $\int_0^{\frac{\pi}{3}} x\sec^2 x\,dx$. **4**

QUESTION 2. Use a *separate* Writing Booklet.

Marks

(a) (i) Express $\sqrt{3} - i$ in modulus–argument form. **4**

(ii) Hence evaluate $\left(\sqrt{3} - i\right)^6$.

(b) (i) Simplify $(-2i)^3$. **4**

(ii) Hence find all complex numbers z such that $z^3 = 8i$. Express your answers in the form $x + iy$.

(c) Sketch the region where the inequalities **3**

$$|z - 3 + i| \leq 5 \quad \text{and} \quad |z + 1| \leq |z - 1|$$

both hold.

(d) Let $w = \dfrac{3 + 4i}{5}$ and $z = \dfrac{5 + 12i}{13}$, so that $|w| = |z| = 1$. **4**

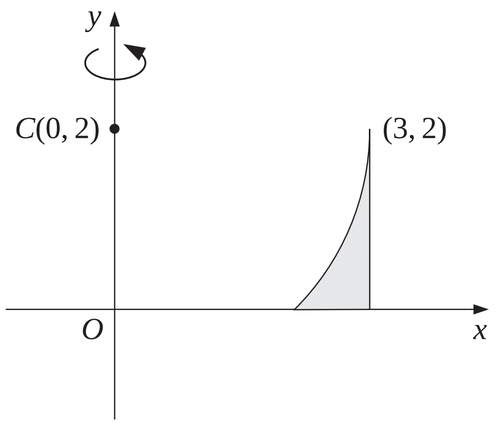
(i) Find wz and $w\bar{z}$ in the form $x + iy$.

(ii) Hence find two distinct ways of writing 65^2 as the sum $a^2 + b^2$, where a and b are integers and $0 < a < b$.

QUESTION 3. Use a *separate* Writing Booklet.

Marks

(a) **4**



In the diagram, the shaded region is bounded by the x axis, the line $x = 3$, and the circle with centre $C(0, 2)$ and radius 3.

Find the volume of the solid formed when the region is rotated about the y axis.

(b) Let $f(x) = 3x^5 - 10x^3 + 16x$. **6**

(i) Show that $f'(x) \geq 1$ for all x.

(ii) For what values of x is $f''(x)$ positive?

(iii) Sketch the graph of $y = f(x)$, indicating any turning points and points of inflection.

(c) In a game, two players take turns at drawing, and immediately replacing, a marble from a bag containing two green and three red marbles. The game is won by player A drawing a green marble, or player B drawing a red marble. Player A draws first. **5**

Find the probability that:

(i) A wins on her first draw;

(ii) B wins on her first draw;

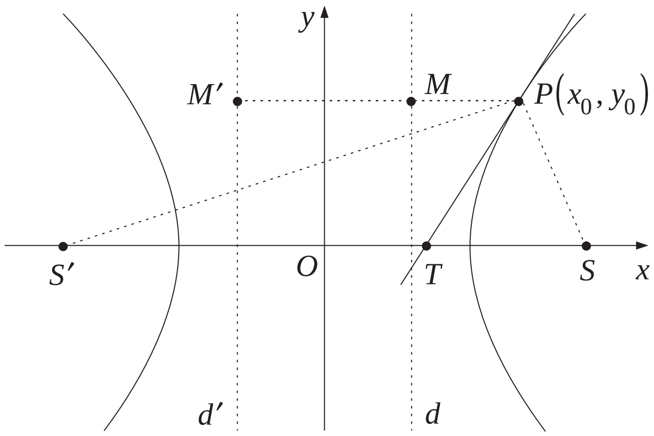
(iii) A wins in less than four of her turns;

(iv) A wins eventually.

QUESTION 4. Use a *separate* Writing Booklet.

Marks

(a) **7**



The point $P(x_0, y_0)$ lies on the hyperbola $\frac{x^2}{9} - \frac{y^2}{7} = 1$. The tangent to the hyperbola at P cuts the x axis at T, and has equation

$$\frac{x_0 x}{9} - \frac{y_0 y}{7} = 1 .$$

The two foci of the hyperbola are S and S', and the two directrices are d and d'. The points M and M' are the closest points to P on the directrices d and d'.

(i) Find the coordinates of the foci.

(ii) Find the equations of the directrices.

(iii) Show that T has coordinates $\left(\frac{9}{x_0}, 0\right)$.

(iv) Using the focus–directrix definition, or otherwise, show that

$$\frac{PS}{PS'} = \frac{TS}{TS'} .$$

(b) (i) Find an expression for $\cot 2A$ in terms of $\tan A$. **8**

(ii) Show that $\tan A$ and $-\cot A$ satisfy the equation

$$x^2 + 2x\cot 2A - 1 = 0 .$$

(iii) Hence, or otherwise, find the exact value of $\tan\frac{\pi}{8}$.

(iv) Hence find the exact value of $\tan\frac{\pi}{16} - \cot\frac{\pi}{16}$.

QUESTION 5. Use a *separate* Writing Booklet.

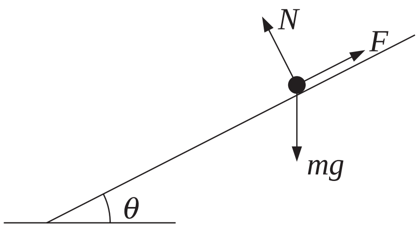
Marks

(a) (i) Sketch the graph of $y = -x^2$ for $-2 \le x \le 2$. **5**

(ii) Hence, without using calculus, sketch the graph of $y = e^{-x^2}$ for $-2 \le x \le 2$.

(iii) The region between the curve $y = e^{-x^2}$, the x axis, the y axis, and the line $x = 2$ is rotated around the y axis to form a solid. Using the method of cylindrical shells, find the volume of the solid.

(b) **3**



A particle of mass m is lying on an inclined plane and does not move. The plane is at an angle θ to the horizontal. The particle is subject to a gravitational force mg, a normal reaction force N, and a frictional force F parallel to the plane, as shown in the diagram.

Resolve the forces acting on the particle, and hence find an expression for $\dfrac{F}{N}$ in terms of θ.

(c) Suppose that b and d are real numbers and $d \ne 0$. Consider the polynomial **7**

$$P(z) = z^4 + bz^2 + d.$$

The polynomial has a double root α.

(i) Prove that $P'(z)$ is an odd function.

(ii) Prove that $-\alpha$ is also a double root of $P(z)$.

(iii) Prove that $d = \dfrac{b^2}{4}$.

(iv) For what values of b does $P(z)$ have a double root equal to $\sqrt{3}\,i$?

(v) For what values of b does $P(z)$ have real roots?

QUESTION 6. Use a *separate* Writing Booklet.

Marks

(a) The series $1 - x^2 + x^4 - \cdots + x^{4n}$ has $2n+1$ terms. **7**

(i) Explain why

$$1 - x^2 + x^4 - \cdots + x^{4n} = \frac{1+x^{4n+2}}{1+x^2}.$$

(ii) Hence show that

$$\frac{1}{1+x^2} \le 1 - x^2 + x^4 - \cdots + x^{4n} \le \frac{1}{1+x^2} + x^{4n+2}.$$

(iii) Hence show that, if $0 \le y \le 1$, then

$$\tan^{-1} y \le y - \frac{y^3}{3} + \frac{y^5}{5} - \cdots + \frac{y^{4n+1}}{4n+1} \le \tan^{-1} y + \frac{1}{4n+3}.$$

(iv) Deduce that $0 < \left(1 - \frac{1}{3} + \frac{1}{5} - \cdots + \frac{1}{1001}\right) - \frac{\pi}{4} < 10^{-3}$.

(b) A ball of mass 2 kilograms is thrown vertically upward from the origin with an initial speed of 8 metres per second. The ball is subject to a downward gravitational force of 20 newtons and air resistance of $\left(v^2/5\right)$ newtons in the opposite direction to the velocity, v metres per second. Hence, until the ball reaches its highest point, the equation of motion is **8**

$$\ddot{y} = -\frac{v^2}{10} - 10,$$

where y metres is its height.

(i) Using the fact that $\ddot{y} = v\dfrac{dv}{dy}$, show that, while the ball is rising,

$$v^2 = 164e^{-y/5} - 100.$$

(ii) Hence find the maximum height reached.

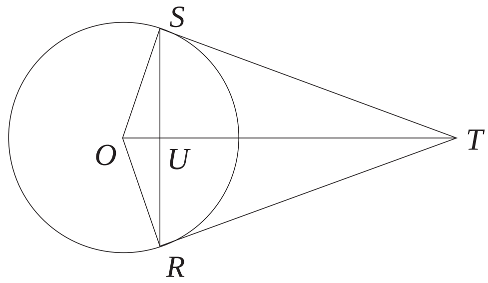
(iii) Using the fact that $\ddot{y} = \dfrac{dv}{dt}$, find how long the ball takes to reach this maximum height.

(iv) How fast is the ball travelling when it returns to the origin?

QUESTION 7. Use a *separate* Writing Booklet.

Marks

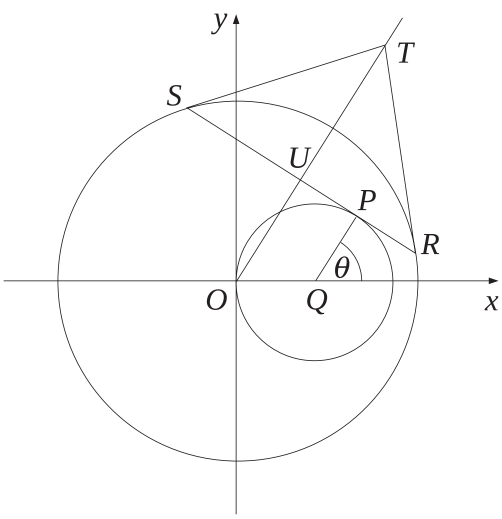
(a) **2**



The points R and S lie on a circle with centre O and radius 1. The tangents to the circle at R and S meet at T. The lines OT and RS meet at U, and are perpendicular.

Show that $OU \times OT = 1$.

(b) **7**



The circle $(x-r)^2 + y^2 = r^2$, with centre $Q(r, 0)$ and radius r, lies inside the circle $x^2 + y^2 = 1$, with centre O and radius 1. The point $P(r + r\cos\theta, r\sin\theta)$ lies on the inner circle, and P and O do not coincide. The tangent to the inner circle at P meets the outer circle at R and S, and the tangents to the outer circle at R and S meet at T. The lines OT and RS meet at U, and are perpendicular.

(i) Show that OT is parallel to QP.

(ii) Show that the equation of RS is $x\cos\theta + y\sin\theta = r(1+\cos\theta)$.

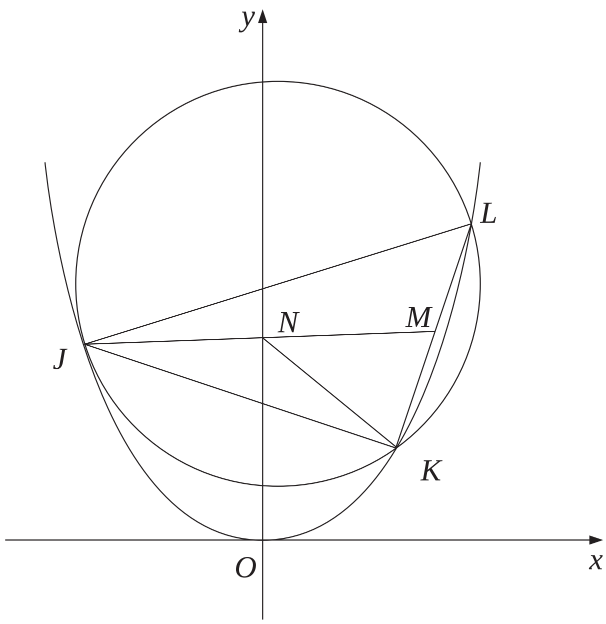
(iii) Find the length of OU.

(iv) By using the result of part (a), show that T lies on the curve $r^2y^2 + 2rx = 1$.

QUESTION 7. (Continued)

Marks

(c) **6**



The parabola $x^2 = 4ay$ touches the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at J, and cuts it at K and L. The midpoint of KL is M, and the line JM cuts the y axis at N, as shown on the diagram.

(i) Find a quartic equation whose roots are the x coordinates of J, K, and L.

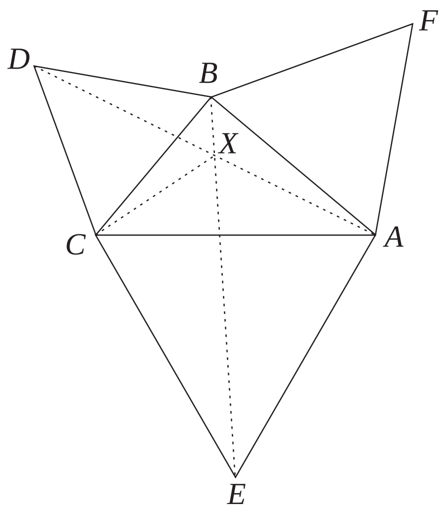
(ii) Show that $JN = NM$.

(iii) Hence show that the area of ΔJKN is one-quarter of the area of ΔJKL.

QUESTION 8. Use a *separate* Writing Booklet.

Marks

(a) **10**



Triangle ABC is scalene. External equilateral triangles ABF, BCD, and CAE are constructed on the sides of triangle ABC as shown.

Lines AD and BE meet at X.

Copy or trace this diagram into your Writing Booklet.

(i) Show that $\angle BCE = \angle DCA$.

(ii) Show that $\Delta BCE \equiv \Delta DCA$.

(iii) Show that $BDCX$ is a cyclic quadrilateral.

(iv) Show that $\angle BXD = \angle DXC = \angle CXE = \angle EXA = \dfrac{\pi}{3}$.

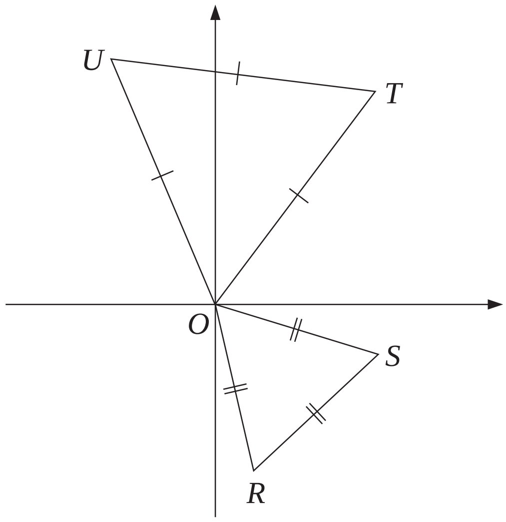
(v) Show that CF passes through X.

(vi) Show that $AD = BE = CF$.

QUESTION 8. (Continued)

Marks

(b) **5**



The diagram shows points O, R, S, T, and U in the complex plane. These points correspond to the complex numbers 0, r, s, t, and u respectively. The triangles ORS and OTU are equilateral. Let $\omega = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

(i) Explain why $u = \omega t$.

(ii) Find the complex number r in terms of s.

(iii) Using complex numbers, show that the lengths of RT and SU are equal.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x > 0$