BOARD OF STUDIES
NEW SOUTH WALES

HIGHER SCHOOL CERTIFICATE EXAMINATION

1999
MATHEMATICS
4 UNIT (ADDITIONAL)

Time allowed—Three hours
(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 12.
- Board-approved calculators may be used.
- Answer each question in a SEPARATE Writing Booklet.
- You may ask for extra Writing Booklets if you need them.

QUESTION 1 Use a SEPARATE Writing Booklet. **Marks**

(a) Evaluate $\int_0^1 xe^{-x^2}\,dx$. **2**

(b) Using the substitution $u = e^x$, or otherwise, find $\int \frac{e^x\,dx}{\sqrt{1-e^{2x}}}$. **2**

(c) Find $\int \frac{4x^3 - 2x^2 + 1}{2x-1}\,dx$. **3**

(d) (i) Find constants a, b and c such that **4**

$$\frac{x^2+2x}{(x^2+4)(x-2)} = \frac{ax+b}{x^2+4} + \frac{c}{x-2}.$$

(ii) Hence find $\int \frac{x^2+2x}{(x^2+4)(x-2)}\,dx$.

(e) Use integration by parts to evaluate $\int_0^{\frac{\pi}{2}} x^2 \sin x\,dx$. **4**

QUESTION 2 Use a SEPARATE Writing Booklet. **Marks**

(a) Let $z = 3 + 2i$ and $w = -1 + i$. Express the following in the form $a + ib$, where a and b are real numbers: **3**

(i) zw

(ii) $\dfrac{2}{iw}$.

(b) Let $\alpha = 1 + i\sqrt{3}$. **4**

(i) Find the exact value of $|\alpha|$ and $\arg \alpha$.

(ii) Find the exact value of α^{11} in the form $a + ib$, where a and b are real numbers.

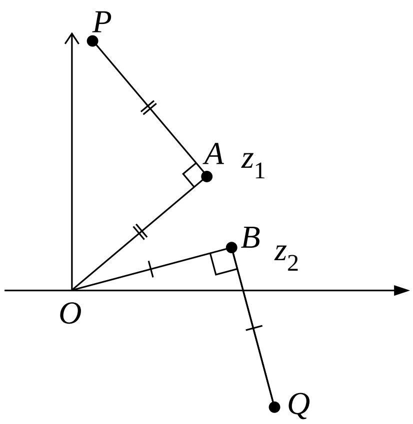
(c) Sketch the region in the Argand diagram where the two inequalities $|z - i| \le 2$ and $0 \le \arg(z + 1) \le \dfrac{\pi}{4}$ both hold. **2**

(d) Consider the equation $2z^3 - 3z^2 + 18z + 10 = 0$. **2**

(i) Given that $1 - 3i$ is a root of the equation, explain why $1 + 3i$ is another root.

(ii) Find all roots of the equation.

(e) **4**



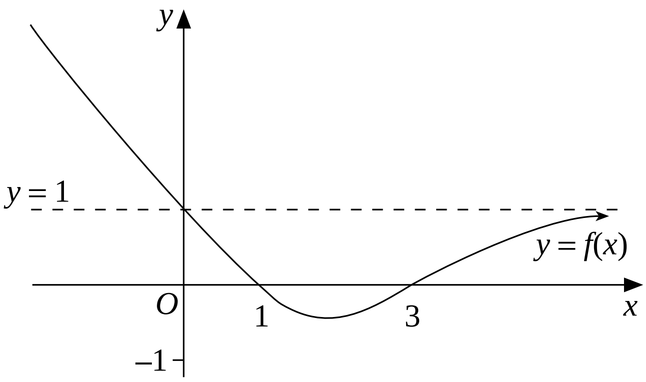
The points A and B in the complex plane correspond to complex numbers z_1 and z_2 respectively. Both triangles OAP and OBQ are right-angled isosceles triangles.

(i) Explain why P corresponds to the complex number $(1 + i)z_1$.

(ii) Let M be the midpoint of PQ. What complex number corresponds to M?

QUESTION 3 Use a SEPARATE Writing Booklet. **Marks**

(a) **6**



The diagram shows the graph of the function $y = f(x)$. The graph has a horizontal asymptote at $y = 1$.

Draw separate half-page sketches of the graphs of the following functions:

(i) $y = |\, f(x) \,|$

(ii) $y = \dfrac{1}{f(x)}$

(iii) $y = \ln f(x)$.

(b) Consider the ellipse E with equation **9**

$$\frac{x^2}{5^2} + \frac{y^2}{3^2} = 1$$

and let $P = (x_0, y_0)$ be an arbitrary point on E.

(i) Calculate the eccentricity of E.

(ii) Find the coordinates of the foci of E and the equations of the directrices of E.

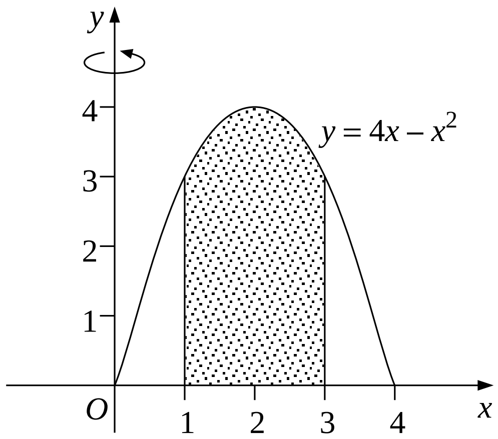
(iii) Show that the equation of the tangent at P is

$$\frac{x_0 x}{5^2} + \frac{y_0 y}{3^2} = 1.$$

(iv) Let the tangent at P meet a directrix at a point L. Show that $\angle PFL$ is a right angle where F is the corresponding focus.

QUESTION 4 Use a SEPARATE Writing Booklet. **Marks**

(a) **4**



The shaded area shown on the diagram between the curve $y = 4x - x^2$, the x axis, $x = 1$ and $x = 3$, is rotated about the y axis to form a solid. Use the method of cylindrical shells to find the volume of the solid.

(b) (i) Suppose the polynomial $P(x)$ has a double root at $x = \alpha$. Prove that $P'(x)$ also has a root at $x = \alpha$. **6**

(ii) The polynomial $A(x) = x^4 + ax^2 + bx + 36$ has a double root at $x = 2$. Find the values of a and b.

(iii) Factorise the polynomial $A(x)$ of part (ii) over the real numbers.

(c) (i) Determine the domain of the function $\sin^{-1}(3x + 1)$. **5**

(ii) Sketch the graph of the function $y = \sin^{-1}(3x + 1)$.

(iii) Solve $\sin^{-1}(3x + 1) = \cos^{-1}x$.

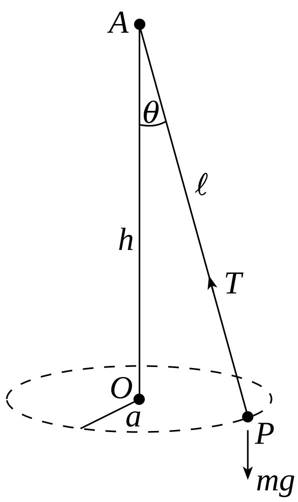
QUESTION 5 Use a SEPARATE Writing Booklet. **Marks**

(a) The roots of $x^3 + 5x^2 + 11 = 0$, are α, β and γ. **3**

(i) Find the polynomial equation whose roots are α^2, β^2 and γ^2.

(ii) Find the value of $\alpha^2 + \beta^2 + \gamma^2$.

(b) **4**



A conical pendulum consists of a bob P of mass m kg and a string of length ℓ metres. The bob rotates in a horizontal circle of radius a and centre O at a constant angular velocity of ω radians per second. The angle OAP is θ and $OA = h$ metres. The bob is subject to a gravitational force of mg newtons and a tension in the string of T newtons.

(i) Write down the magnitude, in terms of ω, of the force acting on P towards centre O.

(ii) By resolving forces, show that $\omega^2 = \dfrac{g}{h}$.

Question 5 continues on page 7

QUESTION 5 (Continued) **Marks**

(c) At time t a wasp population consists of $w(t)$ workers and $r(t)$ reproductives. For the first s days of the wasp season the population produces workers only and after s days the population produces reproductives only. **8**

(i) For $0 \leq t \leq s$, suppose that the equations determining the number of workers are

$$\frac{dw}{dt} = k_1 w \quad \text{and} \quad w(0) = 1\,,$$

where k_1 is a positive constant.

Find an expression for $w(s)$.

(ii) For $t \geq s$, suppose that the equations determining the number of reproductives are

$$\frac{dr}{dt} = k_2 w(s) \quad \text{and} \quad r(s) = 0\,,$$

where k_2 is a positive constant.

Show that $r(t) = k_2 e^{k_1 s}(t - s)$ for $t \geq s$.

(iii) If $k_1 = 0{\cdot}04$, find the value of s which maximises $r(100)$.

QUESTION 6 Use a SEPARATE Writing Booklet.

(a) (i) Let x be a fixed, non-zero number satisfying $x > -1$. Use the method of mathematical induction to prove that **4**

$$(1 + x)^n > 1 + nx$$

for $n = 2, 3, \ldots$.

(ii) Deduce that $\left(1 - \frac{1}{2n}\right)^n > \frac{1}{2}$ for $n = 2, 3, \ldots$.

Question 6 continues on page 8

QUESTION 6 (Continued) **Marks**

(b) A ball of unit mass is projected vertically upwards from ground level with initial speed U. Assume that air resistance is kv, where v is the ball's speed and k is a positive constant. **11**

We wish to consider the ball's motion as it falls back to ground level. Let y be the displacement of the ball measured *vertically downwards* from the point of maximum height, t be the time elapsed after the ball has reached maximum height, and g be the acceleration due to gravity.

(i) Explain why $v(0) = 0$, and $\dfrac{dv}{dt} = g - kv$ while the ball is in motion.

(ii) Deduce that $v = \dfrac{g}{k}\left(1 - e^{-kt}\right)$ for $t \geq 0$.

(iii) By writing $\dfrac{dv}{dt} = v\dfrac{dv}{dy}$, deduce from part (i) that

$$\frac{g}{k}\log_e\left(\frac{g-kv}{g}\right) + v = -ky\ .$$

(iv) Using parts (ii) and (iii), deduce that $t = \dfrac{v+ky}{g}$.

(v) You are given that the ball reaches maximum height

$$h = \frac{1}{k}\left(U - \frac{g}{k}\log_e\left(\frac{g+kU}{g}\right)\right)$$

in time $t_h = \dfrac{1}{k}\log_e\left(\dfrac{g+kU}{g}\right)$.

(Do NOT prove these results.)

Deduce that the total time T that the ball is in the air is $T = \dfrac{U+V}{g}$, where V is the final speed that the ball reaches when returning to ground level.

(vi) If air resistance is ignored, the total time T_0 that the ball is in the air is $T_0 = \dfrac{U+V_0}{g}$, where V_0 is the final speed the ball then reaches when returning to ground level. By considering V_0 and V, determine which is larger: T or T_0.

QUESTION 7 Use a SEPARATE Writing Booklet. **Marks**

(a) (i) Graph $y = \ln x$ and draw the tangent to the graph at $x = 1$. **10**

(ii) By considering the appropriate area under the tangent, deduce that

$$\int_1^{\frac{3}{2}} \ln x \, dx \leq \frac{1}{8}.$$

(iii) By considering the graph of $y = \ln x$, explain why

$$\int_{k-\frac{1}{2}}^{k+\frac{1}{2}} \ln x \, dx \leq \ln k \quad \text{for} \quad k = 2, 3, 4, \ldots .$$

(iv) Deduce that

$$\int_1^n \ln x \, dx \leq \frac{1}{8} + \ln 2 + \ln 3 + \cdots + \ln(n-1) + \frac{1}{2}\ln n \quad \text{for} \quad n = 2, 3, 4, \ldots .$$

(v) Assuming that $\int_1^n \ln x \, dx = n \ln n - n + 1$, deduce that

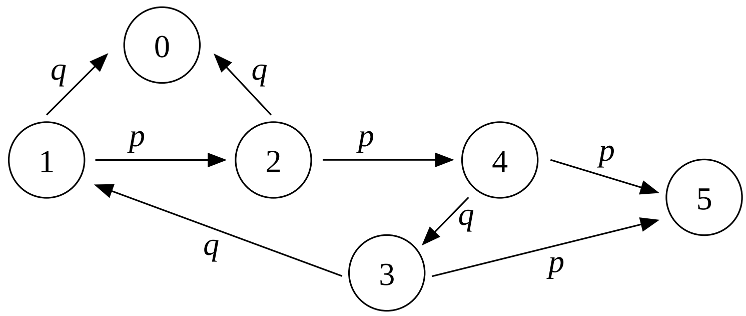
$$n! \geq e^{\frac{7}{8}} n^n \sqrt{n}\, e^{-n} \quad \text{for} \quad n = 2, 3, 4, \ldots .$$

Question 7 continues on page 10

QUESTION 7 (Continued) **Marks**

(b) A player has one token and needs exactly five tokens to win a prize. He plays a game where he can vary the number of tokens he bets. At each stage he either doubles the number of tokens he bets or loses the tokens he bets. The probability that he doubles the number of tokens he bets is p and the probability that he loses the number of tokens he bets is $q = 1 - p$. His strategy is to reach his goal of exactly five tokens as quickly as possible. **5**

The diagram shows the possible outcomes in terms of number of tokens and the probabilities associated with each stage.



(i) Starting with one token, what is the probability that he loses all of his tokens without ever having four tokens?

(ii) What is the probability that he obtains four tokens once and then loses all of his tokens without ever having four tokens again?

(iii) If $p = \frac{1}{2}$, find the probability that he wins a prize.

QUESTION 8 Use a SEPARATE Writing Booklet. **Marks**

(a) Let $\rho = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$. The complex number $\alpha = \rho + \rho^2 + \rho^4$ is a root of the quadratic equation $x^2 + ax + b = 0$, where a and b are real. **8**

(i) Prove that $1 + \rho + \rho^2 + \dots + \rho^6 = 0$.

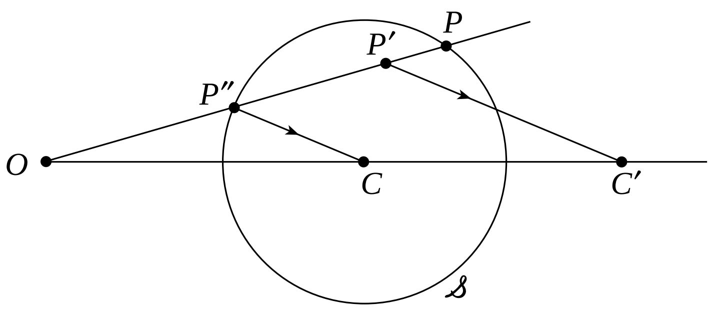
(ii) The second root of the quadratic equation is β. Express β in terms of positive powers of ρ. Justify your answer.

(iii) Find the values of the coefficients a and b.

(iv) Deduce that

$$-\sin\frac{\pi}{7} + \sin\frac{2\pi}{7} + \sin\frac{3\pi}{7} = \frac{\sqrt{7}}{2} .$$

(b) **7**



In the diagram, $\mathcal{S}$ is a circle, centre C, and O is a fixed point outside the circle. The point P is a variable point on $\mathcal{S}$ and P'' is the other point of intersection of OP with $\mathcal{S}$. The point P' is on OP such that $OP \cdot OP' = k^2$ where k is a constant. The point C' is on OC and $P''C \parallel P'C'$.

(i) Explain why $OP \cdot OP''$ is a constant.

(ii) Deduce that $\dfrac{OP''}{OP'}$ is a constant.

(iii) Show that C' is a fixed point.

(iv) Describe fully the locus of P'.

End of paper

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \, \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$