BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1998
MATHEMATICS
3 UNIT (ADDITIONAL) AND 3/4 UNIT (COMMON)

Time allowed—Two hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

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QUESTION 1. Use a SEPARATE Writing Booklet. **Marks**

(a) Differentiate $2x\tan^{-1}x$. **2**

(b) Find the acute angle between the lines $3y = 2x + 8$ and $y = 5x - 9$. **2**

(c) Evaluate $\lim\limits_{x\to 0}\dfrac{\sin 3x}{5x}$. **1**

(d) Given that $\log_2 7 = 2{\cdot}807$ (to three decimal places), find $\log_2 14$. **1**

(e) Let α, β and γ be the roots of the polynomial $2x^3 - 14x - 1 = 0$. Find $\alpha\beta\gamma$. **2**

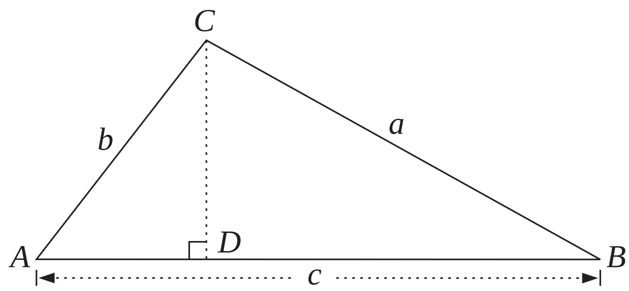
(f) Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \sin^2 x \; dx$. **4**

QUESTION 2. Use a SEPARATE Writing Booklet. **Marks**

(a) Find the quotient, $Q(x)$, and the remainder, $R(x)$, when the polynomial $P(x) = x^4 - x^2 + 1$ is divided by $x^2 + 1$. **3**

(b) Paul plans to contribute to a retirement fund. He will invest \$500 on each birthday from age 25 to 64 inclusive. That is, he will make 40 contributions to the fund. The retirement fund pays interest on the investments at the rate of 8% per annum, compounded annually. How much money will be in Paul's fund on his 65th birthday? **4**

(c) **5**



The triangle ABC has sides of length a, b and c, as shown in the diagram. The point D lies on AB, and CD is perpendicular to AB.

(i) Show that $a \sin B = b \sin A$.

(ii) Show that $c = a \cos B + b \cos A$.

(iii) Given that $c^2 = 4ab \cos A \cos B$, show that $a = b$.

QUESTION 3. Use a SEPARATE Writing Booklet. **Marks**

(a) Use the method of mathematical induction to prove that $4^n + 14$ is a multiple of 6 for $n \geq 1$. **3**

(b) (i) Express $\sin 4t + \sqrt{3}\cos 4t$ in the form $R\sin(4t + \alpha)$, where α is in radians. **4**

(ii) Hence, or otherwise, find the general solution of the equation $\sin 4t + \sqrt{3}\cos 4t = 0$ in exact form.

(c) A particle moves in a straight line and its position at time t is given by **5**

$$x = 1 + \sin 4t + \sqrt{3}\cos 4t\,.$$

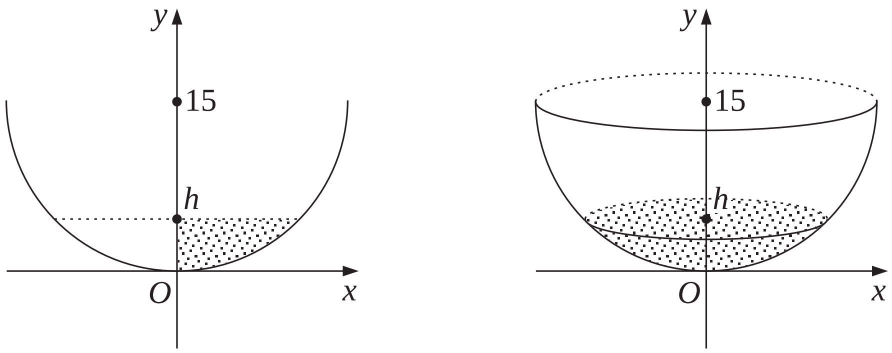
(i) Prove that the particle is undergoing simple harmonic motion about $x = 1$.

(ii) Find the amplitude of the motion.

(iii) When does the particle first reach maximum speed after time $t = 0$?

QUESTION 4. Use a SEPARATE Writing Booklet. **Marks**

(a) **5**



The left-hand diagram shows the lower half of the circle $x^2 + (y - 15)^2 = 15^2$. The shaded area in this diagram is bounded by the semicircle, the line $y = h$, and the y axis.

(i) Show that the volume V formed when the shaded area is rotated around the y axis is given by

$$V = 15\pi h^2 - \frac{\pi h^3}{3}.$$

A semicircle is rotated around the y axis to form a hemispherical bowl of radius 15 cm, as shown in the right-hand diagram.

(ii) The bowl is filled with water at a constant rate of 3 $\text{cm}^3\ \text{s}^{-1}$. Find the rate at which the water level is rising when the water level is 6 cm.

(b) Consider the function $f(x) = 1 + \dfrac{3}{(x-2)}$ for $x > 2$. **5**

(i) Give the equations of the horizontal and vertical asymptotes for $y = f(x)$.

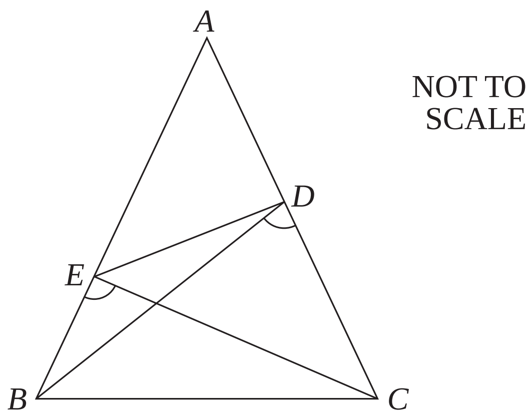
(ii) Find the inverse function $f^{-1}(x)$.

(iii) State the domain of $f^{-1}(x)$.

Question 4 continues on page 7

QUESTION 4. (Continued) **Marks**

(c) **2**



ABC is an acute-angled triangle. *D* is a point on *AC*, *E* is a point on *AB*, and $\angle BEC = \angle BDC$, as shown in the diagram.

Sonya was asked to prove that $\angle AED = \angle ACB$. She provided a two-step proof but did not give reasons.

(i) State a reason for her correct statement that *EDCB* is a cyclic quadrilateral.

(ii) State a reason why she could then correctly conclude that $\angle AED = \angle ACB$.

QUESTION 5. Use a SEPARATE Writing Booklet. **Marks**

(a) Write $8 + 27 + \cdots + n^3$ using Σ notation. **1**

(b) The polynomial $P(x) = 4x^3 + 2x^2 + 1$ has one real root in the interval $-1 < x < 0$. **6**

(i) Sketch the graph of $y = P(x)$ for x between -1 and 1. Clearly label any stationary points.

(ii) Let $x = -\frac{1}{4}$ be a first approximation to the root. Apply Newton's method once to obtain another approximation to the root.

(iii) Explain why the application of Newton's method in part (ii) was NOT effective in improving the approximation to the root.

(c) A tank contains 10 tagged fish and 50 untagged fish. On each day, 4 fish are selected at random from the tank and placed together in a separate tank for observation. Later the same day the 4 fish are returned to the original tank. **5**

(i) What is the probability of selecting no tagged fish on a given day?

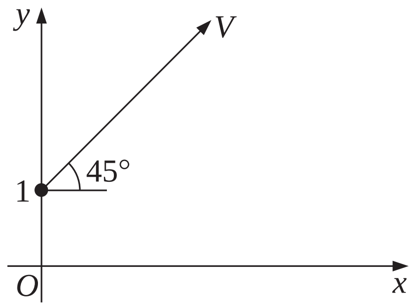
(ii) What is the probability of selecting at least one tagged fish on a given day?

(iii) Calculate the probability of selecting no tagged fish on every day for 7 given days.

(iv) What is the probability of selecting no tagged fish on exactly 2 of the 7 days?

QUESTION 6. Use a SEPARATE Writing Booklet. **Marks**

(a) **8**



A particle is projected from the point (0, 1) at an angle of 45° with a velocity of V metres per second. The equations of motion of the particle are

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g\,.$$

(i) Using calculus, derive the expressions for the position of the particle at time t. Hence show that the path of the particle is given by

$$y = 1 + x - g\frac{x^2}{V^2}\,.$$

A volleyball player serves a ball with initial speed V metres per second and angle of projection 45°. At that moment the bottom of the ball is 1 metre above the ground and its horizontal distance from the net is 9·3 metres. The ball just clears the net, which is 2·3 metres high.

(ii) Show that the initial speed of the ball is approximately 10·3 metres per second. (Take g = 9·8 m s^{-2}.)

(iii) What is the horizontal distance from the net to the point where the ball lands?

(b) The acceleration a m s^{-2} of a particle P moving in a straight line is given by **4**

$$a = 3\left(1 - x^2\right),$$

where x metres is the displacement of the particle to the right of the origin. Initially the particle is at the origin and is moving with a velocity of 4 m s^{-1}.

(i) Show that the velocity v m s^{-1} of the particle is given by

$$v^2 = 16 + 6x - 2x^3.$$

(ii) Will the particle ever return to the origin? Justify your answer.

QUESTION 7. Use a SEPARATE Writing Booklet. **Marks**

(a) (i) Use the binomial theorem to obtain an expansion for **2**

$$(1+x)^{2n}+(1-x)^{2n},$$

where n is a positive integer.

(ii) Hence evaluate $1 + {}^{20}C_2 + {}^{20}C_4 + \cdots + {}^{20}C_{20}$.

(b) (i) Use the substitution $y=\sqrt{x}$ to find **7**

$$\int \frac{dx}{\sqrt{x(1-x)}} .$$

(ii) Use the substitution $z = x - \frac{1}{2}$ to find another expression for

$$\int \frac{dx}{\sqrt{x(1-x)}} .$$

(iii) Use the results of parts (i) and (ii) to express $\sin^{-1}(2x-1)$ in terms of $\sin^{-1}\left(\sqrt{x}\right)$ for $0 < x < 1$.

(c) Find all real x such that **3**

$$|4x-1| > 2\sqrt{x(1-x)} .$$

End of paper

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STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$